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LINEAR PRESERVERS FOR SYLVESTER AND FROBENIUS BOUNDS ON MATRIX RANK

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ABSTRACT. Let A and B be $n \times n$ matrices. A classical result about the rank function is Sylvester's inequality which states that the rank of the product of AB is at most $\min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$ and at least $\operatorname{rank}(A) + \operatorname{rank}(B) - n$. A generalization of Sylvester's inequality is Frobenius's inequality which states that

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(ABC) + \operatorname{rank}(B).$$

In this paper we investigate the structure of linear operators that preserve those ordered pairs or triples of matrices which satisfy one of the extreme cases in these inequalities.

1. Introduction. Let $\mathbf{F}$ be any field, and let $\mathcal{M}_n(\mathbf{F})$ denote the space of all $n \times n$ matrices with entries from $\mathbf{F}$. Let $\rho(A)$ denote the rank of A. Let $E_{i,j}$ be the matrix with a "1" in the (i,j) position and zero elsewhere.

Definition 1.1. If $T : \mathcal{M}_n(\mathbf{F}) \to \mathcal{M}_n(\mathbf{F})$ is a linear operator, we say that T is a (U,V)-*operator* provided there exist nonsingular matrices $U, V \in \mathcal{M}_n(\mathbf{F})$ such that either

1. $T(X) = UXV$ for all $X \in \mathcal{M}_n(\mathbf{F})$ or
2. $T(X) = UX^tV$ for all $X \in \mathcal{M}_n(\mathbf{F})$,

where X^t denotes the transpose of X.

Note that it follows that T is a (U,V)-operator if and only if T is a composition of operators of type 1 above and the transpose operator.

Some classical inequalities concerning the rank of sums and products are:

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The rank sum inequality.

$$|\rho(A)-\rho(B)| \leq \rho(A+B) \leq \rho(A)+\rho(B);$$

Sylvester's inequality.

$$\rho(A)+\rho(B)-n \leq \rho(AB) \leq \min\{\rho(A),\rho(B)\};$$

and

Frobenius's inequality.

$$\rho(AB)+\rho(BC) \leq \rho(ABC)+\rho(B).$$

Here A, B, C are arbitrary matrices from $M_n(\mathbf{F})$.

Definition 1.2. Given a set, $\mathcal{F}$, of ordered pairs of matrices in $\mathcal{M}_n(\mathbf{F}) \times \mathcal{M}_n(\mathbf{F})$ we say that $T : \mathcal{M}_n(\mathbf{F}) \to \mathcal{M}_n(\mathbf{F})$ *preserves* the set $\mathcal{F}$ if $(A, B) \in \mathcal{F}$ implies that $(T(A), T(B)) \in \mathcal{F}$. Similarly, if $\mathcal{F}$ is a set of ordered triples then we say that T *preserves* $\mathcal{F}$ if and only if $(A, B, C) \in \mathcal{F}$ implies that $(T(A), T(B), T(C)) \in \mathcal{F}$.

In this paper we shall investigate linear operators which preserve pairs or triples of matrices which attain one of the extremes of the inequalities above.

Let

$$\mathcal{Q}_1 = \Big\{(A,B) \mid \rho(A+B) = \rho(A)+\rho(B)\Big\};$$
$$\mathcal{Q}_2 = \Big\{(A,B) \mid \rho(A+B) = |\rho(A)-\rho(B)|\Big\};$$
$$\mathcal{Q}_3 = \Big\{(A,B) \mid \rho(AB) = \min\{\rho(A),\rho(B)\}\Big\};$$
$$\mathcal{Q}_4 = \Big\{(A,B) \mid \rho(AB) = \rho(A)+\rho(B)-n\Big\};$$

and

$$\mathcal{Q}_5 = \Big\{(A,B,C) \mid \rho(AB)+\rho(BC) = \rho(ABC)+\rho(B)\Big\}.$$

It was shown in [**1, 3, 6**] that linear operators that preserve $\mathcal{Q}_1$ or $\mathcal{Q}_2$ are (U, V)-operators. Here we investigate linear operators that preserve $\mathcal{Q}_3$, $\mathcal{Q}_4$, or $\mathcal{Q}_5$.

In order to characterize linear preservers for these extreme rank conditions, we need the following lemma which is an easy corollary from Dieudonné [**5**], see also [**2**, Section 2.1].

Lemma 1.3 [**2, 5**]. *Let $\mathbf{F}$ be an arbitrary field and $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ an invertible linear transformation. If T preserves the set of rank-n matrices, or the set of rank-1 matrices, then T is a (U, V)-operator.*

2. Preservers of the set $\mathcal{Q}_3$.

Throughout this section $\mathbf{F}$ will denote an arbitrary field. We begin with a couple of lemmas.

Lemma 2.1. *If $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ preserves the set $\mathcal{Q}_3$ and T is invertible, then T preserves the set of rank-1 matrices.*

Proof. Suppose that T^{-1} does not preserve rank-1 matrices. Then there is some matrix A such that $\rho(A) = k$, $k > 1$, and $\rho(T(A)) = 1$. Since similarity operators preserve $\mathcal{Q}_3$, we may assume without loss of generality that $A = \begin{bmatrix} A_1 \\ O \end{bmatrix}$ where A_1 is $k \times n$, and $T(A) = \begin{bmatrix} \mathbf{a}^t \\ O \end{bmatrix}$; here $\mathbf{a}^t$ denotes a certain nonzero row of the matrix $T(A)$.

Now, if $\mathcal{H}$ is a space of matrices such that for each nonzero $H \in \mathcal{H}$, $HT(A) \neq O$, we must have that $\dim \mathcal{H} \leq n$. (The dimension of the complement of $\mathcal{H}$ is greater than or equal to $n(n-1)$ since all matrices with zero first column and arbitrary columns from 2nd until nth annihilate $T(A)$.)

Let $\mathcal{K} = \{B = [B_1\, O] \in M_n(\mathbf{F}) \mid B_1 \text{ is } n \times k\}$. Then $\dim \mathcal{K} = kn$. Let $B \in \mathcal{K}$. Then $\rho(BA) = \rho(B) = \min(\rho(A), \rho(B))$. Thus, $(B, A) \in \mathcal{Q}_3$. It follows that $(T(B), T(A)) \in \mathcal{Q}_3$, so that $\rho(T(B)T(A)) = \min(\rho(T(B)), \rho(T(A))) = 1$. Thus, for each $C \in T(\mathcal{K})$, $\rho(CT(A)) = 1$, or $CT(A) \neq O$. Therefore, from the above observation, $\dim T(\mathcal{K}) \leq n$.

But T is invertible so that $\dim T(\mathcal{K}) = nk$, a contradiction. Thus T^{-1}, and hence T, preserves the set of rank-1 matrices. $\square$

Lemma 2.2. *Let $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ be defined by $T(X) = UXV$ for some invertible matrices U and V. Then T preserves the set $\mathcal{Q}_3$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix $P \in M_n(\mathbf{F})$ and nonzero scalar $\alpha \in \mathbf{F}$.*

Proof. It is easy to check that the transformation $T(X) = \alpha PXP^{-1}$ preserves the set $\mathcal{Q}_3$.

It is enough to consider transformations of the form $X \to XD$, where D is an arbitrary invertible matrix, instead of $T(X) = UXV$ since the similarity transformation preserves $\mathcal{Q}_3$ and $U^{-1}T(X)U = XVU = XD$. To prove the lemma we need to show that the matrix $D = (d_{ij})$ is scalar.

1. Let us show first that $d_{ii} \neq 0$ for all i, $i = 1, \dots, n$. For arbitrary i we consider the matrices $A_1 = E_{i,i}$, $B_1 = E_{i,j}$, for some $j \neq i$. Thus $(A_1, B_1) \in \mathcal{Q}_3$ since $\rho(A_1B_1) = 1 = \rho(A_1) = \rho(B_1)$. The matrix D is invertible, so we have that $\rho(A_1D) = 1$, $\rho(B_1D) = 1$, $\rho(A_1DB_1D) = \rho(A_1DB_1)$. Hence, $A_1DB_1 \neq O$. On the other hand,

$$A_1D = d_{i1}E_{i,1} + \cdots + d_{in}E_{i,n}.$$

Thus $A_1DB_1 = d_{ii}E_{i,j}$. Therefore, $d_{ii} \neq 0$ for all i, $i = 1, \dots, n$.

2. Let us assume now that there exists i, j, $i \neq j$, such that $d_{ij} \neq 0$. Then consider the matrices $A_2 = E_{j,j} - (d_{jj}/d_{ij})E_{j,i}$, $B_2 = E_{j,i}$. We have $A_2B_2 = E_{j,i}$. Therefore, $\rho(A_2) = \rho(B_2) = \rho(A_2B_2) = 1$. Hence, $(A_2, B_2) \in \mathcal{Q}_3$. Therefore, $(A_2D, B_2D) \in \mathcal{Q}_3$. The matrix D is invertible. Thus, $\rho(A_2D) = 1$ and $\rho(B_2D) = 1$. Then $\rho(A_2DB_2D) = 1$. Hence, $\rho(A_2DB_2D) = \rho(A_2DB_2) = 1$. On the other hand,

$$\begin{aligned} A_2DB_2 &= \left(E_{j,j} - \frac{d_{jj}}{d_{ij}}\,E_{j,i}\right)DE_{j,i} = E_{j,j}DE_{j,i} - \frac{d_{jj}}{d_{ij}}\,E_{j,i}DE_{j,i} \\ &= d_{jj}E_{j,i} - \frac{d_{jj}}{d_{ij}}\,(d_{i1}E_{j,1} + \cdots + d_{in}E_{j,n})E_{ji} \\ &= d_{jj}E_{j,i} - \frac{d_{jj}}{d_{ij}}\,d_{ij}E_{j,j}E_{j,i} \\ &= d_{jj}E_{j,i} - d_{jj}E_{j,i} = O. \end{aligned}$$

Thus $\rho(A_2DB_2) = 0$, a contradiction. Thus the matrix D is diagonal.

3. It remains to check that $D = \operatorname{diag}(d_{11}, \dots, d_{nn})$ is indeed a scalar matrix.

Assume that D is not scalar. Then there exists an index i such that $d_{ii} \neq d_{i+1\,i+1}$. Let us consider the block-diagonal matrices

$$A_3 = \begin{bmatrix} I_{i-1} & O & O \\ O & L & O \\ O & O & I_{n-i-1} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} I_{i-1} & O & O \\ O & M & O \\ O & O & I_{n-i-1} \end{bmatrix}$$

where L and M are the following 2×2-matrices:

$$L = \begin{bmatrix} d_{i+1\,i+1} & d_{i\,i} \\ 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix},$$

and I_k denotes the identity matrix of size k.

Then

$$A_3B_3 = \begin{bmatrix} I_{i-1} & O & O \\ O & N & O \\ O & O & I_{n-i-1} \end{bmatrix},$$

where $N = \left[\begin{smallmatrix} d_{i+1\,i+1} - d_{i\,i} & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Thus, we have $\rho(A_3) = n-1$, $\rho(B_3) = n-1$, $\rho(A_3B_3) = n-1$ if and only if $d_{ii} \neq d_{i+1\,i+1}$, i.e., $(A_3, B_3) \in Q_3$. On the other hand,

$$A_3DB_3 = \begin{bmatrix} d_{11} & & & & & & & \\ & \ddots & & & & & & \\ & & d_{i-1\,i-1} & & & & & \\ & & & 0 & 0 & & & \\ & & & 0 & 0 & & & \\ & & & & & d_{i+2\,i+2} & & \\ & & & & & & \ddots & \\ & & & & & & & d_{nn} \end{bmatrix},$$

implying that $\rho(A_3DB_3D) = n-2$. Hence, $(A_3D, B_3D) \notin Q_3$. This is a contradiction. $\square$

Theorem 2.3. *If* $\mathbf{F}$ *is an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *is an invertible linear transformation, then* T *preserves the set* $\mathcal{Q}_3$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* $P \in M_n(\mathbf{F})$ *and nonzero scalar* $\alpha \in \mathbf{F}$.

Proof. By Lemma 2.1, T preserves the set of rank-1 matrices. By assumptions T is invertible. Thus, by Lemma 1.3, we have that T is a (U, V)-operator. By Lemma 2.2, if T has the form $T(X) = UXV$, then $T(X) = \alpha PXP^{-1}$ for some invertible matrix P.

Suppose $T(X) = UX^tV$. Since similarity preserves $\mathcal{Q}_3$ we may assume that $T(X) = X^tD$ where $D = VU^{-1}$ is invertible. Suppose that $k \neq i$. Then $(D^{-1})^t E_{i,j} E_{j,k} = (D^{-1})^t E_{i,k}$, i.e., $((D^{-1})^t E_{ij}, E_{j,k}) \in \mathcal{Q}$, but $((D^{-1})^t E_{i,j})^t D E_{j,k}^t D = E_{j,i} E_{k,j} = O$, so that $(T((D^{-1})^t E_{i,j}), T(E_{j,k})) \notin \mathcal{Q}_3$. Thus $T(X) = UXV$ does not preserve $\mathcal{Q}_3$. $\square$

Finally, we remark that linear preservers of $\mathcal{Q}_3$ may be singular and nontrivial even over algebraically closed fields.

Example 2.4. Let $\mathbf{F}$ be an arbitrary field, and let the linear transformation $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ be defined by $T(E_{1,1}) = E_{1,1}$, $T(E_{1,2}) = E_{1,2} + E_{2,1}$, and $T(E_{i,j}) = O$ for all $(i, j) \neq (1, 1)$ or $(1, 2)$. Let $A, B \in M_n(\mathbf{F})$, say $A = \begin{bmatrix} a & b & * \\ * & * & * \end{bmatrix}$ and $B = \begin{bmatrix} c & d & * \\ * & * & * \end{bmatrix}$. Then

$$T(A)T(B) = \begin{bmatrix} a & b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} ac + bd & ad & 0 \\ bc & bd & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is routine to show that T preserves $\mathcal{Q}_3$ since any pair in the image of T is in $\mathcal{Q}_3$.

3. Preservers of the set $\mathcal{Q}_4$.

Lemma 3.1. *If* $\mathbf{F}$ *is an arbitrary field and the linear transformation* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *preserves the set* $\mathcal{Q}_4$, *then* T *preserves the set of rank-*n *matrices.*

Proof. Let $A = O$, and let B be any nonsingular matrix. Then, $\rho(A) = 0$ and $\rho(B) = n$. Also, $\rho(AB) = 0$, so that $\rho(AB) = \rho(A)+$

$\rho(B) - n$. It follows that $\rho(T(A)T(B)) = \rho(T(A)) + \rho(T(B)) - n$. That is, $0 = 0 + \rho(T(B)) - n$. It follows that $\rho(T(B)) = n$. That is, T preserves nonsingular matrices. □

Corollary 3.2. *Let* $\mathbf{F}$ *be an algebraically closed field. Assume that the linear transformation* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *preserves the set* $\mathcal{Q}_4$*. Then* T *is invertible.*

Proof. By Lemma 3.1 the transformation T preserves the set of invertible matrices. Linear preservers of invertible matrices over algebraically closed fields are nonsingular, see [**7**, Lemma 2.3] for the complex case and [**4**, Theorem 2] for an arbitrary case. Hence, T is bijective. □

Lemma 3.3. *Let* $\mathbf{F}$ *be an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *defined by* $T(X) = UXV$ *for some invertible matrices* U *and* V*. Then* T *preserves the set* $\mathcal{Q}_4$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* P *and nonzero scalar* $\alpha \in \mathbf{F}$.

Proof. It is easy to see that transformation $T(X) = \alpha PXP^{-1}$ preserves $\mathcal{Q}_4$.

Similarity preserves $\mathcal{Q}_4$. Thus, as in the proof of Lemma 2.2, without loss of generality we assume that $T(X) = XD$ for some nonsingular matrix D. It is enough to show that D is a scalar matrix.

1. We first show that D is diagonal. In order to do this we consider the following matrices:

For any $1 \le i \le n$ we denote $J_i = I - E_{i,i}$. Let us take the matrices $A_i = E_{i,i}$, $B_i = J_i$. We denote

$$D_i = B_i D = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \vdots \\ \mathbf{d}_{i-1} \\ 0 \\ \mathbf{d}_{i+1} \\ \vdots \\ \mathbf{d}_n \end{bmatrix};$$

here $\mathbf{d}_k$ is the kth row of the matrix D. One has that $\rho(A_iB_i) = 0 = \rho(A_i)+\rho(B_i)-n$ so that $(A_i, B_i) \in \mathcal{Q}_4$. It follows that $\rho(A_iDB_iD) = O$. Since the ith row of A_iDB_iD is zero, and the ith row of $\mathbf{d}_iD_i$ is the ith row of $A_iDB_iD = 0$, we have that $\mathbf{d}_iD_i$ is zero. So the ith row of D is orthogonal to all columns of matrix D_i. One has that $\rho(D_i) = n-1$ since D is invertible. But orthogonality gives a linear relation between $(n-1)$ nonzero rows of matrix D_i. Thus this relation is trivial, i.e., $d_{i,j} = 0$ for all $j \neq i$. Since D is nonsingular we have that $d_{i,i} \neq 0$. That is, D is a nonsingular diagonal matrix.

2. In order to prove that D is scalar, we consider $A_i' = E_{i,i} + E_{i,i+1}$, $B_i' = E_{1,1} + \cdots + E_{i,i} - E_{i+1,i} + E_{i+2,i+2} + \cdots + E_{n,n}$. Then $A_i'B_i' = E_{i,i} - E_{i,i} = O$, $\rho(A_i') + \rho(B_i') = 1 + (n-1) = n$. So we have that $(A_i', B_i') \in \mathcal{Q}_4$. Thus, $(A_i'D, B_i'D) \in \mathcal{Q}_4$. Since $\rho(A_i'D) = \rho(A_i')$ and $\rho(B_i'D) = \rho(B_i')$, it follows that $\rho(A_i'DB_i'D) = 0$. Therefore, $A_i'DB_i' = O$. On the other hand, one has

$$\begin{aligned} A_i'DB_i' &= (E_{i,i} + E_{i,i+1})(d_{11}E_{1,1} + \cdots + d_{nn}E_{n,n}) \\ &\quad \times (E_{1,1} + \cdots + E_{i,i} - E_{i+1,i} + E_{i+2,\,i+2} + \cdots + E_{n,n}) \\ &= (d_{ii}E_{i,i} + d_{i+1\,i+1}E_{i,i+1}) \\ &\quad \times (E_{1,1} + \cdots + E_{i,i} - E_{i+1,i} + E_{i+2,\,i+2} + \cdots + E_{n,n}) \\ &= (d_{ii} - d_{i+1\,i+1})E_{i,i}. \end{aligned}$$

Hence, $d_{ii} = d_{i+1\,i+1}$ for all $i = 1, \ldots, n$. Thus D is a scalar matrix. □

Theorem 3.4. *Let $\mathbf{F}$ be an arbitrary field. Then the bijective linear transformation $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ preserves the set $\mathcal{Q}_4$ if and only if $T(X) = \alpha PXP^{-1}$ for some invertible matrix $P \in M_n(\mathbf{F})$ and nonzero scalar $\alpha \in \mathbf{F}$.*

Proof. It is easy to check that if $T(X) = \alpha PXP^{-1}$ for some invertible $P \in M_n(\mathbf{F})$ then T preserves $\mathcal{Q}_4$.

By Lemma 3.1, T preserves the set of nonsingular matrices. Thus by Lemma 1.3, T has the form $T(X) = UXV$ since we assume its invertibility. By Lemma 3.3, if T has the form $T(X) = UXV$, then $UV = D$ for some nonsingular scalar matrix D.

Suppose $T(X) = UX^tV$. Since similarity preserves $\mathcal{Q}_3$ we may assume that $T(X) = X^tD$ where $D = VU^{-1}$ is invertible. Note that $J_i^t = (I - E_{i,i})^t = J_i$ for all i, $i = 1, \ldots, n$. It is easily seen that $((D^{-1})^tE_{i,j}, J_j) \in \mathcal{Q}_4$, but $((((D^{-1})^tE_{i,j})^t)D, J_j^tD) \notin \mathcal{Q}_4$ since $((D^{-1})^tE_{i,j})^tDJ_j^tD = E_{j,i}J_jD = E_{j,i}D \neq O$. Thus $T(X) = UX^tV$ does not preserve $\mathcal{Q}_4$. $\square$

Corollary 3.5. *Let* $\mathbf{F}$ *be an algebraically closed field. Then the linear transformation* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *preserves the set* $\mathcal{Q}_4$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* $P \in M_n(\mathbf{F})$ *and nonzero scalar* $\alpha \in \mathbf{F}$.

Proof. By Corollary 3.2, T is invertible. Hence Theorem 3.4 concludes the proof. $\square$

4. Preservers of the set $\mathcal{Q}_5$.

Lemma 4.1. *Let* $\mathbf{F}$ *be an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *a bijective linear transformation that maps* $\mathcal{Q}_5$ *into* $\mathcal{Q}_5$. *Then* T *preserves invertible matrices.*

Proof. Consider the triple A, B, C, where $A = O$, $B \in M_n(\mathbf{F})$ is arbitrary, $C \in M_n(\mathbf{F})$ is invertible. Then it is straightforward to check that $(A, B, C) \in \mathcal{Q}_5$. Then $(T(A), T(B), T(C)) \in \mathcal{Q}_5$, that is

$$\rho(T(A)T(B)) + \rho(T(B)T(C)) = \rho(T(A)T(B)T(C)) + \rho(T(B)).$$

However, $T(A) = O$ since $A = O$ and T is linear. Thus one has

$$\rho(T(B)T(C)) = \rho(T(B)) \tag{1}$$

for all matrices B. Since T is bijective, it follows that $T(C)$ is invertible. Indeed, $T(B)$ runs through all $M_n(\mathbf{F})$ as far as B does. If $T(C)$ is singular, then it is a zero divisor in $M_n(\mathbf{F})$. Thus there exists a nonzero matrix B such that $T(B)T(C) = O$ and equality (1) does not hold. It is a contradiction. $\square$

Our next lemmas will show that preservers of $\mathcal{Q}_5$ are indeed invertible.

Lemma 4.2. *If* $\mathbf{F}$ *is an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *is a linear transformation which preserves the set* $\mathcal{Q}_5$*, then there are no rank-n matrices in* $\ker T$ *unless* $T \equiv O$*.*

Proof. Suppose T preserves $\mathcal{Q}_5$ and $T(A) = O$ for some A with $\rho(A) = n$. Then $\rho(AB)+\rho(BA) = \rho(ABA)+\rho(B)$ for any $B \in M_n(\mathbf{F})$. Thus $(A, B, A) \in \mathcal{Q}_5$, and hence $(O, T(B), O) \in \mathcal{Q}_5$, which implies that $T(B) = O$. Thus, $T \equiv O$. $\square$

Lemma 4.3. *If* $\mathcal{F}$ *is any field and* A *is an* $m \times n$ *matrix over* $\mathcal{F}$ *of rank-k, then, for some positive integers* k_1 *and* k_2 *such that* $k_1+k_2 = k$*,* A *is similar to a matrix of the form*

$$\begin{bmatrix} X & O \\ O_{k-k_1,k} & O \\ Y & O \\ O_{m-k-k_2,k} & O \end{bmatrix}$$

where X *is* $k_1 \times k$ *and* Y *is* $k_2 \times k$*. Necessarily,* $\rho(X) = k_1$ *and* $\rho(Y) = k_2$*.*

Proof. Let Q be a matrix such that Q^tA^t is in reduced row echelon form. Necessarily, Q^tA^t has all zeros in rows $k + 1, \cdots, n$. Thus AQ has all zeros in columns $k + 1, \cdots, n$. But then $B = Q^{-1}AQ$ has all zeros in columns $k + 1, \cdots, n$. So $B = \left[\begin{smallmatrix} B_1 & O \\ B_2 & O \end{smallmatrix}\right]$ where B_1 is $k \times k$. Let P be the $k \times k$ matrix such that PB_1 is in reduced row echelon form. Let R be the $(n - k) \times k$ matrix such that

$$\begin{bmatrix} I_k & O \\ R & I_{n-k} \end{bmatrix} (P \oplus I_{n-k})B = C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} PB_1 & O \\ RPB_1 + B_2 & O \end{bmatrix}$$

so that if j is a pivot column of PB_1, then the jth column of RPB_1+B_2 has all zero entries. Finally, let S be the $(n - k) \times (n - k)$ matrix such that SC_2 is in reduced row echelon form. Then,

$$(I_k \oplus S)C = D = \begin{bmatrix} D_1 & O \\ O_{k-k_1,k} & O \\ D_2 & O \\ O_{n-k-k_2,k} & O \end{bmatrix}$$

where D_1 is $k_1 \times k$ and D_2 is $k_2 \times k$ for some nonnegative integers k_1 and k_2 (k_1 is the rank of B_1).

Now,

$$\begin{aligned}(I_k \oplus S)\begin{bmatrix} I_k & O \\ R & I_{n-k}\end{bmatrix}&(P \oplus I_{n-k})Q^{-1}AQ(P \oplus I_{n-k})^{-1} \\ &\times \begin{bmatrix} I_k & O \\ R & I_{n-k}\end{bmatrix}^{-1}(I_k \oplus S)^{-1} \\ &= D(P^{-1} \oplus I_{n-k})\begin{bmatrix} I_k & O \\ -R & I_{n-k}\end{bmatrix}(I_k \oplus S^{-1}) \\ &= \begin{bmatrix} D_1P^{-1} & O \\ O_{k-k_1,k} & O \\ D_2P^{-1} & O \\ O_{n-k-k_2,k} & O\end{bmatrix}\end{aligned}$$

has the desired form where $X = D_1P^{-1}$ and $Y = D_2P^{-1}$. □

Lemma 4.4. *If* $\mathbf{F}$ *is an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *is a linear transformation which preserves* $\mathcal{Q}_5$*, then either* $T \equiv O$ *or* T *is invertible.*

Proof. Suppose $T \not\equiv 0$, $A \in \ker T$ and $\rho(A) \geq \rho(Z)$ for all $Z \in \ker T$. Let $\rho(A) = k$ and suppose $k \neq 0$. By Lemma 4.2, $k < n$. Since every similarity operator preserves $\mathcal{Q}_5$, by Lemma 4.3 we may assume that

$$A = \begin{bmatrix} A_1 & A_2 & O & O \\ O & O & O & O \\ A_3 & A_4 & O & O \\ O & O & O & O\end{bmatrix}$$

where A_1 is $k_1 \times k_1$, A_4 is $k_2 \times k_2$, $k_1 + k_2 = k$ and $k + k_2 \leq n$.

Case 1. $k_1 = k$. Here $A = \begin{bmatrix} A_1 & O \\ O & O\end{bmatrix}$. Let (i, j) be a pair such that $\det A[\{1, \cdots, k\} \setminus \{i\} \mid \{1, \cdots, k\} \setminus \{j\}] \neq 0$. Let $B = E_{k+1,j} + E_{i,k+1}$. Then $\rho(AB) = \rho(BA) = 1$ and $\rho(ABA) = 0$, so that $(A, B, A) \in \mathcal{Q}_5$. Thus, $T(B) = O$. Expanding along the last row we obtain

$$\begin{aligned}\det(A + B)&[\{1, \cdots, k+1\} \mid \{1, \cdots, k+1\}] \\ &= \pm\det(A + B)[\{1, \cdots, k\} \mid \{1, \cdots, k+1\}\setminus\{j\}],\end{aligned}$$

and then, expanding along the last column, we get

$$\begin{aligned}
&\det(A+B)[\{1,\cdots,k+1\} \mid \{1,\cdots,k+1\}] \\
&\qquad\qquad = \pm\det(A+B)[\{1,\cdots,k\}\backslash\{i\} \mid \{1,\cdots,k\}\backslash\{j\}] \\
&\qquad\qquad = \det A[\{1,\cdots,k\} \setminus \{i\} \mid \{1,\cdots,k\} \setminus \{j\}] \neq 0.
\end{aligned}$$

That is, $\rho(A+B) > k$ and $T(A+B) = O$, a contradiction to the choice of A.

Case 2. $k_1 < k$. Here

$$A = \begin{bmatrix} A_1 & A_2 & O & O \\ O & O & O & O \\ A_3 & A_4 & O & O \\ O & O & O & O \end{bmatrix}$$

and A_1 is $k_1 \times k_1$. Let $B = E_{k,k} + E_{k,k+1} + E_{k+1,k} + E_{k+1,k+1}$. Then, $\rho(AB) = \rho(BA) = 1$, and $\rho(ABA) \leq 1$. Now, by the Frobenius inequality, $2 = \rho(AB) + \rho(BA) \leq \rho(ABA) + \rho(B) = \rho(ABA) + 1$. Thus, $\rho(ABA) \geq 1$. Thus $\rho(ABA) = 1$, and hence $(A, B, A) \in \mathcal{Q}_5$. Consequently $T(B) = O$. Expanding the determinant along the last column three times and using its additivity by $(k+1)$st row we have

$$\begin{aligned}
&\det(A+B)[\{1,\cdots,k_1,k,\cdots,k+k_2\} \mid \{1,\cdots,k+1\}] \\
&\quad = -\det(A+B)[\{1,\cdots,k_1,k+1,\cdots,k+k_2\} \mid \{1,\cdots,k\}] \\
&\quad\quad + \det(A+B)[\{1,\cdots,k_1,k,k+2,\cdots,k+k_2\} \mid \{1,\cdots,k\}] \\
&\quad = -(\det A[\{1,\cdots,k_1,k+1,\cdots,k+k_2\} \mid \{1,\cdots,k\}] \\
&\quad\quad + \det A[\{1,\cdots,k_1,k+2,\cdots,k+k_2\} \mid \{1,\cdots,k-1\}]) \\
&\quad\quad + \det A[\{1,\cdots,k_1,k+2,\cdots,k+k_2\} \mid \{1,\cdots,k-1\}] \\
&\quad = -\det A[\{1,\cdots,k_1,k+1,\cdots,k+k_2\} \mid \{1,\cdots,k\}] \neq 0,
\end{aligned}$$

since $\rho(A) = k$. That is, $\rho(A+B) > k$ and $T(A+B) = O$, a contradiction to the choice of A.

Since we have reached a contradiction in each case, we conclude that $k = 0$ and the lemma follows. $\square$

Lemma 4.5. *Let* $\mathbf{F}$ *be an arbitrary field,* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *and* $T(X) = UXV$ *for some invertible matrices* U *and* V*. Then* T *preserves the set* $\mathcal{Q}_5$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* $P \in M_n(\mathbf{F})$ *and nonzero scalar* $\alpha \in \mathbf{F}$*.*

Proof. Let us consider arbitrary matrices $(Y, Z) \in \mathcal{Q}_3$. If $\rho(Y) \leq \rho(Z)$, then $\rho(YZ) = \rho(Y)$. Thus, $\rho(OY) + \rho(YZ) = \rho(OYZ) + \rho(Y)$, so that $(O, Y, Z) \in \mathcal{Q}_5$. Thus, $\rho(T(O)T(Y)) + \rho(T(Y)T(Z)) = \rho(T(O)T(Y)T(Z)) + \rho(T(Y))$. That is, $\rho(T(Y)T(Z)) = \rho(T(Y))$, and since $T(X) = UXV$, $\rho(T(Y)) \leq \rho(T(Z))$. Thus, $(T(Y), T(Z)) \in \mathcal{Q}_3$. If $\rho(Z) \leq \rho(Y)$, $(Y, Z, O) \in \mathcal{Q}_5$, and similar to the above argument, $(T(Y), T(Z)) \in \mathcal{Q}_3$. Thus, T preserves $\mathcal{Q}_3$. By Theorem 2.3 the lemma follows. $\square$

Theorem 4.6. *Let* $\mathbf{F}$ *be an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *a bijective linear transformation. Then* T *preserves the set* $\mathcal{Q}_5$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* $P \in M_n(\mathbf{F})$ *and nonzero scalar* $\alpha \in \mathbf{F}$*.*

Proof. If $T(X) = \alpha PXP^{-1}$ for some invertible $P \in M_n(\mathbf{F})$, then clearly T preserves $\mathcal{Q}_5$.

By Lemma 4.1, T preserves the set of nonsingular matrices. Thus, by Lemma 1.3, T is a (U, V)-operator.

Suppose $T(X) = UX^tV$. Since similarity preserves $\mathcal{Q}_3$ we may assume that $T(X) = X^tD$ where $D = VU^{-1}$ is invertible. It is easily seen that $((D^{-2})^tE_{i,j}, I, J_j) \in \mathcal{Q}_5$, but $(T((D^{-2})^tE_{i,j}), T(I), T(J_j)) \notin \mathcal{Q}_5$ since $((D^{-2})^tE_{i,j})^tDIDJ_j^tD = E_{j,i}J_jD = E_{j,i}D \neq O$. Thus, $T(X) = UX^tV$ does not preserve $\mathcal{Q}_5$. Thus, by Lemma 4.5, the theorem follows. $\square$

Corollary 4.7. *If* $\mathbf{F}$ *is an arbitrary field and* $T : M_n(\mathbf{F}) \to M_n(\mathbf{F})$ *is a linear transformation, then* T *preserves the set* $\mathcal{Q}_5$ *if and only if* $T(X) = \alpha PXP^{-1}$ *for some invertible matrix* $P \in M_n(\mathbf{F})$ *and scalar* $\alpha \in \mathbf{F}$*.*

Proof. By Lemma 4.4, $T \equiv O$ (here $\alpha = 0$) or T is invertible. By Theorem 4.6 the result follows. □

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