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A CHARACTERIZATION OF BOUNDARY CONDITIONS FOR REGULAR STURM-LIOUVILLE PROBLEMS WHICH HAVE THE SAME LOWEST EIGENVALUES

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ABSTRACT. In this paper we characterize the self-adjoint boundary conditions for the regular Sturm-Liouville problems which have the same lowest bound. In addition, we answer the equal cases of the inequalities among the minimal eigenvalues of the Sturm-Liouville problems [**4**].

1. Introduction. Let

$$\{\lambda_n(e^{i\theta}K):\ n \in \mathbf{N}_0 = \{0,1,2,\dots,\}\} \tag{1.1}$$

denote the eigenvalues, listed in nondecreasing order, of the Sturm-Liouville problem (SLP) consisting of the equation

$$-(py')'+qy = \lambda wy \qquad \text{on} \quad I := [a,b] \quad \text{with } -\infty < a < b < \infty, \tag{1.2}$$

and the coupled self-adjoint boundary condition (BC)

$$Y(b) = e^{-i\theta}KY(a), \tag{1.3}$$

where $i = \sqrt{-1}, -\pi < \theta < \pi$,

$$\begin{gathered} Y(t) = \begin{bmatrix} y(t) \\ y^{[1]}(t) \end{bmatrix}, \\ K \in \mathrm{SL}(2,\mathbf{R}) := \left\{ K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} : k_{ij} \in \mathbf{R},\ \det(K) = 1 \right\} \end{gathered} \tag{1.4}$$

and

$$p^{-1},\ q,\ w \in L(I,\mathbf{R}), \quad p,\ w > 0 \text{ a.e.}. \tag{1.5}$$

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Here $y^{[1]} := py'$ denotes the quasi-derivative of y, $L(I, \mathbf{R})$ denotes the set of real-valued Lebesgue integrable functions on I and $\mathbf{R}$ the set of real numbers.

For any $K \in \mathrm{SL}\,(2, \mathbf{R})$, let $\{\nu_n : n \in \mathbf{N}_0\}$ and $\{\gamma_n : n \in \mathbf{N}_0\}$ denote the eigenvalues of the following separated BC's

$$y(a) = 0, \qquad k_{22} y(b) - k_{12} y^{[1]}(b) = 0 \tag{1.6}$$

and

$$y^{[1]}(a) = 0, \qquad k_{21} y(b) - k_{11} y^{[1]}(b) = 0, \tag{1.7}$$

respectively. Eastham, Kong, Wu and Zettl [**4**] established a most general result of the inequalities among the eigenvalues of SLPs. This result generalizes some well known classical results of Weidmann [**6**], Eastham [**3**] et al., e.g., [**1, 2, 8**], for some special cases of the matrix K. These general inequalities can be written as follows.

Theorem 1.1. *Let* $K \in \mathrm{SL}\,(2, \mathbf{R})$.

(a) *If* $k_{11} > 0$ *and* $k_{12} \le 0$, *then* $\lambda_0(K)$ *is simple, and for any* $\theta \in (-\pi, \pi)$, $\theta \ne 0$, *we have*

$$\begin{aligned} \nu_0 \le \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\gamma_0, \nu_1\} \\ \le \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\gamma_1, \nu_2\} \le \cdots ; \end{aligned} \tag{1.8}$$

(b) *If* $k_{11} \le 0$ *and* $k_{12} < 0$, *then* $\lambda_0(K)$ *is simple, and for any* $\theta \in (-\pi, \pi)$, $\theta \ne 0$, *we have*

$$\begin{aligned} \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \le \{\gamma_0, \nu_0\} \\ \le \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \le \{\gamma_1, \nu_1\} \le \cdots ; \end{aligned} \tag{1.9}$$

(c) *If neither case* (a) *nor case* (b) *applies to* K, *then either case* (a) *or case* (b) *applies to* $-K$.

The purpose of this paper is to discuss the eigenvalue equalities problem related to (1.8)–(1.9), that is, to ascertain the conditions under which the equality $\lambda_0(K) = \nu_0$ or $\lambda_0(K) = \gamma_0$ holds in (1.8)–(1.9). To this end, we consider the following more general problem: characterize the self-adjoint boundary conditions of the SLP's which have the same lowest bound. That is, let L_0 denote the minimal operator associated with the SL expression. It is known [**6**, page 109] that the operator L_0 is symmetric and L_0 and all of its self-adjoint extensions are bounded below. Given a real constant μ_0 satisfying

$$\mu_0 \leq \Lambda_0(L_0) := \inf\{(L_0 y, y),\ y \in D(L_0), ||y|| = 1\}, \tag{1.10}$$

where $\Lambda_0(L_0)$ is called the lower bound of the operator L_0, we characterize all self-adjoint extensions, L, of L_0 such that their lower bound $\Lambda_0(L) = \mu_0$. This problem may be called the *bound-limited* self-adjoint extension problem. Particularly, when $\mu_0 = \Lambda_0(L_0)$, we call it the *bound-preserving* self-adjoint extension problem.

In [**7**], we provided a complete solution to the bound-preserving self-adjoint extension problem. In the present paper, we will present a close answer to the bound-limited self-adjoint extension problem. Through characterizing the necessary and sufficient condition for an operator to be a bound-limited self-adjoint extension of L_0, all possible forms of the bound-limited self-adjoint extensions of L_0 will be discriminated via a complete classification of self-adjoint BC's. When specialized to the eigenvalue equalities problem, these then naturally yield conditions under which the equalities hold among the minimal eigenvalues in (1.8)–(1.9).

The method used here is different from [**7**]. Based on a direct sum decomposition of the domain of the maximal operator associated with SL expression, we can directly characterize all positive self-adjoint extensions of L_0 when $\Lambda_0(L_0) > 0$. We will show that the positive self-adjoint extensions of L_0 are tightly related to the bound-limited self-adjoint extensions of L_0, Theorem 3.3. Thus, the crucial point of the present research is to find all possible bound-limited self-adjoint extensions of L_0 among the positive self-adjoint extensions of L_0.

This paper is organized as follows. In Section 2 we summarize some of the basic results needed in later discussion and notations. Section 3 contains the main results, characterizing all bound-limited self-adjoint extensions of L_0. In Section 4 we provide all of all possible explicit BC's for the bound-limited self-adjoint extensions. Finally, in Section 5 we obtain all matrices K such that $\lambda_0(K) = \nu_0$ or $\lambda_0(K) = \gamma_0$.

2. Notations and preliminaries. Let l be the differential expression associated with the SL kind differential equation (1.2) defined by

$$ly := w^{-1}[-(py')' + qy]. \tag{2.1}$$

We assume that (1.5) holds throughout the paper.

The operators associated with the differential expression l are studied in the weighted Hilbert space

$$L^2(I, w) \text{ with inner product } (y, z) = \int_I y(t)\bar{z}(t)w(t)\,dt.$$

Associated with the expression l, two differential operators $L_{\max}$ and L_0 respectively, called the *maximal operator* and the *minimal operator*, are defined as follows, see, for example, [**5, 6**]: let

(2.2)

$$D(L_{\max}) = \{y \in L^2(I, w) : y, y^{[1]} \in AC(I) \quad \text{and} \quad ly \in L^2(I, w)\},$$

$$D(L_0) = \{y \in D(L_{\max}) : Y(a) = 0 = Y(b)\}. \tag{2.3}$$

Then

$$L_{\max}y = ly, \qquad y \in D(L_{\max}),$$
$$L_0 y = ly, \qquad y \in D(L_0).$$

It is known [**5**] that $D(L_{\max})$ and $D(L_0)$ all are dense in $L^2(I, w)$; therefore, $L_{\max}$ has a unique adjoint $L^*_{\max}$, $L_0 = L^*_{\max}$, and L_0 is a semi-bounded symmetric operator with lower bound $\Lambda_0(L_0)$.

Denote

$$\hat{J}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$[y, z](t) = Y^T(t)\hat{J}_2\bar{Z}(t), \qquad \langle y, z\rangle(t) = Y^T(t)J_2\bar{Z}(t).$$

For any $y, z \in D(L_{\max})$, it is noted that Green's formula [**5**] and the Dirichlet formula [**7**] are respectively expressed as:

$$\int_a^b [(ly)\overline{z} - y(\overline{lz})]w(t)dt = [y, z](b) - [y, z](a) \tag{2.4}$$

and

$$\int_a^b [(ly)\overline{z} + y(\overline{lz})]w(t)\,dt - 2\int_a^b [p(t)y'\overline{z}' + q(t)y\overline{z}]\,dt = -\langle y,z\rangle(b) + \langle y,z\rangle(a). \tag{2.5}$$

Lemma 2.1. *Let L_F denote the Friedrichs extension of L_0. Then $L_F y = ly$, $y \in D(L_F)$, where*

$$D(L_F) = \{y \in D(L_{\max}) : y(a) = 0 = y(b)\}. \tag{2.6}$$

Proof. See [**8**, Section 5]. □

Under the assumption $\mu_0 < \lambda_0(L_0)$, let φ_1 and φ_2 be the real-valued solutions of the equation $ly = \mu_0 y$ determined by the initial conditions

$$\Phi_1(a) := \begin{bmatrix} \varphi_1(a) \\ \varphi_1^{[1]}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi_2(b) \\ \varphi_2^{[1]}(b) \end{bmatrix} =: \Phi_2(b). \tag{2.7}$$

Note that the solutions φ_1 and φ_2 are linearly independent. (Otherwise, there exists a constant c such that $\varphi_1 = c\varphi_2$. Then, we have $\varphi_1(b) = 0$ and μ_0 is an eigenvalue of the Friedrichs extension L_F. It is known [**8**] that L_F is a bound-preserving self-adjoint extension of L_0. Therefore, the spectral set $\sigma(L_F) \subset [\lambda_0(L_0), \infty)$ and hence $\mu_0 \geq \lambda_0(L_0)$. This contradicts the prerequisite assumption.) Furthermore, by the Green formula and (2.7), we easily deduce that

$$\varphi_1(b) = -\varphi_2(a), \quad \varphi_1(t) > 0 \quad \text{and} \quad \varphi_2(t) < 0, \quad t \in I. \tag{2.8}$$

Lemma 2.2. *Under the assumption $\mu_0 < \lambda_0(L_0)$, each $y \in D(L_{\max})$ can be uniquely represented as*

$$y = y_F + c_1\varphi_1 + c_2\varphi_2, \qquad y_F \in D(L_F), \tag{2.9}$$

where $c_1 = y(b)/\varphi_1(b)$ and $c_2 = y(a)/\varphi_2(a)$. Furthermore, for any y_F and $z_F \in D(L_F)$, we have

$$(L_F y_F, z_F) = \int_I [p(t)y_F'\overline{z}_F' + q(t)y_F\overline{z}_F]\,dt =: (y_F, z_F)_D. \tag{2.10}$$

Proof. Equation (2.9) is similar to that of [**7**, Lemma 2.4] and therefore omitted. Equation (2.10) follows from Lemma 2.1 and by making use of the Dirichlet formula (2.5). □

Remark 2.3. If $\lambda_0(L_0) > 0$, then $(\cdot,\cdot)_D$ on the linear manifold $D(L_F)$ forms an inner product and $D(L_0)$ is densely defined in $D(L_F)$ with respect to this inner product.

3. Bound limited self-adjoint extensions. In this section based on the direct sum decomposition of $D(L_{\max})$, see Lemma 2.2, we first characterize all positive self-adjoint extensions of L_0 when $\Lambda_0(L_0) > 0$. Then, we will identify the bound-limited self-adjoint extensions of L_0 from the positive self-adjoint extensions of L_0.

Our purpose is to find the bound-limited self-adjoint extensions of L_0 with $\mu_0 < \Lambda_0(L_0)$ by means of the positive self-adjoint extensions of L_0. Therefore we need to make the following assumption

$$\Lambda_0(L_0) > 0 \quad \text{and} \quad \mu_0 = 0. \tag{3.1}$$

This assumption will facilitate our subsequent discussion but does not actually impose any limitation for L_0 because, if necessary, we can consider the differential expression $l_{\mu_0} := l - \mu_0$ instead of l.

For any $y \in D(L_{\max})$, we denote

$$\mathbf{Y} = \begin{bmatrix} Y(a) \\ Y(b) \end{bmatrix}, \qquad \tilde{J}_4 = \begin{bmatrix} \hat{J}_2 & 0 \\ 0 & -\hat{J}_2 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 0 & \varphi_2(a) & 0 & 0 \\ \varphi_2(a) & 2\varphi_2^{[1]}(a) & 0 & 2 \\ 0 & 0 & 0 & \varphi_1(b) \\ 0 & 2 & \varphi_1(b) & 2\varphi_1^{[1]}(b) \end{bmatrix}. \tag{3.2}$$

Lemma 3.1. *Let* (3.1) *hold. Then, for any* $y \in D(L_{\max})$*, the following identities hold*

$$2\mathrm{Im}(L_{\max} y, y) = -i\mathbf{Y}^* \tilde{J}_4 \mathbf{Y} \tag{3.3}$$

and

$$2\mathrm{Re}(L_{\max}y,y)=2(y_F,y_F)_D+\varphi_2(a)\mathbf{Y}^*A^{-1}\mathbf{Y} \tag{3.4}$$

where $y_F=y-(y(b)/\varphi_1(b))\varphi_1-(y(a)/\varphi_2(a))\varphi_2$ *belongs to* $D(L_F)$*, see* (2.9)*, and* $(\cdot,\cdot)_D$ *is defined as in* (2.10)*.*

Proof. It is easy to verify from the Green formula (2.4) that the identity (3.3) holds. So, we only need to prove the identity (3.4). By (2.4), (2.9) and (2.10), for any $y\in D(L_{\max})$ we have

$$y=y_F+c_1\varphi_1+c_2\varphi_2,\quad y_F\in D(L_F), \tag{3.5}$$

and

(3.6)

$$\begin{aligned}(L_{\max}y,y)&=(L_{\max}y_F,y_F+c_1\varphi_1+c_2\varphi_2)\\&=(y_F,y_F)_D+[y_F,c_1\varphi_1+c_2\varphi_2](b)-[y_F,c_1\varphi_1+c_2\varphi_2](a)\\&=(y_F,y_F)_D+y_F^{[1]}(a)[\overline{c_1\varphi_1(a)+c_2\varphi_2(a)}]\\&\quad-y_F^{[1]}(b)[\overline{c_1\varphi_1(b)+c_2\varphi_2(b)}]\\&=(y_F,y_F)_D+y_F^{[1]}(a)\varphi_2(a)\bar c_2+y_F^{[1]}(b)\varphi_2(a)\bar c_1.\end{aligned}$$

If we write $\Gamma(y)=(y_F^{[1]}(a),y_F^{[1]}(b),c_1,c_2)^T$, then

$$2\mathrm{Re}\,(L_{\max}y,y)=2(y_F,y_F)_D+\varphi_2(a)\Gamma^*(y)J_4\Gamma(y).$$

Furthermore, from (2.6) and (2.7) we have

(3.7)

$$\begin{aligned}y(a)&=c_2\varphi_2(a),&\quad y(b)&=c_1\varphi_1(b),\\y^{[1]}(a)&=y_F^{[1]}(a)+c_1+c_2\varphi_2^{[1]}(a),&\quad y^{[1]}(b)&=y_F^{[1]}(b)+c_1\varphi_1^{[1]}(b)+c_2.\end{aligned}$$

This then implies $\mathbf{Y}=\Delta^*\Gamma(y)$, where

$$\Delta=\begin{bmatrix}0&1&0&0\\0&0&0&1\\0&1&\varphi_1(b)&\varphi_1^{[1]}(b)\\\varphi_2(a)&\varphi_2^{[1]}(a)&0&1\end{bmatrix}. \tag{3.8}$$

Simple calculations show that

$$A = \Delta^* J_4 \Delta, \qquad A^{-1} = \Delta^{-1} J_4 \Delta^{-1^*} \tag{3.9}$$

and (3.4) holds. This completes the proof of Lemma 3.1. □

The following theorem characterizes all positive self-adjoint extensions of L_0 under the assumption (3.1).

Theorem 3.2. *Let $\Lambda_0(L_0) > 0$ hold and the matrix A be defined by (3.2). An operator L is a positive self-adjoint extension of L_0 if and only if there exists a 2×4 matrix M such that*

$$\operatorname{rank} M = 2, \qquad M\tilde{J}_4 M^* = 0, \tag{3.10}$$

(3.11)
MAM^* is a positive definite or positive semi-definite matrix

and $Ly = L_{\max} y$, $y \in D(L)$, where

$$D(L) = \{y \in D(L_{\max}) : \ M\mathbf{Y} = 0\}. \tag{3.12}$$

Proof. Let us suppose that the operator L is a positive self-adjoint extension of L_0, that is, L is self-adjoint and satisfies $(Ly, y) \geq 0$ for all $y \in D(L)$. From the self-adjointness of L and [**8**, Section 4], there exists a 2×4 matrix M such that (3.10) and (3.12) are satisfied. On the other hand, we can show that $(Ly, y) \geq 0$ is equivalent to

$$\mathbf{Y}^* A^{-1} \mathbf{Y} \leq 0, \qquad \text{for all } y \in D(L). \tag{3.13}$$

Obviously, from (2.8), (3.1) and (3.4) we only need to prove (3.13). If it is not true, there then is a function y_1 in $D(L)$ satisfying $\mathbf{Y}_1^* A^{-1} \mathbf{Y}_1 =: 2\varepsilon_1 > 0$. From the definition of the Friedrichs extension, cf. [**8**, Section 5], and the representation (2.9) of y_1, $y_1 = y_{1F} + c_1\varphi_1 + c_2\varphi_2$, $y_{1F} \in D(L_F)$, we conclude that $D(L_0)$ is densely defined in $D(L_F)$ with respect to the inner product $(\cdot,\cdot)_D$ and, for the positive number $-\varphi_2(a)\varepsilon_1/2$, there exists a function y_0 in $D(L_0)$ such that

$(y_{1F}-y_0, y_{1F}-y_0)_D \le -\varphi_2(a)\varepsilon_1/2$. Since $D(L)$ is a extension manifold of $D(L_0)$, $y_1 - y_0 \in D(L)$ and

$$\begin{aligned}0 &\le (L(y_1-y_0), y_1-y_0)\\ &= (y_{1F}-y_0, y_{1F}-y_0)_D + (1/2)\varphi_2(a)\mathbf{Y}_1^* A^{-1}\mathbf{Y}_1\\ &\le (1/2)\varphi_2(a)\varepsilon_1 < 0.\end{aligned}$$

This contradiction shows that (3.13) holds. Furthermore, note that the mapping $\mathbf{Y} : D(L_{\max}) \to \mathbf{C}^4$ (the set of 4-dimensional column vectors on $\mathbf{C}$) is linear and surjective. If we write $\tilde{J}_4 M^*$ as $[\alpha_1, \alpha_2]$, that is, $\tilde{J}_4 M^* = [\alpha_1, \alpha_2]$, where α_1, $\alpha_2 \in \mathbf{C}^4$, then, from (3.10) and (3.12), we easily see that

$$D(L) = \{y \in D(L_{\max}) : \ \mathbf{Y} \in \operatorname{span}\{\alpha_1, \alpha_2\}\}. \tag{3.14}$$

This, combined with (3.13), yields that

$$\begin{bmatrix}\alpha_1^*\\ \alpha_2^*\end{bmatrix} A^{-1}[\alpha_1, \alpha_2] = M\tilde{J}_4^* A^{-1}\tilde{J}_4 M^*$$

is a negative definite or negative semi-definite matrix. Simple calculations show that

$$A = -\varphi_2(a)^2 \tilde{J}_4 A^{-1}\tilde{J}_4^*,$$

which shows that (3.11) holds. Thus, the necessary part of Theorem 3.2 is proved.

Conversely, if there is a 2×4 matrix M that satisfies (3.10), (3.11) and $D(L)$ satisfies (3.12), then, from [**8**, Section 4], we conclude that the operator L is self-adjoint. In addition, if we write $\tilde{J}_4 M^* = [\alpha_1, \alpha_2]$, then by (3.13) and (3.14) we can conclude that $(Ly, y) \ge 0$ for all $y \in D(L)$. This shows that L is a positive self-adjoint extension of L_0. We complete the proof of Theorem 3.2. □

The following theorem characterizes all bound-limited self-adjoint extensions of L_0 by means of the positive self-adjoint extensions of L_0.

Let

$$\Phi = \begin{bmatrix} 0 & 1 & \varphi_1(b) & \varphi_1^{[1]}(b)\\ \varphi_2(a) & \varphi_2^{[1]}(a) & 0 & 1\end{bmatrix}. \tag{3.15}$$

Let $\operatorname{diag}(A)$ denote all diagonal elements of the square matrix A and $\det(A)$ the determinant of A.

Theorem 3.3. *Under the assumption that the constant $\mu_0 < \Lambda_0(L_0)$, an operator L is a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = \mu_0$ if and only if there exists a 2×4 matrix M such that*

$$\operatorname{rank} M = 2, \qquad M\tilde{J}_4 M^* = 0, \tag{3.16}$$

$$\operatorname{diag}(MAM^*) \geq 0, \qquad \det(M\Phi^*) = 0 \tag{3.17}$$

and $Ly = L_{\max} y$, $y \in D(L)$, where

$$D(L) = \{y \in D(L_{\max}) : \ M\mathbf{Y} = 0\}. \tag{3.18}$$

Proof. Without loss of generality, we may assume that $\Lambda_0(L_0) > 0$ and $\mu_0 = 0$. Let L be a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = 0$. Then, L is a positive self-adjoint operator. By Theorem 3.2, there is a 2×4 matrix M such that (3.16) and (3.18) are satisfied and MAM^* is a positive definite or positive semi-definite matrix. Furthermore, the condition $\Lambda_0(L) = \mu_0 = 0$ shows $0 \in \sigma_P(L)$ (the set of eigenvalues of L) and there exist constants c_1 and c_2 such that $\varphi_0 := c_1\varphi_1 + c_2\varphi_2 (\not\equiv 0)$ is the eigenfunction corresponding to 0. This, combined with (2.7) and (3.18), yields $\det(M\Phi^*) = 0$. Let $\alpha_0 = (c_1, c_2)$ satisfy $\alpha_0 \neq 0$ and $\alpha_0 M\Phi^* = 0$ and let

$$\Delta = \begin{bmatrix} \Delta_{11} \\ \Phi \end{bmatrix} \quad \text{with} \quad \Delta_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here Δ is defined by (3.8). From (3.9) and (3.10) we obtain

$$\begin{aligned} \alpha_0 MAM^*\alpha_0^* &= \alpha_0 M\Delta^* J_4 \Delta M^* \alpha_0^* \\ &= [\,\alpha_0 M\Delta_{11}^*, \quad \alpha_0 M\Phi^*\,]\, J_4 \begin{bmatrix} \Delta_{11} M^* \alpha_0^* \\ \Phi M^* \alpha_0^* \end{bmatrix} \\ &= \alpha_0 M\Delta_{11}^* J_2 \Phi M^* \alpha_0^* + \alpha_0 M\Phi^* J_2 \Delta_{11} M^* \alpha_0^* \\ &= 0. \end{aligned}$$

This concludes $\operatorname{rank} MAM^* \leq 1$. In this case, applying the characterization of positive semi-definite matrix, we obtain that MAM^* is a positive semi-definite matrix if and only if diag $(MAM^*) \geq 0$. This completes the proof of the necessary part.

Conversely, by (3.17) and the above proof, we can conclude that $0 \in \sigma_P(L)$ and MAM^* is a positive semi-definite matrix. This, together with Theorem 3.2, implies the sufficiency of Theorem 3.3. □

If we consider all self-adjoint extension operators $L(M_1, M_2)$, see, e.g., [**8**, Section 4], defined by $L(M_1, M_2)y = ly$, $y \in D(L(M_1, M_2))$ with

$$D(L(M_1, M_2)) = \{y \in D(L_{\max}) : M_1 Y(a) + M_2 Y(b) = 0\}, \tag{3.19}$$

where the 2×2 matrices M_1 and M_2 satisfy

$$\operatorname{rank}(M_1, M_2) = 2 \quad \text{and} \quad M_1 \hat{J}_2 M_1^* - M_2 \hat{J}_2 M_2^* = 0; \tag{3.20}$$

then, Theorem 3.3 can be equivalently restated as follows.

Theorem 3.3′. *Let the constant* $\mu_0 < \Lambda_0(L_0)$ *and* $L(M_1, M_2)$ *be a self-adjoint operator. Then* $L(M_1, M_2)$ *satisfies* $\Lambda_0(L(M_1, M_2)) = \mu_0$ *if and only if the matrix* $M := (M_1, M_2)$ *satisfies*

$$\det(M\Phi^*) = 0 \tag{3.21}$$

and

$$\operatorname{diag}(MAM^*) \geq 0. \tag{3.22}$$

4. Explicit boundary conditions. The characteristic theorem (Theorem 3.3) of bound-limited self-adjoint extensions of L_0 can be applied to provide a more detailed description of bound-limited self-adjoint extensions via all possible explicit BC's.

For our purpose here it is convenient to divide the self-adjoint BC's of L which is a self-adjoint extension of L_0 into two disjoint subclasses, cf. [**8**, Section 4]:

(a) *Separated self-adjoint BC's.* These can be parameterized as follows

$$\cos\alpha y(a) - \sin\alpha y^{[1]}(a) = 0, \quad 0 \leq \alpha < \pi; \tag{4.1}$$

$$\cos\beta y(b) - \sin\beta y^{[1]}(b) = 0, \quad 0 < \beta \leq \pi; \tag{4.2}$$

(b) *Coupled self-adjoint BC's.* These can be formulated as in (1.3).

Theorem 4.1 (**Separated BC's**). *If the operator L is deduced from the separated self-adjoint boundary condition and the constant $\mu_0 < \Lambda_0(L_0)$, then L is a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = \mu_0$ if and only if α and β satisfy one of the following three cases*:

(i) *If* $\sin\alpha = 0$, *then* $\cot\beta = \varphi_1^{[1]}(b)/\varphi_1(b)$;

(ii) *If* $\sin\beta = 0$, *then* $\cot\alpha = \varphi_2^{[1]}(a)/\varphi_2(a)$;

(iii) *If* $\sin\alpha\sin\beta \neq 0$, *then* $\cot\alpha > \varphi_2^{[1]}(a)/\varphi_2(a)$, $\cot\beta < \varphi_1^{[1]}(b)/\varphi_1(b)$ *and*

$$\cot\alpha = \frac{-\cot\beta\varphi_1(b)\varphi_2^{[1]}(a) + \varphi_2^{[1]}(a)\varphi_1^{[1]}(b) - 1}{\varphi_2(a)(\varphi_1^{[1]}(b) - \cot\beta\varphi_1(b))}. \tag{4.3}$$

Proof. In this case

$$M_1 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix}. \tag{4.4}$$

If we write

$$k_2(\alpha) = \cos\alpha\varphi_2(a) - \sin\alpha\varphi_2^{[1]}(a) \tag{4.5}$$

and

$$k_1(\beta) = \cos\beta\varphi_1(b) - \sin\beta\varphi_1^{[1]}(b),$$

then

$$\begin{aligned} M\Phi^* &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \varphi_2(a) \\ 1 & \varphi_2^{[1]}(a) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 & 0 \\ \cos\beta & -\sin\beta \end{bmatrix} \begin{bmatrix} \varphi_1(b) & 0 \\ \varphi_1^{[1]}(b) & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\sin\alpha & k_2(\alpha) \\ k_1(\beta) & -\sin\beta \end{bmatrix} \end{aligned} \tag{4.6}$$

and

(4.7) MAM^*

$$= \begin{bmatrix} -\sin\alpha\varphi_2(a) & k_2(a) - \sin\alpha\varphi_2^{[1]}(a) & 0 & -2\sin\alpha \\ 0 & -2\sin\beta & -\sin\beta\varphi_1(b) & k_1(\beta) - \sin\beta\varphi_1^{[1]}(b) \end{bmatrix} M^*$$
$$= \begin{bmatrix} -2k_2(\alpha)\sin\alpha & 2\sin\alpha\sin\beta \\ 2\sin\alpha\sin\beta & -2k_1(\beta)\sin\beta \end{bmatrix}.$$

By Theorem 3.3, we have

(4.8) $\det(M\Phi^*)$

$$\begin{aligned} &= \sin\alpha\sin\beta - k_2(\alpha)k_1(\beta) \\ &= \sin\alpha\sin\beta - (\cos\alpha\varphi_2(a) - \sin\alpha\varphi_2^{[1]}(a))(\cos\beta\varphi_1(b) - \sin\beta\varphi_1^{[1]}(b)) \\ &= \sin\alpha\sin\beta(1 - \varphi_2^{[1]}(a)\varphi_1^{[1]}(b)) + \cos\alpha\sin\beta\varphi_2(a)\varphi_1^{[1]}(b) \\ &\quad - \cos\alpha\cos\beta\varphi_2(a)\varphi_1(b) + \sin\alpha\cos\beta\varphi_1(b)\varphi_2^{[1]}(a) \\ &= 0 \end{aligned}$$

and

$$k_2(\alpha)\sin\alpha \leq 0 \quad \text{and} \quad k_1(\beta)\sin\beta \leq 0. \tag{4.9}$$

If $\sin\alpha = 0$, from (2.9) and (4.5), then (4.8) is equivalent to $\cot\beta = \varphi_1^{[1]}(b)/\varphi_1(b)$ and (4.9) holds. This proves (i). Also, if $\sin\beta = 0$, we can prove (ii).

If $\sin\alpha\sin\beta \neq 0$, then (4.8) implies that $k_2(\alpha) \neq 0$ and $k_1(\beta) \neq 0$, and then (4.3) is satisfied. Furthermore, from (2.8) and (4.5), we see that (4.9) is equivalent to

$$\cot\alpha > \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)} \quad \text{and} \quad \cot\beta < \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}.$$

This completes the proof of Theorem 4.1. □

Theorem 4.2 (**Coupled BC's**). *Let $K \in \mathrm{SL}(\mathbf{R})$. If the operator L is deduced from the coupled self-adjoint boundary condition and the constant $\mu_0 < \Lambda_0(L_0)$, then L is a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = \mu_0$ if and only if the real numbers k_{ij}, $1 \le i, j \le 2$, and θ satisfy one of the following three cases*:

(i) *If* $k_{12} = 0$, *then*

$$k_{21} = \frac{-\varphi_1^{[1]}(b)k_{11} - \varphi_2^{[1]}(a)k_{22} + 2\cos\theta}{\varphi_2(a)}; \tag{4.10}$$

(ii) *If* $k_{12} > 0$, *then* $k_{11} \le -(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$, $\varphi_1^{[1]}(b)k_{21} \ge [(\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_2(a)]k_{22}$ *and*

$$k_{21} = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\, k_{11} - \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{22} + \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_1(b)\varphi_2(a)}\, k_{12} + \frac{2\cos\theta}{\varphi_2(a)}; \tag{4.11}$$

(iii) *If* $k_{12} < 0$, *then* $k_{11} \ge -(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$, $\varphi_1^{[1]}(b)k_{21} \le [(\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_2(a)]k_{22}$ *and* (4.11) *is satisfied.*

Proof. In this case we may denote by $M_1 = K$ and $M_2 = -e^{i\theta} I_2$. If we write

$$k_1(a) = k_{11}\varphi_2(a) + k_{12}\varphi_2^{[1]}(a) \quad \text{and} \quad k_2(a) = k_{21}\varphi_2(a) + k_{22}\varphi_2^{[1]}(a), \tag{4.12}$$

then

$$\begin{aligned} M\Phi^* &= \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} 0 & \varphi_2(a) \\ 1 & \varphi_2^{[1]}(a) \end{bmatrix} - e^{i\theta} \begin{bmatrix} \varphi_1(b) & 0 \\ \varphi_1^{[1]}(b) & 1 \end{bmatrix} \\ &= \begin{bmatrix} k_{12} & k_1(a) \\ k_{22} & k_2(a) \end{bmatrix} - \begin{bmatrix} e^{i\theta}\varphi_1(b) & 0 \\ e^{i\theta}\varphi_1^{[1]}(b) & e^{i\theta} \end{bmatrix} \\ &= \begin{bmatrix} k_{12} - e^{i\theta}\varphi_1(b) & k_1(a) \\ k_{22} - e^{i\theta}\varphi_1^{[1]}(b) & k_2(a) - e^{i\theta} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} &MAM^* \\ &= \begin{bmatrix} k_{12}\varphi_2(a) & k_1(a) + k_{12}\varphi_2^{[1]}(a) & 0 & 2k_{12} - e^{i\theta}\varphi_1(b) \\ k_{22}\varphi_2(a) & k_2(a) + k_{22}\varphi_2^{[1]}(a) - 2e^{i\theta} & -e^{i\theta}\varphi_1(b) & 2k_{22} - 2e^{i\theta}\varphi_1^{[1]}(b) \end{bmatrix} M^* \\ &= \begin{bmatrix} 2k_{12}k_1(a) & k_{12}k_2(a) + k_{22}k_1(a) - 2e^{-i\theta}k_{12} + \varphi_1(b) \\ k_{12}k_2(a) + k_{22}k_1(a) - 2e^{i\theta}k_{12} + \varphi_1(b) & 2k_{22}k_2(a) - 2(e^{-i\theta} + e^{i\theta})k_{22} + 2\varphi_1^{[1]}(b) \end{bmatrix}. \end{aligned}$$

Note that

(4.13)

$$k_{11}k_{22} - k_{12}k_{21} = 1 \quad \text{and} \quad k_{12}k_2(a) - k_{22}k_1(a) = -\varphi_2(a) = \varphi_1(b).$$

By Theorem 3.3, we have

$$\begin{aligned}(4.14)\quad &\det(M\Phi^*)\\ &= (k_{12} - e^{i\theta}\varphi_1(b))(k_2(a) - e^{i\theta}) - k_1(a)(k_{22} - e^{i\theta}\varphi_1^{[1]}(b))\\ &= -e^{i\theta}\varphi_1(b)k_2(a) + e^{i\theta}\varphi_1^{[1]}(b)k_1(a) - e^{i\theta}k_{12} + e^{i2\theta}\varphi_1(b) + \varphi_1(b)\\ &= -e^{i\theta}\varphi_1(b)\left[k_2(a) - \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_1(a) + \frac{1}{\varphi_1(b)}\,k_{12} - 2\cos\theta\right]\\ &= 0\end{aligned}$$

and

$$(4.15)\quad k_{12}k_1(a) \geq 0 \quad \text{and} \quad \lambda_{22} := k_{22}k_2(a) - 2\cos\theta k_{22} + \varphi_1^{[1]}(b) \geq 0.$$

Furthermore, from (4.14) and (4.12) we obtain

$$(4.16)\qquad k_2(a) = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_1(a) - \frac{1}{\varphi_1(b)}\,k_{12} + 2\cos\theta$$

and

$$(4.17)\quad k_{21} = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_{11} - \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\,k_{22} + \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_1(b)\varphi_2(a)}\,k_{12} + \frac{2\cos\theta}{\varphi_2(a)}.$$

From (4.12), (4.13), (4.15) and (4.16), we have

(4.18)

$$\begin{aligned}\lambda_{22} &= k_{22}\left[\frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_1(a) - \frac{1}{\varphi_1(b)}\,k_{12} + 2\cos\theta\right] - 2\cos\theta k_{22} + \varphi_1^{[1]}(b)\\ &= \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_1(a)k_{22} - \frac{1}{\varphi_1(b)}\,k_{12}k_{22} + \varphi_1^{[1]}(b)\\ &= \frac{1}{\varphi_1(b)}\,[\varphi_2(a)\varphi_1^{[1]}(b)k_{11}k_{22} + (\varphi_2^{[1]}(a)\varphi_1^{[1]}(b) - 1)k_{12}k_{22}\\ &\qquad\qquad + \varphi_1(b)\varphi_1^{[1]}(b)]\\ &= \frac{k_{12}}{\varphi_1(b)}\,[-\varphi_2(a)\varphi_1^{[1]}(b)k_{21} + (\varphi_2^{[1]}(a)\varphi_1^{[1]}(b) - 1)k_{22}].\end{aligned}$$

If $k_{12} = 0$, from (4.15), (4.17) and (4.18), we prove (i). If $k_{12} > 0$, by (2.8), (4.15) and (4.18), we have

$$(4.19)\quad k_{11} \le -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{12} \quad \text{and} \quad \varphi_1^{[1]}(b) k_{21} \ge \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_2(a)}\, k_{22}.$$

This, together with (4.17), proves (ii). Also, if $k_{12} < 0$, we can similarly prove (iii). So, we complete the proof of Theorem 4.2. □

If $\theta = 0$ in (1.3), then corresponding BC's are called the real coupled self-adjoint BC's. In this case, as was seen in the proof of Theorem 4.2, we have the following corollary.

Corollary 4.3 (**Real coupled BC's**). *Let* $K \in \mathrm{SL}(\mathbf{R})$. *If the operator* L *is deduced from the real coupled self-adjoint boundary condition,* $\theta = 0$, *and* $\mu_0 < \Lambda_0(L_0)$, *then* L *is a bound-limited self-adjoint extension of* L_0 *with* $\Lambda_0(L) = \mu_0$ *if and only if the real numbers* k_{ij}, $1 \le i, j \le 2$, *satisfy one of the following three cases*:

(i) *If* $k_{12} = 0$, *then*

$$(4.20)\qquad k_{21} = \frac{-\varphi_1^{[1]}(b)k_{11} - \varphi_2^{[1]}(a)k_{22} + 2}{\varphi_2(a)};$$

(ii) *If* $k_{12} > 0$, *then* $k_{11} \le -(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$, $\varphi_1^{[1]}(b)k_{21} \ge [(\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_2(a)]k_{22}$ *and*

$$(4.21)\quad k_{21} = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\, k_{11} - \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{22} + \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_1(b)\varphi_2(a)}\, k_{12} + \frac{2}{\varphi_2(a)};$$

(iii) *If* $k_{12} < 0$, *then* $k_{11} \ge -(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$, $\varphi_1^{[1]}(b)k_{21} \le [(\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_2(a)]k_{22}$ *and* (4.21) *is satisfied.*

Let us finish this section with a concrete example to show the bound limited self-adjoint extensions L of L_0.

Example 4.4. Consider the Fourier expression:

$$(4.22)\qquad ly = -y'', \quad \text{with} \quad t \in I = [0, 1] \quad \text{in} \quad L^2[0, 1].$$

Clearly, l is a regular SL kind differential expression on I. It is easily calculated that $\Lambda_0(L_0) = \pi^2$. Given a constant $\mu_0 = \pi^2/4$ $(< \Lambda_0(L_0))$, applying Theorems 4.1 and 4.2 we will find all bound limited self-adjoint extensions L of L_0 such that $\Lambda_0(L) = \pi^2/4$. In this case it is not hard to see that $\varphi_1 = 2/\pi \sin(\pi t/2)$ and $\varphi_2 = -1/\pi \cos(\pi t/2)$ are the solutions of the equation $ly = (\pi^2/4)y$ which satisfy the condition (2.7) and

$$\varphi_1(1) = \frac{2}{\pi}, \quad \varphi_1'(1) = 0, \quad \varphi_2(0) = -\frac{2}{\pi}, \quad \varphi_2'(0) = 0. \tag{4.23}$$

If the operator L is deduced from the separated self-adjoint boundary condition, by (4.23) and Theorem 4.1, then L is a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = \pi^2/4$ if and only if α and β satisfy one of the following three cases:

(i) $\sin\alpha = \cos\beta = 0$;

(ii) $\sin\beta = \cos\alpha = 0$;

(iii) $\cos\alpha > 0$, $\cot\alpha = -(\pi^2/4)\tan\beta$.

Thus, by (4.1)–(4.2), the following separated BC's

$$\begin{gathered} y(0) = 0 = y'(1); \quad y(1) = 0 = y'(0); \\ \cos\alpha y(0) - \sin\alpha y'(0) = 0 = \sin\alpha y(1) + (\pi^2/4)\sin\alpha y'(1); \end{gathered} \tag{4.24}$$

$\cos\alpha > 0$, together with the expression l, define the operators that satisfy $\Lambda_0(L) = \pi^2/4$.

If the operator L is deduced from the coupled self-adjoint boundary condition, by (4.23), then (4.11) is equivalent to

$$k_{21} = \frac{\pi^2}{4}\, k_{12} - \pi\cos\theta \tag{4.26}$$

and therefore $K \in \mathrm{SL}(2, \mathbf{R})$ is equivalent to

$$\begin{aligned} k_{11}k_{22} &= 1 + k_{12}k_{21} = 1 + k_{12}\left(\frac{\pi^2}{4}\, k_{12} - \pi\cos\theta\right) \\ &= \left(\frac{\pi}{2}\, k_{12} - \cos\theta\right)^2 + \sin^2\theta. \end{aligned} \tag{4.27}$$

By (4.23) and Theorem 4.2, L is a bound-limited self-adjoint extension of L_0 with $\Lambda_0(L) = \pi^2/4$ if and only if the real numbers k_{ij}, $1 \le i$, $j \le 2$, and θ ($\in (-\pi, \pi)$) satisfy one of the following three cases:

(i) $k_{12} = 0$, $k_{21} = -\pi \cos\theta$, $k_{11} = 1/k_{22}$;

(ii) $k_{12} > 0$, $k_{11} \le 0$, $k_{22} \le 0$, (4.26) and (4.27) hold;

(iii) $k_{12} < 0$, $k_{11} \ge 0$, $k_{22} \ge 0$, (4.26) and (4.27) hold.

Thus, by (1.3), under one of the above three cases, the following coupled BC's

$$KY(a) = e^{i\theta}Y(b),$$

together with the expression l, define the operators that are the bound limited self-adjoint extensions L of L_0 with $\Lambda_0(L) = \pi^2/4$.

5. Equal cases: $\lambda_0(K) = \upsilon_0$ or $\lambda_0(K) = \gamma_0$. In this section let us specialize the above Theorems 4.1–4.2 to the eigenvalue equalities problem related to (1.8)–(1.9). Thus we will search for the self-adjoint BC's under which there hold the equalities on the minimal eigenvalues, see Theorem 1.1. That is, we are try to find the possible matrices K such that $\lambda_0(K) = \upsilon_0$ or $\lambda_0(K) = \gamma_0$, where υ_0 and γ_0 are the minimal eigenvalues for (1.6) and (1.7) respectively.

Theorem 5.1. *Let φ_1 and φ_2 denote the solutions of the equation $ly = \nu_0 y$ which satisfy the initial conditions* (2.7) *and $k_{12} \neq 0$. Then $\lambda_0(K) = \nu_0$ if and only if the real numbers k_{ij}, $1 \le i$, $j \le 2$, satisfy the following conditions*:

$$(5.1) \qquad k_{12} = \varphi_1(b), \quad k_{22} = \varphi_1^{[1]}(b), \quad \varphi_1^{[1]}(b)k_{11} - \varphi_1(b)k_{21} = 1,$$

$$(5.2) \quad k_{11} \le \varphi_2^{[1]}(a) \quad \text{and} \quad \varphi_1^{[1]}(b)[\varphi_1^{[1]}(b)k_{11} + \varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 2] \ge 0.$$

Remark 5.2. Note that the self-adjoint BC in (1.6) becomes the Dirichlet BC when $k_{12} = 0$. In this case, the matrices K satisfying $\lambda_0(K) = \lambda_0^D := \Lambda_0(L_0)$ may be btained in [**7**, Theorem 5.2].

Proof. Since υ_0 is the minimal eigenvalue for (1.6) and $k_{12} \neq 0$, then $\nu_0 < \Lambda_0(L_0)$, see [**8**, Section 4]. By Theorem 4.1 and (1.6) we have

$$\alpha = 0, \quad \cot\beta = \frac{k_{22}}{k_{12}} = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}. \tag{5.3}$$

We proceed to apply Theorem 4.2 to υ_0 in distinguishing two possible cases.

(i) $k_{12} > 0$. In this case by $\det(K) = 1$, (2.8) and (4.21) we have

(5.4)

$$\begin{aligned} k_{21} &= \frac{k_{22}}{k_{12}}\, k_{11} - \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{22} + \frac{k_{22}}{k_{12}}\, \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{12} - \frac{1}{\varphi_1(b)\varphi_2(a)}\, k_{12} + \frac{2}{\varphi_2(a)} \\ &= \frac{1 + k_{12}k_{21}}{k_{12}} + \frac{1}{\varphi_2(a)^2}\, k_{12} + \frac{2}{\varphi_2(a)} \\ &= k_{21} + \frac{1}{k_{12}} + \frac{1}{\varphi_2(a)^2}\, k_{12} + \frac{2}{\varphi_2(a)}. \end{aligned}$$

This deduces to $k_{12} = -\varphi_2(a)$ and $k_{22} = \varphi_1^{[1]}(b)$. Furthermore, we see that $k_{11} \leq -(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$ is equivalent to $k_{11} \leq \varphi_2^{[1]}(a)$ and $\varphi_1^{[1]}(b)k_{21} \geq [(\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_2(a)]k_{22}$ is equivalent to

$$\begin{aligned} 0 &\leq \varphi_1^{[1]}(b)k_{21} - \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_2(a)}\, \varphi_1^{[1]}(b) \\ &= -\frac{\varphi_1^{[1]}(b)}{\varphi_2(a)}\, [-\varphi_2(a)k_{21} + \varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1] \\ &= -\frac{\varphi_1^{[1]}(b)}{\varphi_2(a)}\, [\varphi_1^{[1]}(b)k_{11} + \varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 2]. \end{aligned}$$

So, together with (2.8) and $\det(K) = 1$, (5.1) and (5.2) hold.

(ii) $k_{12} < 0$. In this case, from Corollary 4.3 (iii) and (4.21), the equation (5.4) does not hold as $k_{12} < 0$ and, therefore, there no exists the matrix K satisfying $\lambda_0(K) = \nu_0$.

By the above proof, it is easy to verify the sufficiency, thus completing the proof. $\square$

Theorem 5.3. *Let φ_1 and φ_2 denote the solutions of the equation $ly = \gamma_0 y$ which satisfy the initial conditions* (2.7). *Then $\lambda_0(K) = \gamma_0$ if and only if the real numbers k_{ij}, $1 \le i,\ j \le 2$, satisfy one of the following four conditions*:

(i) *If $k_{11} \neq 0$ and $k_{12} = 0$, then $k_{11} = 1/k_{22} = \varphi_2^{[1]}(a)$, $k_{21} = (\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_1(b)$;*

(ii) *If $k_{11} \neq 0$ and $k_{12} < 0$, then $k_{21}(k_{22} + \varphi_1^{[1]}(b)) \geq 0$,*

$$
\begin{gathered}
k_{11} = \varphi_2^{[1]}(a), \quad k_{21} = \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_1(b)}, \\
k_{22}\varphi_2^{[1]}(a) - k_{12}\frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_1(b)} = 1;
\end{gathered} \tag{5.5}
$$

(iii) *If $k_{11} \neq 0$ and $k_{12} > 0$, then $k_{12} \geq -\varphi_2(a)$, $k_{21}(k_{22}+\varphi_1^{[1]}(b)) \leq 0$ and* (5.5) *is satisfied*;

(iv) *If $k_{11} = 0$, then $k_{12} = -1/k_{21} = -\varphi_2(a)$ and $k_{22} \leq \varphi_1^{[1]}(b)$.*

Proof. Since γ_0 is the minimal eigenvalue for (1.7), then $\gamma_0 < \Lambda_0(L_0)$, see [**8**, Section 4]. By Theorem 4.1 we have the following two cases:

(5.6)
$$
(1) \quad k_{11} \neq 0: \quad \varphi_2^{[1]}(a) > 0, \quad \cot\beta = \frac{k_{21}}{k_{11}} = \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)} - \frac{1}{\varphi_1(b)\varphi_2^{[1]}(a)};
$$

(5.7)
$$
(2) \quad k_{11} = 0: \quad \varphi_2^{[1]}(a) = 0.
$$

Based on the above two cases, we now apply Theorem 4.2 through distinguishing four cases:

Case i. $k_{11} \neq 0$ and $k_{12} = 0$. In this case by (2.8), (4.20) and (5.6), we obtain

$$
k_{21} - \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\,k_{11} = -\frac{1}{\varphi_1(b)\varphi_2^{[1]}(a)}\,k_{11} = -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\,k_{22} + \frac{2}{\varphi_2(a)}.
$$

Thus

$$\frac{k_{11}}{\varphi_2^{[1]}(a)} + \varphi_2^{[1]}(a)k_{22} = 2.$$

Note that $k_{11}k_{22} = 1$. Then $k_{11} = 1/k_{22} = \varphi_2^{[1]}(a)$. This and (5.6) prove (i).

Case ii. $k_{11} \neq 0$ and $k_{12} < 0$. In this case by (2.8), (5.6), (4.21) and $\det(K) = 1$, we have

(5.8)

$$\begin{aligned}
&k_{21} - \frac{\varphi_1^{[1]}(b)}{\varphi_1(b)}\, k_{11} \\
&\quad = -\frac{1}{\varphi_1(b)\varphi_2^{[1]}(a)}\, k_{11} \\
&\quad = -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{22} + \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)} \left[\frac{\varphi_1^{[1]}(b)}{\varphi_1(b)} - \frac{1}{\varphi_1(b)\varphi_2^{[1]}(a)}\right] k_{12} + \frac{2}{\varphi_2(a)} \\
&\quad = -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, k_{22} + \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)} \cdot \frac{k_{21}}{k_{11}}\, k_{12} + \frac{2}{\varphi_2(a)} \\
&\quad = -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)} k_{22} + \frac{\varphi_2^{[1]}(a)}{\varphi_2(a)}\, (k_{22} - \frac{1}{k_{11}}) + \frac{2}{\varphi_2(a)} \\
&\quad = -\frac{\varphi_2^{[1]}(a)}{\varphi_2(a)k_{11}} + \frac{2}{\varphi_2(a)}.
\end{aligned}$$

Therefore,

$$\frac{1}{\varphi_2^{[1]}(a)}\, k_{11} + \varphi_2^{[1]}(a)\, \frac{1}{k_{11}} = 2.$$

This implies that $k_{11} = \varphi_2^{[1]}(a)$ and $k_{21} = (\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1)/\varphi_1(b)$. Moreover, by (iii) of Corollary 4.3 and (5.6), we see that $k_{11} \geq$

$-(\varphi_2^{[1]}(a)/\varphi_2(a))k_{12}$ is equivalent to $k_{12} \le -\varphi_2(a)$ and

$$\begin{aligned} 0 &\ge \varphi_1^{[1]}(b)k_{21} - \frac{\varphi_1^{[1]}(b)\varphi_2^{[1]}(a) - 1}{\varphi_2(a)}\, k_{22} \\ &= \varphi_1^{[1]}(b)k_{21} + \left[\frac{\varphi_1^{[1]}(b)}{\varphi_1(b)} - \frac{1}{\varphi_1(b)\varphi_2^{[1]}(a)}\right]\varphi_2^{[1]}(a)k_{22} \\ &= \varphi_1^{[1]}(b)k_{21} + \frac{k_{21}}{k_{11}}\, k_{11}k_{22} \\ &= k_{21}(k_{22} + \varphi_1^{[1]}(b)). \end{aligned}$$

Note that $\varphi_2(a) < 0$ and $k_{12} < 0$ implies $k_{12} \le -\varphi_2(a)$. Thus, by $\det(K) = 1$ we prove (ii).

Case iii. $k_{11} \ne 0$ and $k_{12} > 0$. In this case, the assertion can be justified similarly to that of case ii.

Case iv. $k_{11} = 0$. In this case by $\det(K) = 1$ we then know that $k_{12} \ne 0$. From (5.7) and (4.21) we have

$$k_{21} = -\frac{1}{\varphi_1(b)\varphi_2(a)}\, k_{12} + \frac{2}{\varphi_2(a)}.$$

Note that $k_{12}k_{21} = -1$. So, $k_{12} = -\varphi_2(a) = -1/k_{21} > 0$. Therefore, by Corollary 4.3 (ii), we obtain $k_{22} \le \varphi_1^{[1]}(b)$ and prove case iv.

By the above proof, the sufficiency is clear, thus completing the proof of Theorem 5.3. □

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