

REPRESENTATION RESULTS FOR OPERATORS
GENERATED BY A QUASI-DIFFERENTIAL
MULTI-INTERVAL SYSTEM IN A
HILBERT DIRECT SUM SPACE

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ABSTRACT. We study the spectral structure of operators generated as direct sums of self-adjoint extensions of quasi-differential minimal operators on a multi-interval set (self-adjoint vector-operators). Special attention is given to the ordered spectral representation for such operators.

1. Introduction.

1.1 *Problem overview.* The modern theory of quasi-differential v-operators originates from the fundamental work of Gesztesy and Kirsch [10], where these authors considered a Schrödinger operator generated by the Hamiltonian

$$(1) \quad H = -\frac{d^2}{dx^2} + \left(s^2 - \frac{1}{4}\right) \frac{1}{\cos^2 x}, \quad s > 0.$$

It is clear that the potential in (1) has a countable number of singularities on \mathbf{R} , leading to spoiling of the local integrability. In order to overcome this difficulty, operators T_i are constructed, generated by the same Hamiltonian (1) in the coordinate spaces

$$L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right),$$

$i \in \mathbf{Z}$, and then the direct sum operator $\bigoplus_{i \in \mathbf{Z}} T_i$ is considered in the space

$$\bigoplus_{i \in \mathbf{Z}} L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right).$$

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The work [10] gave birth to various generalizations of the problem. In 1992, Everitt and Zettl [9] studied direct sums of minimal and maximal operators generated by arbitrary formally self-adjoint expressions in Hilbert spaces considered on arbitrary intervals (maximal and minimal v -operators). Later in 2000, v -operators were also considered in complete locally convex spaces by Ashurov and Everitt in [2], which was a natural generalization of their basic work [1]. Since 1992, quasi-differential v -operators have mostly been investigated in connection with their non-spectral properties, such as introduction of minimal and maximal v -operators and their relationship (it was shown that adjoint of the minimal v -operator is maximal in a Hilbert space, see [9], and the analogous result with the modification for Frechet spaces was obtained in [2]). Everitt and Markus have developed the theory of self-adjoint extensions for v -operators with the employment of symplectic geometry. In connection with this, see their recent memoirs [7] and [8]. The modern theory of differential v -operators has a connection with the theory of differential operators on graphs. In some cases certain boundary conditions lead to considering a differential operator on a graph as a direct sum operator. The most modern results pertinent to the spectral theory of differential operators on graphs were obtained in the works of Carlson [4, 5] and Kurasov and Stenberg [11].

Since the theory of quasi-differential v -operators in a Hilbert space is quite young and the most recent studies concerned mostly problems connected with their common theory, a very small attention was given to its spectral aspects. Some results, describing position of spectra of v -operators were presented in 1985 in [10] and the most recent results belong to Sobhy El-Sayed Ibrahim [10, 11]. Some spectral properties of abstract self-adjoint v -operators were studied by M.S. Sokolov in [16, 17] and R.R. Ashurov, M.S. Sokolov in [3]. Nevertheless, a rigorous structural spectral theory for such operators has not been developed yet.

The attention of the current work is mainly focused on representation results for an abstract and, in particular, quasi-differential self-adjoint v -operator.

1.2 *Quasi-differential operators and v -operators.* Basic concepts of quasi-differential operators are described in [7, 9]. A good reference for operators with real coefficients is the book of Naimark [15].

Let us have a number $n \in \mathbf{N}$, $n \geq 2$, and an arbitrary interval $I \subseteq \mathbf{R}$. Let $Z_n(I)$ be a set of Shin-Zettl matrices. These are matrices $A = \{a_{rs}\}$, $a_{rs} : I \rightarrow \mathbf{C}$ of the order $n \times n$, such that for almost all $x \in I$:

$$\begin{cases} \text{(i)} & a_{rs} \in L_{loc}(I) \quad r, s = \overline{1, n}; \\ \text{(ii)} & a_{r, r+1}(x) \neq 0 \quad r = \overline{1, n-1}; \\ \text{(iii)} & a_{rs} = 0 \quad s = \overline{r+2, n}; \quad r = \overline{1, n-2}. \end{cases}$$

Consider a function $f : I \rightarrow \mathbf{C}$. Its quasi-derivatives relative to a Shin-Zettl matrix A are defined by

$$\begin{cases} \text{(i)} & f_A^{[0]} := f; \\ \text{(ii)} & f_A^{[r]} := 1/(a_{r, r+1}) \left[(d/dx) f_A^{[r-1]} - \sum_{s=1}^r a_{rs} f_A^{[s-1]} \right], \quad r = \overline{1, n-1}; \\ \text{(iii)} & f_A^{[n]} := (d/dx) f_A^{[n-1]} - \sum_{s=1}^n a_{ns} f_A^{[s-1]}. \end{cases}$$

Let us introduce a linear manifold $D(A) \subset AC_{loc}(I)$:

$$D_A(I) := \{ f : I \rightarrow \mathbf{C} \mid f_A^{[r-1]} \in AC_{loc}(I), \quad r = \overline{1, n} \}.$$

It is possible to see that $f \in D_A(I)$ implies $f_A^{[n]} \in L_{loc}(I)$, and it is possible to prove that $D_A(I)$ is dense in $L_{loc}(I)$.

Relative to a matrix $A \in Z_n(I)$, we have the quasi-differential expression $M_A[f] = i^n f_A^{[n]}$, $f \in D_A(I)$.

A matrix $A^+ \in Z_n(I)$ designates a Lagrange adjoint matrix to A if $A^+ := -L_n^{-1} A^* L_n$, where A^* is the adjoint matrix, and $L_n = \{l_{rs}\}$ is an $(n \times n)$ -matrix, defined as:

$$l_{r, n+1-r} = \begin{cases} (-1)^{r-1} & r = \overline{1, n}; \\ 0 & \text{for other } r, s. \end{cases}$$

Using this notation we suppose that in this work we deal only with Lagrange symmetric (formally self-adjoint) expressions, that is, $M_{A^+}[f] = M_A[f] = \tau(f)$, where τ is an alternative notation for a Lagrange symmetric expression.

For a quasi-differential expression $M_A[f]$, the Lagrange formula is known ($[\alpha, \beta] \subseteq I$ – an arbitrary compact subinterval of I):

$$(2) \int_{\alpha}^{\beta} \{ \overline{g(x)M_A[f](x)} - f(x)\overline{M_{A^+}[g(x)]} \} dx = [f, g]_A(\beta) - [f, g]_A(\alpha),$$

where $f \in D_A$, $g \in D_{A^+}$, $[f, g]_A(\beta)$ and $[f, g]_A(\alpha)$ may be derived from:

$$[f, g]_A(x) = i^n \sum_{i=1}^n (-1)^{i-1} f_A^{[i-1]}(x) \overline{g_{A^+}^{[n-i]}(x)}, \quad x \in I.$$

Let $\omega > 0$ be a weight function from $L_{\text{loc}}(I)$, $\omega : I \rightarrow \mathbf{R}$. The Hilbert space $L^2(I : \omega)$ is formed as usual.

We define maximal and minimal operators as follows:

Definition 1.1. Operators T_{\max} and T_{\min} are called respectively *maximal* and *minimal* operators if they are generated by $\tau(f)$ on the domains $D(T_{\max})$ and $D(T_{\min})$:

$$D(T_{\max}) = \{f : I \rightarrow \mathbf{C} \mid f \in D_A(I); \omega^{-1}\tau(f) \in L^2(I : \omega)\},$$

$$T_{\max}f = \omega^{-1}\tau(f), \quad (f \in D(T_{\max}));$$

$$D(T_{\min}) = \{f \mid f \in D(T_{\max}); [f, g]_A(b) - [f, g]_A(a) = 0 \ (g \in D(T_{\max}))\},$$

$$T_{\min}f = \omega^{-1}\tau(f), \quad (f \in D(T_{\min})),$$

where $[f, g]_A(b)$ and $[f, g]_A(a)$ are limits, which necessarily exist, of bilinear forms from (2), that is, $\lim_{\beta \rightarrow b} [f, g]_A(\beta) = [f, g]_A(b)$ and $\lim_{\alpha \rightarrow a} [f, g]_A(\alpha) = [f, g]_A(a)$.

The following general theorem is known for the operators T_{\max} and T_{\min} :

Theorem 1.2. *For the operators T_{\max} and T_{\min} and their domains the following facts are valid:*

(a) $D(T_{\min}) \subseteq D(T_{\max})$. Domains $D(T_{\min})$ and $D(T_{\max})$ are dense in $L^2(I : \omega)$;

(b) The operator T_{\min} is closed and symmetric, the operator T_{\max} is closed in $L^2(I : \omega)$;

(c) $T_{\min}^* = T_{\max}$ and $T_{\max}^* = T_{\min}$.

All self-adjoint extensions of T_{\min} appear to be the contractions of T_{\max} .

Let Ω be a finite or a countable set of indices. On Ω , we have an Everitt-Markus-Zettl multi-interval quasi-differential system $\{I_i, \tau_i; \omega_i\}_{i \in \Omega}$. This EMZ system generates a family of the weighted Hilbert spaces $\{L^2(I_i : \omega_i) = L_i^2\}_{i \in \Omega}$ and families of minimal $\{T_{\min,i}\}_{i \in \Omega}$ and maximal $\{T_{\max,i}\}_{i \in \Omega}$ operators. Consider a respective family $\{T_i\}_{i \in \Omega}$ of self-adjoint extensions.

We introduce the system Hilbert space $\mathbf{L}^2 = \oplus_{i \in \Omega} L_i^2$ consisting of vectors $\mathbf{f} = \oplus_{i \in \Omega} f_i$, such that $f_i \in L_i^2$ and

$$\|\mathbf{f}\|^2 = \sum_{i \in \Omega} \|f_i\|_i^2 = \sum_{i \in \Omega} \int_{I_i} |f_i|^2 \omega_i dx < \infty,$$

where $\|\cdot\|_i^2$ are the norms in L_i^2 . In the space \mathbf{L}^2 consider the operator $T : D(T) \subseteq \mathbf{L}^2 \rightarrow \mathbf{L}^2$, defined on the domain

$$D(T) = \left\{ \mathbf{f} \in \bigoplus_{i \in \Omega} D(T_i) \subseteq \mathbf{L}^2 : \sum_{i \in \Omega} \|T_i f_i\|_i^2 < \infty \right\}$$

by $T\mathbf{f} = \oplus_{i \in \Omega} T_i f_i$.

Definition 1.3. The operator $T = \oplus_{i \in \Omega} T_i$ is called a *quasi-differential v-operator* generated by the self-adjoint extensions T_i , or simply a *vector-operator* (or shortly a *v-operator*). If Ω is infinite, the *v-operator* T is called *infinite*. The operators T_i are called *coordinate operators*. For $\Omega' \subset \Omega$, the operator $\oplus_{k \in \Omega'} T_k$ is called a *sub-v-operator* of the *v-operator* $\oplus_{i \in \Omega} T_i$.

The following abstract preliminaries may be found, for instance, in the books [6, 13].

Fix $i \in \Omega$. For each T_i there exists a unique resolution of the identity E_λ^i and a unitary operator U_i , making the isometrically isomorphic mapping of the Hilbert space L_i^2 onto the space $L^2(M_i, \mu_i)$, where the operator T_i is represented as a multiplication operator. Below, we remind the structure of the mapping U_i .

We call $\phi \in L_i^2$ a *cyclic vector* if for each $z \in L_i^2$ there exists a Borel function f such that $z = f(T_i)\phi$. Generally, there is no a cyclic vector in L_i^2 but there is a collection $\{\phi^k\}$ of them in L_i^2 , such that $L_i^2 = \bigoplus_k L_i^2(\phi^k)$, where $L_i^2(\phi^k)$ are T_i -invariant subspaces in L_i^2 generated by the cyclic vectors ϕ^k . That is, $L_i^2(\phi^k) = \overline{\{f(T_i)\phi^k\}}$, varying the Borel function f , such that $\phi^k \in D(f(T_i))$. There exist unitary operators

$$U^k : L_i^2(\phi^k) \longrightarrow L^2(\mathbf{R}, \mu^k),$$

where $\mu^k(\Delta) = \|E^i(\Delta)\phi^k\|_i^2$ for any Borel set Δ . In $L^2(\mathbf{R}, \mu^k)$, the operator T_i has the form of multiplication by λ , i.e.,

$$\left(U^k T_i |_{L_i^2(\phi^k)} U^{k-1} z \right) (\lambda) = \lambda z(\lambda).$$

Then the operator

$$U_i = \bigoplus_k U^k : \bigoplus_k L_i^2(\phi^k) \longrightarrow \bigoplus_k L^2(\mathbf{R}, \mu^k)$$

makes the spectral representation of the space L_i^2 onto the space $L^2(M_i, \mu_i)$, where M_i is a union of nonintersecting copies of the real line (a *sliced union*) and $\mu_i = \sum_k \mu^k$. That is, $(U_i T_i U_i^{-1} z)(\lambda) = f(\lambda)z(\lambda)$, where $z \in U[D(T_i)]$ and f is a Borel function defined almost everywhere according to the measure μ_i .

A vector $\phi \in L_i^2$ is called *maximal* relative to the operator T_i , if each measure $(E^i(\cdot)x, x)_i$, $x \in L_i^2$, is absolutely continuous relative to the measure $(E^i(\cdot)\phi, \phi)_i$.

For each Hilbert space L_i^2 , there exist a unique (up to unitary equivalence) decomposition $L_i^2 = \bigoplus_k L_i^2(\varphi_i^k)$, where φ_i^1 is maximal in L_i^2 relative to T_i , and a decreasing set of multiplicity sets e_k^i , where e_1^i is the whole line, such that $\bigoplus_k L_i^2(\varphi_i^k)$ is equivalent with $\bigoplus_k L^2(e_k^i, \mu_i)$, where the measure of the ordered representation is defined as $\mu_i(\cdot) =$

$(E^i(\cdot)\varphi_i^1, \varphi_i^1)_i$. A spectral representation of T_i in $\oplus_k L^2(e_k^i, \mu_i)$ is called the *ordered representation* and it is unique, up to a unitary equivalence. Two operators are called *equivalent*, if they create the same ordered representation of their spaces.

2. Spectral properties of the vector-operator T .

2.1 *The spectral representation for the operator T .* In this section we show how the ordinary spectral representation of the v-operator depends on the ordinary spectral representations of its coordinate operators.

Definition 2.1. For $i \in \Omega$, we introduce a *sliced union* of sets M_i (see also preliminaries) as a set M , containing all M_i on different copies of $\cup_{i \in \Omega} M_i$. The sets M_i do not intersect in M but they can *superpose*, i.e., two sets M_i and M_j superpose, if their projections in the set $\cup_{i \in \Omega} M_i$ intersect.

Separate arguments show that the following auxiliary proposition is true.

Proposition 2.2. *Let us have a set of measures $\mu_i, i \in \Omega$, defined on nonintersecting supports. If*

$$\sum_{i \in \Omega} \int_{-\infty}^{\infty} f(\lambda) d\mu_i(\lambda) < \infty,$$

for any Borel function $f(\lambda)$, then the following equality is true:

$$\sum_{i \in \Omega} \int_{-\infty}^{\infty} f(\lambda) d\mu_i(\lambda) = \int_{-\infty}^{\infty} f(\lambda) d \sum_{i \in \Omega} \mu_i(\lambda).$$

Lemma 2.3. *The identity resolution E_λ of the v-operator T equals the direct sum of the coordinate identity resolutions E_λ^i , that is,*

$$E_\lambda = \bigoplus_{i \in \Omega} E_\lambda^i.$$

Proof. Consider $\mathbf{x} \in D(T)$. This holds, if and only if

$$\|T\mathbf{x}\|^2 = \sum_{i \in \Omega} \|T_i x_i\|_i^2 = \sum_{i \in \Omega} \int_{-\infty}^{\infty} \lambda^2 d \|E_\lambda^i x_i\|_i^2 < \infty.$$

Then, using Proposition 2.2 we find out that:

$$\sum_{i \in \Omega} \int_{-\infty}^{\infty} \lambda^2 d \|E_\lambda^i x_i\|_i^2 = \int_{-\infty}^{\infty} \lambda^2 \sum_{i \in \Omega} \|E_\lambda^i x_i\|_i^2.$$

This means that $\mathbf{x} \in D(T)$, if and only if

$$\int_{-\infty}^{\infty} \lambda^2 \sum_{i \in \Omega} \|E_\lambda^i x_i\|_i^2 < \infty$$

and

$$\|T\mathbf{x}\|^2 = \int_{-\infty}^{\infty} \lambda^2 \sum_{i \in \Omega} \|E_\lambda^i x_i\|_i^2.$$

Using the uniqueness property of an identity resolution, the last two equations show that the operator $\oplus_{i \in \Omega} E_\lambda^i$ is the identity resolution of the v -operator T . That is, according to our notations $E_\lambda = \oplus_{i \in \Omega} E_\lambda^i$. The lemma is proved. \square

Lemma 2.4. *For any Borel function f and any vector $\mathbf{x} \in D(f(T))$, the following equality holds: $f(T)\mathbf{x} = [\oplus_{i \in \Omega} f(T_i)]\mathbf{x}$.*

Proof. Let $\mathbf{x} \in D(f(T))$. Then, paying attention to Proposition 2.2 and Lemma 2.3, for any $\mathbf{y} \in \mathbf{L}^2$, we obtain:

$$\begin{aligned} (f(T)\mathbf{x}, \mathbf{y}) &= \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda \mathbf{x}, \mathbf{y}) \\ &= \int_{-\infty}^{\infty} f(\lambda) d \sum_{i \in \Omega} (E_\lambda^i x_i, y_i)_i \\ &= \sum_{i \in \Omega} \int_{-\infty}^{\infty} f(\lambda) d (E_\lambda^i x_i, y_i)_i \\ &= \sum_{i \in \Omega} (f(T_i) x_i, y_i)_i \\ &= ([\oplus_{i \in \Omega} f(T)] \mathbf{x}, \mathbf{y}). \end{aligned}$$

Since \mathbf{y} is arbitrary, we have $f(T)\mathbf{x} = [\oplus_{i \in \Omega} f(T_i)]\mathbf{x}$. This completes the proof of the lemma. \square

For $z_i \in L_i^2$, $i \in \Omega$, define $\tilde{\mathbf{z}}_i = \{0, \dots, 0, z_i, 0, \dots, 0\} \in \mathbf{L}^2$, where z_i is on the i th place.

For each $i \in \Omega$, let $\varepsilon(T_i)$ denote the *subspectrum* of the operator T_i , i.e., the set where all spectral measures of T_i are concentrated. Note that $\overline{\varepsilon(T_i)} = \sigma(T_i)$.

Consider a projecting mapping $P : M \rightarrow \cup_{i \in \Omega} M_i$, see Definition 2.1, such that $P(\varepsilon(T_i)) = \varepsilon(T_i)$.

Definition 2.5. Let $\Omega = \cup_{k=1}^K A_k$, $A_k \cap A_s = \emptyset$ for $k \neq s$ and

$$A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l, P(\varepsilon(T_s)) \cap P(\varepsilon(T_l)) = B_{sl},$$

$$\text{where } \|E^t(B_{sl})\varphi_t\|_t^2 = 0 \text{ for any cyclic } \varphi_t \in L_t^2, t = s, l\}.$$

From all the possible divisions of this type we choose and fix the one which contains the minimal number of A_k . If all the coordinate spectra $\sigma(T_i)$ are simple, we call the number $\Lambda = \min\{K\}$ as the *spectral index* of the v-operator T .

Theorem 2.6. *Let each T_i have a cyclic vector a_i in L_i^2 . Then the v-operator T has Λ cyclic vectors $\{\mathbf{a}_k\}_{k=1}^\Lambda$, having the form $\mathbf{a}_k = \sum_{i \in A_k} \tilde{\mathbf{a}}_i$.*

Proof. First we consider the case of two coordinate operators. Let $s, l \in \Omega$. Then, in order to obtain one cyclic vector in $L_s^2 \oplus L_l^2$ having the form $a_s \oplus a_l$, for any $\mathbf{x} = x_s \oplus x_l \in L_s^2 \oplus L_l^2$ we have to find a Borel function f such that

$$\mathbf{x} = f(T_s \oplus T_l)[a_s \oplus a_l].$$

From Lemma 2.4 it follows that

$$\mathbf{x} = [f(T_s) \oplus f(T_l)][a_s \oplus a_l].$$

On the other hand, we must obtain each space L_p^2 , $p = s, l$, by closing the set $\{f_p(T_p)a_p\}$, letting f_p vary over all the Borel functions such

that $a_p \in D(f_p(T_p))$. If $s, l \in A_k$, then supposing that $f = f_p$ on $P(\varepsilon(T_p))$, we obtain the required function f , since any functions in the isomorphic space L^2 are considered equal on the set of measure zero. Hence, it is clear that, for all $i \in A_k$, we may build a single cyclic vector of the form

$$\mathbf{a}_k = \oplus_{i \in A_k} a_i = \sum_{i \in A_k} \tilde{\mathbf{a}}_i,$$

using the process described above, each time operating with a pair of operators.

We remind that we have the minimal number of A_k . Consider the Hilbert space

$$(3) \quad [\oplus_{i \in A_k} L_i^2] \oplus [\oplus_{j \in A_q} L_j^2], \quad k \neq q.$$

We know that then

$$[\cup_{i \in A_k} P(\varepsilon(T_i))] \cap [\cup_{j \in A_q} P(\varepsilon(T_j))] = B_{kq}$$

has a non-zero spectral measure. From the reasonings described in the beginning of this proof, we see that for joining the cyclic vectors $\mathbf{a}_k = \oplus_{i \in A_k} a_i$ and $\mathbf{a}_q = \oplus_{j \in A_q} a_j$ into the one

$$\mathbf{a}_k + \mathbf{a}_q = \sum_{i \in A_k} \tilde{\mathbf{a}}_i + \sum_{j \in A_q} \tilde{\mathbf{a}}_j,$$

we would have to derive the Hilbert space (3) by closing the set

$$\{f_k(\oplus_{i \in A_k} T_i) \mathbf{a}_k\} \oplus \{f_q(\oplus_{j \in A_q} T_j) \mathbf{a}_q\},$$

with varying the Borel functions f_k and f_q , which coincide on B_{kq} . This is not possible, since the set of such functions is not dense in the isomorphic space L^2 (the isomorphism is understood as in the spectral representation of the space (3)). Hence, we have obtained Λ cyclic vectors

$$\mathbf{a}_k = \sum_{i \in A_k} \tilde{\mathbf{a}}_i \in \mathbf{L}^2, \quad k = \overline{1, \Lambda}$$

and have proven the theorem. \square

Corollary 2.7. *Let each T_i have a single cyclic vector. Then*

1. $\Lambda = 1$ if and only if the coordinate operators T_i , $i \in \Omega$, have almost everywhere, relative to the spectral measure, pairwise non-superposing subspectra.

2 a) $\text{card}(\Omega) < \aleph_0$. $\Lambda = \text{card}(\Omega)$, if and only if all the coordinate operators T_i have pairwise superposing subspectra;

b) $\text{card}(\Omega) = \aleph_0$. $\Lambda = \infty$, if and only if T has an infinite sub- v -operator, the coordinate operators of which have pairwise superposing subspectra.

Proof. The proof directly follows from the reasonings of the proof of Theorem 2.6. \square

In the next section we will rigorously show what a spectral multiplicity of a v -operator is. Nevertheless, this notation is intuitively clear. Running ahead, let us present here two examples, which will show the difference between the spectral index and the spectral multiplicity of the v -operator T .

Example 1. Let us have a three-interval EMZ differential system $\{I_i, \tau_i, 1\}_{i=1}^3$ (an impulse, an impulse and a kinetic energy):

$$\begin{aligned}
 I_1 &= (-\infty, +\infty), & \tau_1 &= \frac{1}{i} \frac{d}{dt}, & D(T_1) &= D(T_{\max,1}); \\
 I_2 &= [0, 1], & \tau_2 &= \frac{1}{i} \frac{d}{dt}, & D(T_2) &= \{f \in D(T_{\max,2}) : \\
 & & & & & f(0) = e^{i\alpha} f(1), \\
 & & & & & \alpha \in [0, 2\pi]\}; \\
 I_3 &= [0, 1], & \tau_3 &= -\left(\frac{d}{dt}\right)^2, & D(T_3) &= \{f \in D(T_{\max,3}) : \\
 & & & & & f(0) = 0, f(1) = 0\}.
 \end{aligned}$$

If $\alpha \notin \cup_{n=1}^{\infty} (2\pi n - \pi^2 n^2)$, it may be shown that the spectra $\sigma(T_i)$ are simple and

$$\varepsilon(T_1) = \mathbf{R}, \quad \varepsilon(T_2) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha), \quad \varepsilon(T_3) = \bigcup_{n=1}^{\infty} (\pi n)^2.$$

Thus, in this example $A_1 = \{1\}$, $A_2 = \{2, 3\}$, which means that $\Lambda = 2$. From the reasonings, presented below in the process of building of the ordered representation (next section) it follows that the operator $\oplus_{i=1}^3 T_i$ has a simple spectrum. So, the spectral index does not equal the spectral multiplicity.

Example 2. We have a three-interval EMZ system $\{I_i, \tau_i, 1\}_{i=1}^3$ (a kinetic energy, a mirror kinetic energy, an impulse):

$$I_1 = [0, +\infty), \quad \tau_1 = - \left(\frac{d}{dt} \right)^2,$$

$$D(T_1) = \{f \in D(T_{\max,1}) : f(0) + kf'(0) = 0, \quad -\infty < k \leq \infty\};$$

$$I_2 = [0, +\infty), \quad \tau_2 = \left(\frac{d}{dt} \right)^2,$$

$$D(T_2) = \{f \in D(T_{\max,2}) : f(0) + sf'(0) = 0, \quad -\infty < s \leq \infty\};$$

$$I_3 = [0, 1], \quad \tau_3 = \frac{1}{i} \frac{d}{dt},$$

$$D(T_3) = \{f \in D(T_{\max,3}) : f(0) = e^{i\alpha} f(1), \quad \alpha \in [0, 2\pi]\}.$$

a) If $k, s \in (-\infty, 0] \cup \{+\infty\}$, then

$$\varepsilon(T_1) = (0, +\infty), \quad \varepsilon(T_2) = (-\infty, 0), \quad \varepsilon(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

For this system we have: $\{1, 2, 3\} = \cup_{k=1}^2 A_k$ and $A_1 = \{1, 2\}$, $A_2 = \{3\}$. Thus, here the spectral index also does not coincide with the spectral multiplicity (which is 1) and equals 2.

b) The case $0 < k, s < +\infty$ leads to the following

$$\varepsilon(T_1) = \left\{ -\frac{1}{k^2} \right\} \cup (0, +\infty), \quad \varepsilon(T_2) = (-\infty, 0) \cup \left\{ \frac{1}{s^2} \right\},$$

$$\varepsilon(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

If

$$\alpha \notin \left[\bigcup_{n=-\infty}^{\infty} \left(2\pi n + \frac{1}{k^2} \right) \right] \cup \left[\bigcup_{n=-\infty}^{\infty} \left(2\pi n - \frac{1}{s^2} \right) \right],$$

we have $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$. That is, $\Lambda = 3$ but $\oplus_{i=1}^3 T_i$ has a simple spectrum.

Example 3. Let us have a vector-operator $\oplus_{i=1}^3 T_i$ with

$$\varepsilon(T_1) = \bigcup_{n \in \mathbf{Z}, n \geq 0} \pi n, \quad \varepsilon(T_2) = \bigcup_{n \in \mathbf{Z}, n \leq 0} \pi n, \quad \varepsilon(T_3) = \bigcup_{n \in \mathbf{Z}, n \neq 0} \pi n.$$

Spectral index equals 3 but spectral multiplicity equals 2.

Definition 2.8. A v-operator $T = \oplus_{i \in \Omega} T_i$ with the simple coordinate spectra $\sigma(T_i)$ is called *distorted* if its spectral index does not equal its spectral multiplicity.

With some loss of technical value but more clearly for applications, Theorem 2.6 may be reformulated as

Corollary 2.9. *Let each T_i have a simple spectrum. Then undistorted v-operator T has Λ -multiple spectrum.*

Let us pass to the general case when each operator T_i has m_i cyclic vectors. There exists a decomposition

$$T = \bigoplus_{i \in \Omega} T_i = \bigoplus_{i \in \Omega} \bigoplus_{k=1}^{m_i} T_i^k = \bigoplus_s T_s,$$

where each T_s has a single cyclic vector. For the v-operator T decomposed as above, we apply Theorem 2.6 and find the spectral index Λ . It is clear that in this case for the spectral index there exists the estimate

$$(4) \quad \Lambda \geq \max\{m_i\}.$$

As it has been stated in the preliminaries, for each operator T_i , there exists the unitary operator U_i such that $U_i : L_i^2 \rightarrow L^2(M_i, \mu_i)$. Hence

$$\bigoplus_{i \in \Omega} U_i : \bigoplus_{i \in \Omega} L_i^2 \longrightarrow \bigoplus_{i \in \Omega} L^2(M_i, \mu_i).$$

Or, in the general case, i.e., when there are T_i with more than one cyclic vector,

$$\bigoplus_{i \in \Omega} U_i : \bigoplus_{i \in \Omega} \bigoplus_{k=1}^{m_i} L_{i,k}^2 \longrightarrow \bigoplus_{i \in \Omega} \bigoplus_{k=1}^{m_i} L^2(\mathbf{R}, \mu_i^k).$$

From Theorem 2.6 it follows that there exists a unitary operator

$$(5) \quad V : \bigoplus_{i \in \Omega} \bigoplus_{k=1}^{m_i} L^2(\mathbf{R}, \mu_i^k) = \bigoplus_s L^2(\mathbf{R}, \mu_s) \longrightarrow \bigoplus_{q=1}^{\Lambda} L^2\left(\mathbf{R}, \sum_{j \in A_q} \mu_j\right).$$

This means that for any v-operator T there exists the unitary operator $V \bigoplus_{i \in \Omega} U_i$, which represents the space \mathbf{L}^2 on the space $L_2(N, \mu)$:

$$V \bigoplus_{i \in \Omega} U_i : \mathbf{L}^2 \longrightarrow L^2(N, \mu),$$

where N is the sliced union of Λ copies of \mathbf{R} and

$$\mu = \sum_{q=1}^{\Lambda} \sum_{j \in A_q} \mu_j,$$

according to the symbols in (5). We finally obtain

Theorem 2.10. *Let the v-operator $T = \bigoplus_{i \in \Omega} T_i$ be undistorted. The unitary operator V is defined as in (5). If unitary operators U_i make spectral representations of the Hilbert spaces L_i^2 on the spaces $L^2(M_i, \mu_i)$, then the unitary operator*

$$W = V \bigoplus_{i \in \Omega} U_i$$

makes the spectral representation of the space \mathbf{L}^2 on the space $L^2(N, \mu)$.

We formulated this theorem for undistorted v-operators, since only for them the operator V will reduce the quantity of spectral measures to the minimal possible. For arbitrary v-operators (distorted and undistorted) we shall present the method of constructing an ordered representation.

2.2 *The ordered spectral representation.* Here we present the process of building the ordered representation for a v-operator. For convenience, we separate this process into units.

Let there be a v-operator $T = \oplus_{i \in \Omega} T_i$.

(A) Let a_i be maximal vectors relative to the operators T_i in L_i^2 . We want to find a maximal vector relative to the v-operator T . We know that the vector $\oplus_{i \in \Omega} a_i$ does not give a single measure, if a set $P(\varepsilon(T_i)) \cap P(\varepsilon(T_j))$ has a non-zero spectral measure for $i \neq j$. Consider restrictions $T_i|_{L_i^2(a_i)} = T'_i$. Since all the operators T'_i have single cyclic vectors a_i , we can divide Ω into A_k , $k = \overline{1, \Lambda}$, see Definition 2.5, and apply Theorem 2.6 for the operator $\oplus_{i \in \Omega} T'_i$. Thus, we have derived Λ vectors $\mathbf{a}^k = \oplus_{j \in A_k} a_j$, which are maximal in the respective spaces $\mathbf{L}^2(\mathbf{a}^k) = \oplus_{j \in A_k} L_j^2(a_j)$. Indeed, this is obvious for the case $\text{card}(A_k) < \aleph_0$. For the infinite case, if arbitrary $\mathbf{y} = \oplus_{j \in A_k} y_j \in \mathbf{L}^2(\mathbf{a}^k)$ and if

$$(6) \quad ([\oplus_{j \in A_k} E^j](\cdot)\mathbf{a}^k, \mathbf{a}^k) = \sum_{j \in A_k} (E^j(\cdot)a_j, a_j)_j = 0,$$

then from the maximality of the vectors a_j for all $j \in A_k$, and since $P(\varepsilon(T'_j)) \cap P(\varepsilon(T'_k))$ has zero spectral measures for $j \neq k$, we obtain

$$\sum_{j \in A_k} (E^j(\cdot)y_j, y_j)_j = ([\oplus_{j \in A_k} E^j](\cdot)\mathbf{y}, \mathbf{y}) = 0,$$

which follows from the convergence to zero of the series with the positive maximal elements (6). Thus, in particular, we have constructed a maximal vector in \mathbf{L}^2 for the case $\Lambda = 1$.

(B) Let now $1 < \Lambda < \infty$. Designate $T^k = \oplus_{j \in A_k} T'_j$. For any two operators T^k and T^s , $k \neq s$, let us introduce the sets $\varepsilon_{k,s} = P(\varepsilon(T^k)) \cap P(\varepsilon(T^s))$ and $\varepsilon_k = P(\varepsilon(T^k)) \setminus \varepsilon_{k,s}$. There exist unitary representations $U^k : \mathbf{L}^2(\mathbf{a}^k) \rightarrow L^2(\mathbf{R}, \mu_{\mathbf{a}^k})$, see formula (5) supposing there $\Lambda = 1$. Consider measures μ_k and $\mu_{k,s}$, defined as $\mu_{k,s}(e) = \mu_{\mathbf{a}^k}(e \cap \varepsilon_{k,s})$ and $\mu_k(e) = \mu_{\mathbf{a}^k}(e \cap \varepsilon_k)$, for any measurable set e . For any operator T^k , with respect to T^s , measures μ_k and $\mu_{k,s}$ are mutually singular and $\mu_k + \mu_{k,s} = \mu_{\mathbf{a}^k}$; therefore,

$$L^2(\mathbf{R}, \mu_{\mathbf{a}^k}) = L^2(\mathbf{R}, \mu_k) \oplus L^2(\mathbf{R}, \mu_{k,s}).$$

It means that (according to our designations):

$$U^{k-1} : L^2(\mathbf{R}, \mu_{\mathbf{a}^k}) \longrightarrow \mathbf{L}^2(\mathbf{a}_k^k) \oplus \mathbf{L}^2(\mathbf{a}_{k,s}^k)$$

and $\mathbf{a}^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k$, where \mathbf{a}_k^k and $\mathbf{a}_{k,s}^k$ form the measures μ_k and $\mu_{k,s}$ respectively. Designate also as $\max\{w, \psi\}$ the vector, which is maximal of the two vectors in the brackets (Note that this designation is valid only for vectors, considered on the same set. In order not to complicate the investigation we assume here that any two vectors are comparable in this sense. In order to achieve this, it is enough to decompose each coordinate operator T_i into the direct sum $T_i^{pp} \oplus T_i^{\text{cont}}$, where the operators have respectively pure point and continuous spectra. Then after redesignation we obtain the equivalent v-operator to the initial v-operator $\oplus T_i$).

Consider first two operators T^1 and T^2 . It is clear that the vector

$$\mathbf{a}^{1\oplus 2} = \mathbf{a}_1^1 \oplus \mathbf{a}_2^2 \oplus \max\{\mathbf{a}_{1,2}^1, \mathbf{a}_{2,1}^2\}$$

is maximal in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2)$. Note that \mathbf{a}_1^1 and \mathbf{a}_2^2 and they both may equal zero. A maximal vector in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2) \oplus \mathbf{L}^2(\mathbf{a}^3)$ will have the form:

$$\mathbf{a}^{1\oplus 2\oplus 3} = \mathbf{a}_{1\oplus 2}^{1\oplus 2} \oplus \mathbf{a}_3^3 \oplus \max\{\mathbf{a}_{1\oplus 2,3}^{1\oplus 2}, \mathbf{a}_{3,1\oplus 2}^3\}.$$

Continuing this process, we obtain a maximal vector in the main space \mathbf{L}^2 :

$$(7) \quad \mathbf{a}^{1\oplus \dots \oplus \Lambda} = \mathbf{a}_{1\oplus \dots \oplus \Lambda-1}^{1\oplus \dots \oplus \Lambda-1} \oplus \mathbf{a}_\Lambda^\Lambda \oplus \max\{\mathbf{a}_{1\oplus \dots \oplus \Lambda-1,\Lambda}^{1\oplus \dots \oplus \Lambda-1}, \mathbf{a}_{\Lambda,1\oplus \dots \oplus \Lambda-1}^\Lambda\}.$$

Formula (7) may be simplified, if we divide the measures $\mu_{\mathbf{a}^k}$ into continuous and pure point, that is, $\mu_{\mathbf{a}^k} = \mu_{\mathbf{a}^k}^{\text{cont}} + \mu_{\mathbf{a}^k}^{pp}$. Then $\mathbf{a}^k = \mathbf{a}^{k,\text{cont}} \oplus \mathbf{a}^{k,pp}$. Relative to any operator T^s , $k \neq s$, we have

$$\mathbf{a}^{k,\text{cont}} = \mathbf{a}_k^{k,\text{cont}} \oplus \mathbf{a}_{k,s}^{k,\text{cont}} \quad \text{and} \quad \mathbf{a}^{k,pp} = \mathbf{a}_k^{k,pp} \oplus \mathbf{a}_{k,s}^{k,pp}.$$

Now we can repeat the process described above in (B), separately for the continuous and the pure point parts. Since measures with the same null set may be considered equivalent, we have

$$\begin{aligned} \max\{w^{\text{cont}}, \psi^{\text{cont}}\} &= \text{either } w^{\text{cont}} \text{ or } \psi^{\text{cont}}, \\ \max\{w^{pp}, \psi^{pp}\} &= \text{either } w^{pp} \text{ or } \psi^{pp}, \end{aligned}$$

for any two vectors w and ψ . We obtain

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda,\text{cont}} = \mathbf{a}^{1,\text{cont}} \oplus \left[\bigoplus_{j=2}^{\Lambda} \mathbf{a}_j^{j,\text{cont}} \right].$$

Similarly,

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda,pp} = \mathbf{a}^{1,pp} \oplus \left[\bigoplus_{j=2}^{\Lambda} \mathbf{a}_j^{j,pp} \right].$$

Since $\max\{w^{\text{cont}}, \psi^{pp}\} = \psi^{pp}$, we finally derive

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda} = \mathbf{a}^{1\oplus\cdots\oplus\Lambda,pp} \oplus \mathbf{a}_{1\oplus\cdots\oplus\Lambda}^{1\oplus\cdots\oplus\Lambda,\text{cont}}.$$

Let $\Lambda = \infty$. We define $\mathbf{a}^{1\oplus\cdots\oplus\Lambda}$ as a vector which satisfies the following equality:

$$\left\| \left[\bigoplus_{i \in \Omega} E^i(\cdot) \right] \mathbf{a}^{1\oplus\cdots\oplus\Lambda} \right\|^2 = \lim_{L \rightarrow \infty} \left\| \left[\bigoplus_{j=1}^L E^j(\cdot) \right] \mathbf{a}^{1\oplus\cdots\oplus L} \right\|^2,$$

since the limit on the right side exists.

(C) The next step is to build the measure of the ordered representation for the v-operator. From Lemma 2.3 and the reasonings above, it follows that such a measure will be

$$\theta(\cdot) = \left(\left[\bigoplus_{i \in \Omega} E^i(\cdot) \right] \mathbf{a}^{1\oplus\cdots\oplus\Lambda}, \mathbf{a}^{1\oplus\cdots\oplus\Lambda} \right).$$

(D) The final step is to construct the canonical multiplicity sets s_n of the v-operator. s_1 is the whole line. s_2 must contain all the spectrum, multiplicity of which exceeds or equals 2. For this purpose, we are primarily to unite all e_2^i . But, nevertheless, $\cup_i e_2^i$ will not include all the sets of multiplicity ≥ 2 since we know that if $P(e_1^i \setminus e_2^i) \cap P(e_1^j \setminus e_2^j)$ has a non-zero spectral measure, all the intersections of this sort will represent the multiplicity 2 and should be included into s_2 (since then it is not possible to construct a single cyclic vector). That is, $s_2 = (\cup_i P(e_2^i)) \cup (\cup \cap (P(e_1^i \setminus e_2^i)))$. Using this idea and the fact that an infinite intersection of measurable sets is a measurable set, by induction we may finally build s_n :

$$(8) \quad s_n = \left[\bigcup_i P(e_n^i) \right] \cup \left[\bigcup_{\sum m_i \geq n} \bigcap P(e_{m_i}^i \setminus e_{m_i+1}^i) \right].$$

Eventually, we are ready to formulate the theorem:

Theorem 2.11. *The measure θ defined in (2.2) is the measure of the ordered representation of the v-operator T . The sets s_n defined in (8) are the canonical multiplicity sets of the ordered representation of the v-operator T . Thus, the spectral representation of the space \mathbf{L}^2 on the space $\oplus_n L^2(s_n, \theta)$ is the ordered representation.*

Let us return to Examples 1 and 2. For each distorted v-operator $T_1 \oplus T_2 \oplus T_3$, a spectral measure will be constructed on the vector $\mathbf{a}^{1 \oplus 2 \oplus 3}$. For the v-operator from Example 3 two spectral measures are constructed on $\mathbf{a}^{1 \oplus 2 \oplus 3}$ and

$$\min\{a_{1,2}^1, a_{2,1}^2\} \oplus \min\{a_{2,3}^2, a_{3,2}^3\} \oplus \min\{a_{3,1}^3, a_{1,3}^1\},$$

where the sense of the minimums is clear.

Now the term ‘distorted v-operator’ is clearly explained by the form of the cyclic vectors for such an operator. The multiplicity set e_2 will be

$$\begin{aligned} [P(\varepsilon(T_1)) \cap P(\varepsilon(T_2))] \cup [P(\varepsilon(T_2)) \cap P(\varepsilon(T_3))] \\ \cup [P(\varepsilon(T_3)) \cap P(\varepsilon(T_1))]. \end{aligned}$$

Using the obtained spectral representation we can construct equivalence classes in families of self-adjoint operators:

Definition 2.12. Two families of self-adjoint operators $\{T_i\}_{i=1}^N$ and $\{S_j\}_{j=1}^L$ are called *equivalent*, if the respective v-operators $\oplus_{i=1}^N T_i$ and $\oplus_{j=1}^L S_j$ are equivalent.

Note that if two families $\{T_i\}_{i=1}^N$ and $\{S_j\}_{j=1}^L$ are equivalent, it is not necessarily that $N = L$ and T_i is equivalent with S_i .

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