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GENERALIZED VERSION OF THE CHARACTERISTIC NUMBER OF TWO SIMULTANEOUS PELL'S EQUATIONS

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ABSTRACT. Let D be a given square-free natural number and N a given non-zero integer. Extending the earlier work of Mohanty and Ramasamy, we present in this paper a generalized version of the characteristic number of the simultaneous Pell's equations $U^2 - DV^2 = N$ and $Z^2 - gV^2 = h$ where g and h are given integers. A numerical example is provided at the end explaining the application of the method developed in this paper. It is shown that the only positive integral solutions common to the two Pell's equations $U^2 - 11V^2 = 5$ and $Z^2 - 17V^2 = -32$ are $U = 7$, $V = 2$ and $Z = 6$.

1. Introduction. Quite recently, Pell's equation with restriction has been studied by several authors like Anglin, Baker, Davenport, Cohn, Mohanty, Ramasamy, Pinch, Ponnudurai, Tzanakis, etc. An elaborate list of references on this subject has been furnished by Tzanakis [**10, 11**]. Baker and Davenport [**3**] determined the common solutions of the equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$ by the method of linear forms in logarithms of algebraic numbers. Anglin [**1**] presented a method for solving a system of Pell's equations with the parameters absolutely less than 1000 and, in [**2**], he considered the system $x^2 - Ry^2 = 1$ and $z^2 - Sy^2 = 1$ with $R < S \leq 200$. For the system in the general case, Tzanakis [**11**] gave a method using elliptic curves and linear forms in elliptic logarithms. Ponnudurai [**9**] dealt with the Pell's equation $U^2 - 11V^2 = -2$ with the restrictions $Y^2 = 5 + 4U$ and $X^2 = 5 + 4V$. In [**6**], the system consisting of the Pell's equations $5y^2 - 20 = x^2$ and $2y^2 + 1 = z^2$ was considered. In [**7**], the concept of the characteristic number of two simultaneous Pell's equations was introduced by Mohanty and Ramasamy. A method for a set of Pell's

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equations involving any square-free natural number D was given by the authors. The utility of this method in solving a class of quartic Thue equations has been demonstrated by Tzanakis [**10**].

One would like to have generalization in two directions, viz. results concerning

1. a general D which governs the Pell's equation on hand and

2. a general prime number or a product of primes employed in the residue classes leading to the determination of quadratic characters.

The work of Mohanty and Ramasamy [**7**] was concerned with the generalization in the first direction and specific cases involving the primes 3 and 5, and their products were considered in that paper as regards the second direction. Extending the earlier work of Mohanty and Ramasamy [**7**], a completely general result, involving a general odd prime or a product of a finite number of odd primes, not necessarily distinct, is presented here. The main result of this study is contained in Theorem 13.

Part I. The Pell's equation $A^2 - DB^2 = 1$.

2. Preliminaries. Let D be a given square-free natural number. It is well known that the Pell's equation

$$A^2 - DB^2 = 1 \tag{1}$$

always has an infinite number of integral solutions, see, e.g., Nagell [**8**]. All the solutions of (1) with positive A and B are obtained by the formula

$$A_r + B_r\sqrt{D} = (a + b\sqrt{D})^r$$

where $r = 1, 2, 3, \dots$, and $a + b\sqrt{D}$ is the fundamental solution of (1), i.e., $A_1 = a$ and $B_1 = b$. Copley [**4**] has given the following relations for the solutions $A_r + B_r\sqrt{D}$ of (1).

$$A_{r+s} = A_rA_s + DB_rB_s \tag{2}$$
$$B_{r+s} = A_rB_s + B_rA_s \tag{3}$$
$$A_{r+1} = A_1A_r + DB_1B_r \tag{4}$$
$$B_{r+1} = A_1B_r + B_1A_r \tag{5}$$
$$A_{2r} = 2A_r^2 - 1 \tag{6}$$
$$B_{2r} = 2A_rB_r \tag{7}$$

Emerson [**5**] gave the relations

$$A_{r+2} = 2aA_{r+1} - A_r \tag{8}$$
$$B_{r+2} = 2aB_{r+1} - B_r. \tag{9}$$

3. Notations and recurrence relations. Let $\alpha_{n,k}$, respectively $\beta_{n,k}$, denote the absolute value of the coefficient of a^k, respectively $a^{k-1}b$, in the expression for A_n, respectively B_n, arising from (1). Then we have the expressions

$$A_n = \alpha_{n,n}a^n - \alpha_{n,n-2}a^{n-2} + \alpha_{n,n-4}a^{n-4} - \cdots \tag{10}$$

and

$$B_n = \beta_{n,n}a^{n-1}b - \beta_{n,n-2}a^{n-3}b + \beta_{n,n-4}a^{n-5}b - \cdots . \tag{11}$$

One may check that $\alpha_{0,0} = 1$, $\alpha_{1,1} = 1$, $\alpha_{1,0} = 0$, $\alpha_{2,2} = 2$, $\alpha_{2,1} = 0$, $\alpha_{2,0} = 1$, etc., and $\beta_{0,0} = 0$, $\beta_{1,1} = 1$, $\beta_{1,0} = 0$, $\beta_{2,2} = 2$, $\beta_{2,1} = 0$, $\beta_{2,0} = 0$, etc.

Using induction, one may establish the following relations.

Theorem 1. 1. $\alpha_{0,0} = 1$, $\beta_{0,0} = 0$.

2. *For* $n > 0$, $k \geq 0$, $k < n$,

$$\alpha_{n,k} \begin{cases} > 0 & \text{if } k \equiv n \pmod 2 \\ = 0 & \text{otherwise} \end{cases} .$$

3. *For all* $n > 0$, $\beta_{n,0} = 0$ *and, for* $n > 0$, $k > 0$, $k < n$,

$$\beta_{n,k} \begin{cases} > 0 & \textit{if } k \equiv n \pmod 2 \\ = 0 & \textit{otherwise} \end{cases}.$$

Theorem 2.

$$\alpha_{n+1,n+1} = 2\alpha_{n,n} \quad \textit{for all } n > 0 \tag{12}$$

$$\alpha_{n,k} = 2\alpha_{n-1,k-1} + \alpha_{n-2,k} \quad \textit{for all } n \geq 3 \textit{ and } 0 < k < n. \tag{13}$$

Theorem 3.

$$\beta_{n+1,n+1} = 2\beta_{n,n} \quad \textit{for all } n > 0 \tag{14}$$

$$\beta_{n,k} = 2\beta_{n-1,k-1} + \beta_{n-2,k} \quad \textit{for all } n \geq 3 \textit{ and } 0 < k < n. \tag{15}$$

The relations between the α's and β's of (10) and (11) are given in the sequel.

Theorem 4.

$$\alpha_{n,n} = \beta_{n,n} \quad \textit{for all } n > 0. \tag{16}$$

Theorem 5.

$$\alpha_{n,n} = \alpha_{n-1,n-1} + \beta_{n-1,n-1}, \quad n > 0, \tag{17}$$

$$\beta_{n,n} = \alpha_{n-1,n-1} + \beta_{n-1,n-1}, \quad n > 0. \tag{18}$$

Theorem 6. *Let* n *and* k *be integers such that* $0 < k < n$.

$$\beta_{n,k} = \alpha_{n-1,k-1} + \beta_{n-1,k-1}, \quad n \geq 2 \tag{19}$$

$$\alpha_{n,k} + \alpha_{n-1,k+1} = \beta_{n,k+2} + \beta_{n,k}, \quad n \geq 2 \tag{20}$$

$$\alpha_{n,k} - \alpha_{n-2,k} = \beta_{n,k} + \beta_{n-2,k}, \quad n \geq 3 \tag{21}$$

$$\alpha_{n,k} = \beta_{n-1,k+1} + \beta_{n,k}, \quad n \geq 2. \tag{22}$$

Corollary 1.

$$\alpha_{n,k} = \alpha_{n-2,k} + 2\alpha_{n-2,k-2} + 2\beta_{n-2,k} + 2\beta_{n-2,k-2} \tag{23}$$
$$\beta_{n,k} = 2\alpha_{n-2,k-2} + 2\beta_{n-2,k-2} + \beta_{n-2,k}. \tag{24}$$

Theorem 7.

$$\begin{aligned}
\alpha_{n,0} &= 1 && \text{when } n \text{ is even,}\\
\alpha_{n,1} &= n && \text{when } n \text{ is odd,}\\
\alpha_{n,2} &= \frac{n^2}{2} && \text{when } n \text{ is even,}\\
\beta_{n,1} &= 1 && \text{when } n \text{ is odd,}\\
\beta_{n,2} &= n && \text{when } n \text{ is even,}\\
\beta_{n,3} &= \frac{(n^2-1)}{2} && \text{when } n \text{ is odd} \geq 3.
\end{aligned}$$

Corollary 2. *When n is odd,*

$$\alpha_{n,1} + 2\beta_{n,1} = \alpha_{n+2,1}.$$

4. Identities and the consequence. Now we derive two identities for the solutions $A_r + B_r\sqrt{D}$ of (1), which are significant from computational point of view.

Theorem 8. *For all integers $n \geq 2$,*

$$A_nA_{n-2} - A_{n-1}^2 = Db^2 \tag{25}$$
$$B_nB_{n-2} - B_{n-1}^2 = -b^2. \tag{26}$$

Proof. It is straightforward to check the validity of the relations for $n = 2$. Assume (25) for all positive integers $n \geq 2$ up to m. Now, using

(8), we get

$$\begin{aligned} A_{m+1}A_{m-1} &= (2aA_m - A_{m-1})A_{m-1} \\ &= 2aA_mA_{m-1} - (A_mA_{m-2} - Db^2) \quad \text{by assumption} \\ &= A_m(2aA_{m-1} - A_{m-2}) + Db^2 \\ &= A_m^2 + Db^2, \quad \text{again using (8).} \end{aligned}$$

This proves (25). Employing (9), a similar proof follows for (26) by induction on n. □

As a consequence of the identity (26) and the expression for B_n given by (11), we have the following

Theorem 9. *Let n be odd. Then the β's satisfy the following relations*:

$$\begin{aligned} \beta_{n,n}\beta_{n-2,n-2} &= \beta_{n-1,n-1}^2; \\ \beta_{n,n}\beta_{n-2,n-4} + \beta_{n,n-2}\beta_{n-2,n-2} &= 2\beta_{n-1,n-1}\beta_{n-1,n-3}; \\ \beta_{n,n}\beta_{n-2,n-6} + \beta_{n,n-2}\beta_{n-2,n-4} + \beta_{n,n-4}\beta_{n-2,n-2} &= 2\beta_{n-1,n-1}\beta_{n-1,n-5} \\ &\quad + \beta_{n-1,n-3}^2; \\ &\vdots \end{aligned}$$

$$\beta_{n,n}\beta_{n-2,3} + \beta_{n,n-2}\beta_{n-2,5} + \beta_{n,n-4}\beta_{n-2,7} + \cdots + \beta_{n,5}\beta_{n-2,n-2}$$
$$= \begin{cases} 2\beta_{n-1,n-1}\beta_{n-1,4} + 2\beta_{n-1,n-3}\beta_{n-1,6} + \cdots + \beta_{n-1,(n+3)/2}^2 \\ \quad \text{if } n \equiv 1 \pmod 4 \\ 2\beta_{n-1,n-1}\beta_{n-1,4} + 2\beta_{n-1,n-3}\beta_{n-1,6} + \cdots + 2\beta_{n-1,(n+5)/2}\beta_{n-1,(n+1)/2} \\ \quad \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\beta_{n,n}\beta_{n-2,1} + \beta_{n,n-2}\beta_{n-2,3} + \beta_{n,n-4}\beta_{n-2,5} + \cdots + \beta_{n,3}\beta_{n-2,n-2}$$
$$= \begin{cases} 2\beta_{n-1,n-1}\beta_{n-1,2} + 2\beta_{n-1,n-3}\beta_{n-1,4} + \cdots + 2\beta_{n-1,(n+3)/2}\beta_{n-1,(n-1)/2} \\ \quad \text{if } n \equiv 1 \pmod 4 \\ 2\beta_{n-1,n-1}\beta_{n-1,2} + 2\beta_{n-1,n-3}\beta_{n-1,4} + \cdots + \beta_{n-1,(n+1)/2}^2 \\ \quad \text{if } n \equiv 3 \pmod 4 \end{cases}$$

$$\beta_{n,n-2}\beta_{n-2,1}+\beta_{n,n-4}\beta_{n-2,3}+\beta_{n,n-6}\beta_{n-2,5}+\cdots+\beta_{n,1}\beta_{n-2,n-2}$$
$$=\begin{cases} 2\beta_{n-1,n-3}\beta_{n-1,2}+2\beta_{n-1,n-5}\beta_{n-1,4}+\cdots+\beta_{n-1,(n-1)/2}^2 \\ \quad \text{if } n\equiv 1 \pmod 4 \\ 2\beta_{n-1,n-3}\beta_{n-1,2}+2\beta_{n-1,n-5}\beta_{n-1,4}+\cdots+2\beta_{n-1,(n+1)/2}\beta_{n-1,(n-3)/2} \\ \quad \text{if } n\equiv 3 \pmod 4 \end{cases}$$

$$\vdots$$

$$\begin{aligned} \beta_{n,7}\beta_{n-2,1}+\beta_{n,5}\beta_{n-2,3}+\beta_{n,3}\beta_{n-2,5}+\beta_{n,1}\beta_{n-2,7} &= 2\beta_{n-1,6}\beta_{n-1,2} \\ &\quad+\beta_{n-1,4}^2; \\ \beta_{n,5}\beta_{n-2,1}+\beta_{n,3}\beta_{n-2,3}+\beta_{n,1}\beta_{n-2,5} &= 2\beta_{n-1,4}\beta_{n-1,2}; \\ \beta_{n,3}\beta_{n-2,1}+\beta_{n,1}\beta_{n-2,3} &= \beta_{n-1,2}^2; \\ \beta_{n,1}\beta_{n-2,1} &= 1. \end{aligned}$$

Theorem 10. *When* $n\equiv k \pmod 2$,

$$\beta_{n,k}=\binom{(n+k)/2}{k}\frac{k}{n+k}\,2^k \tag{27}$$

where $\binom{i}{j}$ *denotes the number of combinations of* i *objects taken* j *at a time.*

Proof. Let $F(x,y)$ be the generating function of the recurrence sequence given by (15) with the boundary conditions $\beta_{0,0}=0$, $\beta_{1,0}=0$, $\beta_{1,1}=1$. Then

$$F(x,y)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\beta_{n,k}x^ny^k=\frac{xy}{1-2xy-x^2}.$$

From this, we obtain the expression

$$\beta_{n,k}=\binom{(n+k-2)/2}{k-1}2^{k-1}=\binom{(n+k)/2}{k}\frac{k}{n+k}\,2^k. \qquad \square$$

5. Transferability of coefficients. Now we establish a remarkable property of the coefficients αs and βs, which is highly useful for reducing the computations to a large extent. When n is odd, A_n and B_n have transferable coefficients in the sense that the coefficients of B_n in (11) can be used to write the expression of A_n in terms of D and b while the coefficients of A_n in (10) can be used to write the expression of B_n in terms of D and b as provided by

Theorem 11. *Let n be an odd integer ≥ 3. Then*

$$A_n = a(\beta_{n,n}D^{(n-1)/2}b^{n-1} + \beta_{n,n-2}D^{(n-3)/2}b^{n-3} + \cdots + \beta_{n,1}) \tag{28}$$

$$B_n = b(\alpha_{n,n}D^{(n-1)/2}b^{n-1} + \alpha_{n,n-2}D^{(n-3)/2}b^{n-3} + \cdots + \alpha_{n,1}) \tag{29}$$

Proof. We observe that

$$A_3 = a(4a^2 - 3) = a(4Db^2 + 1) = a(\beta_{3,3}Db^2 + \beta_{3,1}),$$
$$B_3 = b(4a^2 - 1) = b(4Db^2 + 3) = b(\alpha_{3,3}Db^2 + \alpha_{3,1}).$$

Thus (28) and (29) hold for $n = 3$. Now assume these relations for all integers n up to m. Consider the case when $n = m + 2$. From (2) we have

$$\begin{aligned} A_{m+2} &= (2Db^2 + 1)A_m + 2abDB_m \\ &= a(2Db^2 + 1)(\beta_{m,m}D^{(m-1)/2}b^{m-1} + \beta_{m,m-2}D^{(m-3)/2}b^{m-3} + \cdots \\ &\qquad + \beta_{m,1}) \\ &\quad + 2ab^2D(\alpha_{m,m}D^{(m-1)/2}b^{m-1} + \alpha_{m,m-2}D^{(m-3)/2}b^{m-3} + \cdots \\ &\qquad + \alpha_{m,1}) \end{aligned}$$

by induction assumption. Thus we have

$$\begin{aligned} A_{m+2} &= a\big[2(\alpha_{m,m} + \beta_{m,m})D^{(m+1)/2}b^{m+1} \\ &\qquad + (2\alpha_{m,m-2} + 2\beta_{m,m-2} + \beta_{m,m})D^{(m-1)/2}b^{m-1} \\ &\qquad + \cdots + (2\alpha_{m,1} + 2\beta_{m,1} + \beta_{m,3})Db^2 + \beta_{m,1}\big] \\ &= a\big[\beta_{m+2,m+2}D^{(m+1)/2}b^{m+1} + \beta_{m+2,m}D^{(m-1)/2}b^{m-1} + \cdots \\ &\qquad + \beta_{m+2,3}Db^2 + \beta_{m+2,1}\big] \end{aligned}$$

using (14), (18), (24) and Theorem 7. This proves (28). Similarly, using (3), (12), (17), (23) and the corollary to Theorem 7, we check that (29) holds. $\square$

6. The Functions $a(t)$ and $b(t)$.

Definition 1 (Mohanty and Ramasamy [**7**]). Let t be a natural number. Define

$$a(t) = A_{2^{t-1}} \tag{30}$$

and

$$b(t) = B_{2^{t-1}} \tag{31}$$

These functions have the following properties:

$$a(t+1) = 2(a(t))^2 - 1 \tag{32}$$

$$b(t+1) = 2a(t)b(t) \tag{33}$$

Using the relations (32) and (33), by induction we obtain the following:

$$a(t) \equiv 1 \pmod 8, \quad \text{for all } t \geq 3 \tag{34}$$

$$b(t) \equiv 0 \pmod 8, \quad \text{for all } t \geq 4 \tag{35}$$

$$b(3) \equiv 0 \pmod 4. \tag{36}$$

Lemma 1.

$$\left(\frac{-1}{(a(t))^2 + D(b(t))^2}\right) = \left(\frac{D}{(a(t))^2 + D(b(t))^2}\right) = \begin{cases} -1 & \textit{if } D \equiv 3 \pmod 4,\ t = 1 \textit{ and } a \textit{ is even} \\ +1 & \textit{otherwise,} \end{cases}$$

where (e/f) *denotes the Jacobi symbol.*

Corollary 3.

$$\left(\frac{-D}{(a(t))^2 + D(b(t))^2}\right) = +1 \quad \textit{for all } t \geq 1.$$

7. Crucial factorization theorem. Now we establish the factorization of an expression involving the function $b(t)$ given by (31) and the coefficients β's arising from (11). The factorization which plays a crucial role in the derivation of the characteristic number of a system of Pell's equations in the generalized case is furnished by the following

Theorem 12. *When n is odd,*

(37)

$$\begin{aligned} D(b(t))^2\big(&\beta_{n-1,n-1}D^{(n-3)/2}(b(t))^{n-3}+\beta_{n-1,n-3}D^{(n-5)/2}(b(t))^{n-5}+\cdots \\ &+\beta_{n-1,4}D(b(t))^2+\beta_{n-1,2}\big)^2+1 \\ =\big(&\beta_{n,n}D^{(n-1)/2}(b(t))^{n-1}+\beta_{n,n-2}D^{(n-3)/2}(b(t))^{n-3}+\cdots \\ &+\beta_{n,3}D(b(t))^2+\beta_{n,1}\big) \\ &\times\big(\beta_{n-2,n-2}D^{(n-3)/2}(b(t))^{n-3}+\beta_{n-2,n-4}D^{(n-5)/2}(b(t))^{n-5}+\cdots \\ &+\beta_{n-2,3}D(b(t))^2+\beta_{n-2,1}\big). \end{aligned}$$

Proof. Expanding the expression in the left side of (37), we get

$$\begin{aligned} &\beta_{n-1,n-1}^2D^{n-2}(b(t))^{2n-4}+2\beta_{n-1,n-1}\beta_{n-1,n-3}D^{n-3}(b(t))^{2n-6} \\ &\quad+(2\beta_{n-1,n-1}\beta_{n-1,n-5}+\beta_{n-1,n-3}^2)D^{n-4}(b(t))^{2n-8}+\cdots \\ &\quad+(2\beta_{n-1,6}\beta_{n-1,2}+\beta_{n-1,4}^2)D^3(b(t))^6+2\beta_{n-1,4}\beta_{n-1,2} \\ &\quad\times D^2(b(t))^4+\beta_{n-1,2}^2D(b(t))^2+1 \\ &=\beta_{n,n}\beta_{n-2,n-2}D^{n-2}(b(t))^{2n-4}+(\beta_{n,n}\beta_{n-2,n-4}+\beta_{n,n-2}\beta_{n-2,n-2}) \\ &\qquad\qquad\times D^{n-3}(b(t))^{2n-6} \\ &\quad+(\beta_{n,n}\beta_{n-2,n-6}+\beta_{n,n-2}\beta_{n-2,n-4}+\beta_{n,n-4}\beta_{n-2,n-2}) \\ &\quad\times D^{n-4}(b(t))^{2n-8}+\cdots \\ &\quad+(\beta_{n,5}\beta_{n-2,1}+\beta_{n,3}\beta_{n-2,3}+\beta_{n,1}\beta_{n-2,5})D^2(b(t))^4 \\ &\quad+(\beta_{n,3}\beta_{n-2,1}+\beta_{n,1}\beta_{n-2,3})D(b(t))^2+\beta_{n,1}\beta_{n-2,1} \end{aligned}$$

using Theorem 9. This completes the proof. □

Part II. The Pell's equation $U^2 - DV^2 = N$.

8. Results in brief. Now we consider the general Pell's equation

$$U^2 - DV^2 = N \tag{38}$$

where N is a given non-zero integer. We assume the solvability of (38). For the concept of a class of solutions of (38) and related results, one may see Nagell [**8**]. We consider a class K of solutions of (38) and fix it. Let $u + v\sqrt{D}$ be the fundamental solution of (38) contained in K. Let $U_r + V_r\sqrt{D}$, $r = 0, 1, 2, \dots$, be the solutions of (38) in K. Then

$$U_r + V_r\sqrt{D} = (u + v\,\sqrt{D})\,(a + b\,\sqrt{D})^r.$$

We have the following relations (Mohanty and Ramasamy [**7**]):

$$U_{r+s} = A_s U_r + DB_s V_r \tag{39}$$

$$V_{r+s} = B_s U_r + A_s V_r \tag{40}$$

$$U_{r+2s} \equiv -U_r \pmod{A_s} \tag{41}$$

$$V_{r+2s} \equiv -V_r \pmod{A_s}. \tag{42}$$

9. Characteristic number of a system. Consider a finite number of given odd primes $p_1, p_2, \dots, p_s$, not necessarily distinct. Choose one of these primes, fix it and denote it by p. Let P denote $p_1 p_2 \cdots p_s$. Now we discuss the method of establishing that there cannot exist a non-negative integer n with $n \equiv i \pmod{m}$, $n \neq i$, $0 \leq i < m$ and $V = V_n$ which will satisfy the system of Pell's equations

$$\begin{cases} U^2 - DV^2 = N, \\ Z^2 - g\,V^2 = h \end{cases} \tag{43}$$

where g and h are given integers and m is $2p$, respectively $2P$, or its multiple by a power of 2.

Case (I). First, let m be $2p$ or its multiple by a power of 2 where p is a given odd prime. Using the expression for m, write $n = i + p \cdot 2^t(2\mu + 1)$ where μ is a non-negative integer and t is an appropriately chosen natural number. For example, if $m = 2p$, then $t \geq 1$; if $m = 4p$, then $t \geq 2$, etc. Denote 2^t by k. Using (42) successively, we have $V_n \equiv (-1)^\mu V_{pk+i} \pmod{A_{pk}}$. Using (40), we obtain $V_n \equiv (-1)^\mu U_i B_{pk} \pmod{A_{pk}}$. Hence

$$Z^2 = gU_i^2 B_{pk}^2 + h \pmod{A_{pk}}.$$

In view of (10), (11), (28) and (29), we get

$$\begin{aligned} Z^2 \equiv gU_i^2(b(t+1))^2(\beta_{p,p}(a(t+1))^{p-1} - \beta_{p,p-2}(a(t+1))^{p-3} \\ + \beta_{p,p-4}(a(t+1))^{p-5} - \cdots)^2 + h \\ (\text{mod } a(t+1)(\alpha_{p,p}(a(t+1))^{p-1} - \alpha_{p,p-2}(a(t+1))^{p-3} \\ + \alpha_{p,p-4}(a(t+1))^{p-5} - \cdots)). \end{aligned} \tag{44}$$

Considering (44) modulo $a(t+1)$, we have

$$\begin{aligned} Z^2 &\equiv gU_i^2(b(t+1))^2 + h && (\text{mod } a(t+1)) \\ &\equiv 2(2gU_i^2(a(t))^2 - Dh)\,(b(t))^2 && (\text{mod } (a(t))^2 + D(b(t))^2) \\ &\equiv -4D(gU_i^2 - Dh)\,(b(t))^4 && (\text{mod } (a(t))^2 + D(b(t))^2). \end{aligned} \tag{45}$$

Now

$$\left(\frac{-4D(gU_i^2 - Dh)(b(t))^4}{(a(t))^2 + D(b(t))^2}\right) = \left(\frac{gU_i^2 - Dh}{(a(t))^2 + D(b(t))^2}\right)$$

using Corollary 3.

Next, considering (44) modulo $\alpha_{p,p}(a(t+1))^{p-1} - \alpha_{p,p-2}(a(t+1))^{p-3} + \alpha_{p,p-4}(a(t+1))^{p-5} - \cdots$, we obtain

(46)

$$\begin{aligned} Z^2 \equiv h + gU_i^2(b(t+1))^2(\beta_{p,p}(a(t+1))^{p-1} - \beta_{p,p-2}(a(t+1))^{p-3} \\ + \beta_{p,p-4}(a(t+1))^{p-5} - \cdots)^2 \\ (\text{mod } \alpha_{p,p}(a(t+1))^{p-1} - \alpha_{p,p-2}(a(t+1))^{p-3} \\ + \alpha_{p,p-4}(a(t+1))^{p-5} - \cdots). \end{aligned}$$

Applying the property of transferability of the coefficients as provided by (28) and (29), we get

(47)

$$\begin{aligned} Z^2 &\equiv h + gU_i^2(b(t+1))^2(\alpha_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} \\ &\quad + \alpha_{p,p-2}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \alpha_{p,3}D(b(t+1))^2 + \alpha_{p,1})^2 \\ &\quad (\bmod \beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \beta_{p,p-2}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots \\ &\quad + \beta_{p,3}D(b(t+1))^2 + \beta_{p,1})) \\ &= h + gU_i^2(b(t+1))^2\big[(\alpha_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \alpha_{p,1}) \\ &\qquad - (\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1})\big]^2 \\ &= h + gU_i^2(b(t+1))^2 \times \big[(\alpha_{p,p} - \beta_{p,p})D^{(p-1)/2}(b(t+1))^{p-1} \\ &\quad + (\alpha_{p,p-2} - \beta_{p,p-2})D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + (\alpha_{p,1} - \beta_{p,1})\big]^2. \end{aligned}$$

Invoking (16) and (22), we obtain

(48) $$Z^2 \equiv h + gU_i^2(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \beta_{p-1,2})^2.$$

Now

$$\begin{aligned} &\left(\frac{h + gU_i^2(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \beta_{p-1,2})^2}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) \\ &= \left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) \\ &\quad \times \left(\frac{Dh + gU_i^2D(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \beta_{p-1,2})^2}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right). \end{aligned}$$

Therefore, to determine the quadratic character of the expression in the right side of (48) with respect to $\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}$, we have to consider the quadratic character of

1. D, and
2. $Dh + gU_i^2D(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \beta_{p-1,2})^2$.

Now we shall make use of the crucial factorization theorem provided by (37). Considering

$$Dh + gU_i^2 D(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots + \beta_{p-1,2})^2$$
$$\text{modulo}\, \beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1},$$

and applying (37) we obtain

$$\begin{aligned} &gU_i^2\big[(\beta_{p-2,p-2}D^{(p-3)/2}(b(t+1))^{p-3} + \beta_{p-2,p-4}D^{(p-5)/2}(b(t+1))^{p-5} + \cdots \\ &\quad + \beta_{p-2,3}D(b(t+1))^2\beta_{p-2,1}) \\ &\quad \times (\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \beta_{p,p-2}D^{(p-3)/2}(b(t+1))^{p-3} + \cdots \\ &\qquad + \beta_{p,3}D(b(t+1))^2 + \beta_{p,1}) - 1\big] + Dh, \\ &\equiv Dh + gU_i^2(-1) \\ &\equiv -(gU_i^2 - Dh). \end{aligned}$$

Hence

$$\begin{aligned} &\left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right)\left(\frac{-(gU_i^2 - Dh)}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) \\ &= \left(\frac{-1}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right)\left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) \\ &\quad \times \left(\frac{gU_i^2 - Dh}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right). \end{aligned}$$

Clearly,

$$\left(\frac{-1}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) = +1.$$

If $D = 2$, then

$$\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1} \equiv +1 \pmod 8$$

and so

$$\left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right) = +1.$$

If D is odd, then

$$\begin{aligned}&\left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1}+\cdots+\beta_{p,3}D(b(t+1))^2+\beta_{p,1}}\right)\\&\quad=\left(\frac{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1}+\cdots+\beta_{p,3}D(b(t+1))^2+\beta_{p,1}}{D}\right)\\&\quad=\left(\frac{\beta_{p,1}}{D}\right)=\left(\frac{1}{D}\right)=+1.\end{aligned}$$

Thus in any case we obtain

$$\left(\frac{D}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1}+\cdots+\beta_{p,1}}\right)=+1.$$

Hence,

$$\begin{aligned}&\left(\frac{h+gU_i^2(b(t+1))^2(\beta_{p-1,p-1}D^{(p-3)/2}(b(t+1))^{p-3}+\cdots+\beta_{p-1,2})^2}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1}+\cdots+\beta_{p,1}}\right)\\&\qquad\qquad=\left(\frac{gU_i^2-Dh}{\beta_{p,p}D^{(p-1)/2}(b(t+1))^{p-1}+\cdots+\beta_{p,1}}\right).\end{aligned}$$

Case (II). Next let m be $2P$ or its multiple by a power of 2 where P is the product of given odd primes $p_1, p_2, \ldots, p_s$, not necessarily distinct. Then $n = i + P\cdot 2^t(2\mu+1)$ and we get $V_n = (-1)^\mu U_i B_{Pk} \pmod{A_{Pk}}$ where $k = 2^t$. Hence

$$\begin{aligned}Z^2 \equiv\ & h+gU_i^2(b(t+1))^2(\beta_{P,P}(a(t+1))^{P-1}\\&-\beta_{P,P-2}(a(t+1))^{P-3}+\beta_{P,P-4}(a(t+1))^{P-5}-\cdots)^2\\&(\bmod\ a(t+1)(\alpha_{P,P}(a(t+1))^{P-1}\\&-\alpha_{P,P-2}(a(t+1))^{P-3}+\alpha_{P,P-4}(a(t+1))^{P-5}-\cdots)).\end{aligned}\tag{49}$$

Considering (44) modulo $a(t+1)$, we arrive at

$$\left(\frac{gU_i^2-Dh}{(a(t))^2+D(b(t))^2}\right).$$

In this case, we have to consider the factors of the polynomial $\alpha_{P,P}(a(t+1))^{P-1}-\alpha_{P,P-2}(a(t+1))^{P-3}+\alpha_{P,P-4}(a(t+1))^{P-5}-\cdots)$ or equivalently those of the polynomial $\beta_{P,P}D^{(P-1)/2}(b(t+1))^{P-1}+\cdots+\beta_{P,1}$. The corresponding factors of the polynomials in $\{a(t+1)\}$ and $\{b(t+1)\}$ sequences have transferable coefficients as provided by Theorem 11. Each distinct prime p_j in the set $\{p_1, p_2, \ldots, p_s\}$ contributes a factor F_j of $\beta_{P,P}D^{(P-1)/2}(b(t+1))^{P-1}+\cdots+\beta_{P,1}$ of degree p_j-1. These factors are polynomials in $D(b(t+1))^2$. Letting $p_1, \ldots, p_w$ be the distinct primes in the set $\{p_1, p_2, \ldots, p_s\}$ we have

$$\beta_{P,P}D^{(P-1)/2}(b(t+1))^{P-1}+\cdots+\beta_{P,1} = F_1\ F_2\ \cdots\ F_w\ G$$

where

(50)

$$\begin{aligned} F_j = {} & \beta_{q,q}D^{(q-1)/2}(b(t+1))^{q-1}+\beta_{q,q-2}D^{(q-3)/2}(b(t+1))^{q-3}+\cdots \\ & + \beta_{q,3}D(b(t+1))^2+\beta_{q,1} \end{aligned}$$

with $q=p_j$ and G is a polynomial in $D(b(t+1))^2$ of degree $P-(p_1+\cdots+p_w)+w-1$. If G is reducible, let G_d be an irreducible polynomial dividing G. If G is irreducible, let G_d denote G itself. Once again we shall use Theorem 12.

Let σ denote any F_j or G_d. Considering

$$h+gU_i^2(b(t+1))^2(\alpha_{P,P}D^{(P-1)/2}(b(t+1))^{P-1}+\cdots+\alpha_{P,1})^2$$

modulo σ, we get

$$\begin{aligned} & h+gU_i^2(b(t+1))^2(\alpha_{P,P}D^{(P-1)/2}(b(t+1))^{P-1}+\cdots \\ & \qquad +\alpha_{P,1}-F_1\ F_2\ \cdots\ F_w\ G)^2 \\ & \quad \equiv gU_i^2[F_1\ F_2\ \cdots\ F_w\ G(\beta_{P-2,P-2}D^{(P-3)/2}(b(t+1))^{P-3}+\cdots \\ & \qquad +\beta_{P-2,1})-1]+Dh, \quad \text{because of (37)} \\ & \quad \equiv Dh+gU_i^2(-1) \\ & \quad \equiv -(gU_i^2-Dh) \pmod{\sigma} \quad \text{with} \quad \sigma = F_j \text{ or } G_d. \end{aligned}$$

Thus we are led to the following

Theorem 13. *Let $p_1=p$, $p_2, \ldots, p_s$ be given odd primes, not necessarily distinct. Choose a modulo m, which has the form*

(1) $2^\tau \cdot 2p$ *or*

(2) $2^\tau \cdot 2p_1\, p_2 \cdots p_s$ *with* $\tau \geq 0$.

Let $n = i + p \cdot 2^t(2\mu + 1)$, $t \geq 1$. *If, for each* $t \geq 1$, *in case* (1) *at least one of the Jacobi symbols*

$$\left(\frac{gU_i^2 - Dh}{(a(t))^2 + D(b(t))^2}\right), \left(\frac{gU_i^2 - Dh}{\beta_{p,p} D^{(p-1)/2}(b(t+1))^{p-1} + \cdots + \beta_{p,1}}\right)$$

equals -1, *and in case* (2), *with the notation of* (50) *at least one of*

$$\left(\frac{gU_i^2 - Dh}{(a(t))^2 + D(b(t))^2}\right), \left(\frac{gU_i^2 - Dh}{F_1}\right), \ldots, \left(\frac{gU_i^2 - Dh}{F_w}\right), \left(\frac{gU_i^2 - Dh}{G_d}\right)$$

equals -1, *then the system* (43) *has no solution with* $V = V_n$ *and* $n \equiv i \pmod m$, *except possibly* $V = V_i$.

Definition 2. Since the number $gU_i^2 - Dh$ plays a fundamental role, we call it the characteristic number of the system (43) for given integer i.

10. Numerical example. In this section a numerical example is provided so as to illustrate the application of the method developed in the previous sections. Let us consider the system of Pell's equations

$$\begin{cases} U^2 - 11V^2 = 5 \\ Z^2 - 17V^2 = -32 \end{cases} \tag{51}$$

We determine the common solutions of the two equations in the above system.

The Pell's equation

$$A^2 - 11B^2 = 1 \tag{52}$$

has the fundamental solution $10 + 3\sqrt{11}$. The Pell's equation $U^2 - 11V^2 = 5$ has two non-associated classes of solutions and the fundamental solutions are $4 - \sqrt{11}$, $4 + \sqrt{11}$ respectively. So the general solution of the equation $U^2 - 11V^2 = 5$ in the concerned class is given by

$$U_r + \sqrt{11}\, V_r = (4 - \sqrt{11})\,(10 + 3\sqrt{11})^r \tag{53}$$

or

$$U_r + \sqrt{11}\, V_r = (4 + \sqrt{11})\,(10 + 3\sqrt{11})^r \tag{54}$$

as the case may be. Consider the class of solutions of the Pell's equation $U^2 - 11V^2 = 5$ arising from (53) with the restriction given by $17V^2 - 32 = Z^2$. It is ascertained that $r \equiv 1 \pmod 2$, $r \equiv 1 \pmod 9$, $r \equiv 1 \pmod{11}$ and $r \equiv 1 \pmod{13}$. A sketch of the stepwise calculations is provided here. Since equation (42) implies that $V_{r+22} \equiv -V_r \pmod{89}$, the sequence $V_r \pmod{89}$ is periodic with period 44 and the sequence $17(V_r)^2 - 32 \pmod{89}$ is periodic with period 22. Moreover, $17(V_r)^2 - 32 = (Z_r)^2$, therefore the Jacobi symbol $((17(V_r)^2 - 32)/89)$ must have the value $+1$. However, for $r \equiv 3, 7, 15, 17, 25, 29, 37, 39 \pmod{44}$, a simple calculation shows that the Jacobi symbol has the value -1; hence, these values must be excluded. Next, working analogously with the prime moduli 103093, 1085657 and 353, in place of 89, we further refine our sieving leaving only the case $r \equiv 1 \pmod{88}$. In a similar way we see that $r \equiv 1 \pmod{13}$ and $r \equiv 1 \pmod 9$. The prime moduli we will have to work are respectively 521, 103, 5927, 2587551907171415793360232l in the former case and 2953 in the latter case.

The characteristic number $gU_i^2 - Dh$ of the system (51) for $i = 1$, given by Definition 2, is 1185. We have to choose a modulus and determine the values of t for which the characteristic number will be a quadratic non-residue with respect to some appropriate polynomial. By induction we obtain the following results:

$$a(t+1) \equiv 3 \pmod 4 \text{ for } t = 1 \text{ and } 1 \pmod 4 \text{ for all } t \geq 2,$$
$$b(t+1) \equiv 0 \pmod 4 \text{ for all } t \geq 1.$$

The prime factors of the characteristic number of the system are 3, 5 and 79. We have

$$a(t+1) \equiv 1 \pmod 3 \text{ for all } t \geq 1,$$
$$b(t+1) \equiv 0 \pmod 3 \text{ for all } t \geq 1,$$
$$a(t+1) \equiv 4 \pmod 5 \text{ for } t = 1 \text{ and } 1 \pmod 5 \text{ for all } t \geq 2,$$
$$b(t+1) \equiv 0 \pmod 5 \text{ for all } t \geq 1.$$

The sequences $a(t+1)$ and $b(t+1) \pmod{79}$ are periodic with period 12.

Now we illustrate the method of solving the problem with two different moduli.

(A) *Modulus with a single odd prime.* The modulus $m = 2^\tau \cdot 2 \cdot 13$ ($\tau \geq 0$). Here, with the notations of case (i) of Theorem 13, we have $Z^2 \equiv 1185 \pmod{a(t+1)F}$ where $F = 4096D^6\{b(t+1)\}^{12} + 11264D^5\{b(t+1)\}^{10} + 11520D^4\{b(t+1)\}^8 + 5376D^3\{b(t+1)\}^6 + 1120D^2\{b(t+1)\}^4 + 84D\{b(t+1)\}^2 + 1$. The Jacobi symbol

$$\left(\frac{1185}{a(t+1)}\right) = \left(\frac{3}{a(t+1)}\right) \cdot \left(\frac{5}{a(t+1)}\right) \cdot \left(\frac{79}{a(t+1)}\right).$$

When $t = 1$, we have $a(t+1) \equiv 3 \pmod 4$, $1 \pmod 3$, $4 \pmod 5$ and $41 \pmod{79}$. So

$$\left(\frac{1185}{a(t+1)}\right) = \left(\frac{a(t+1)}{3}\right) \cdot \left(\frac{a(t+1)}{5}\right) \cdot \left(\frac{a(t+1)}{79}\right) = \left(\frac{41}{79}\right) = -1.$$

Consequently $t = 1$ is impossible. Henceforth we consider the case $t \geq 2$. Now $a(t+1) \equiv 1 \pmod 4$, $1 \pmod 3$ and $1 \pmod 5$. We have

$$\left(\frac{a(t+1)}{3}\right) = +1 \quad \text{and} \quad \left(\frac{a(t+1)}{5}\right) = +1.$$

Therefore $(1185/a(t+1))$ reduces to $(a(t+1)/79)$. When $t \equiv 1, 2, 3, 4, 6, 8 \pmod{12}$, we have respectively $a(t+1) \equiv 41, 43, 63, 37, 66, 12 \pmod{79}$. Since each one of these values of $a(t+1)$ is a quadratic non-residue of 79, it follows that the relation $Z^2 = 17(V_r)^2 - 32$ is impossible for these values of t. Next we have

$$\left(\frac{1185}{F}\right) = \left(\frac{3}{F}\right) \cdot \left(\frac{5}{F}\right) \cdot \left(\frac{79}{F}\right).$$

Because of the relations $F \equiv 1 \pmod 4$, $1 \pmod 3$, $1 \pmod 5$ for all $t \geq 1$, we obtain

$$\left(\frac{1185}{F}\right) = \left(\frac{F}{79}\right).$$

When $t \equiv 0, 5, 7, 9, 10, 11 \pmod{12}$, we have $F \equiv 75, 24, 47, 15, 70, 27 \pmod{79}$. Since these values are quadratic non-residues of 79, we see that the relation $Z^2 = 17(V_r)^2 - 32$ does not hold for these values of t.

(B) *Modulus with an odd prime and the square of another odd prime.* The modulus $m = 2^\tau \cdot 2 \cdot 3^2 \cdot 11$, $\tau \geq 0$. With the notations of case (ii) of Theorem 13, we have $Z^2 \equiv 1185 \pmod{a(t+1)\, F_1\, F_2\, G_1\, G_2\, G_3}$ where

$$\begin{aligned}
F_1 &= 4D\{b(t+1)\}^2 + 1, \\
F_2 &= 1024D^5\{b(t+1)\}^{10} + 2304D^4\{b(t+1)\}^8 + 1792D^3\{b(t+1)\}^6 \\
&\quad + 560D^2\{b(t+1)\}^4 + 60D\{b(t+1)\}^2 + 1, \\
G_1 &= 64D^3\{b(t+1)\}^6 + 96D^2\{b(t+1)\}^4 + 36D\{b(t+1)\}^2 + 1, \\
G_2 &= 1048576D^{10}\{b(t+1)\}^{20} + 5505024D^9\{b(t+1)\}^{18} \\
&\quad + 12320768D^8\{b(t+1)\}^{16} + 15302656D^7\{b(t+1)\}^{14} \\
&\quad + 11493376D^6\{b(t+1)\}^{12} + 5326848D^5\{b(t+1)\}^{10} \\
&\quad + 1487104D^4\{b(t+1)\}^8 + 232256D^3\{b(t+1)\}^6 \\
&\quad + 17440D^2\{b(t+1)\}^4 + 480D\{b(t+1)\}^2 + 1
\end{aligned}$$

and G_3 is a polynomial in $b(t+1)$ of degree 60. We have to consider the Jacobi symbols

$$\left(\frac{1185}{a(t+1)}\right), \left(\frac{1185}{F_1}\right), \left(\frac{1185}{F_2}\right), \left(\frac{1185}{G_1}\right), \left(\frac{1185}{G_2}\right) \text{ and } \left(\frac{1185}{G_3}\right).$$

The results pertaining to $(1185/a(t+1))$ obtained in (A) are applicable here also. We get

$$\left(\frac{1185}{F_1}\right) = \left(\frac{F_1}{79}\right), \left(\frac{1185}{F_2}\right) = \left(\frac{F_2}{79}\right), \left(\frac{1185}{G_1}\right) = \left(\frac{G_1}{79}\right)$$

and

$$\left(\frac{1185}{G_2}\right) = \left(\frac{G_2}{79}\right).$$

When $t \equiv 1, 6, 9, 10 \pmod{12}$, we have $F_1 \equiv 6, 41, 43, 37 \pmod{79}$; when $t \equiv 2, 3, 4, 7, 9, 11 \pmod{12}$, we have $F_2 \equiv 3, 48, 24, 66, 75, 70 \pmod{79}$; when $t \equiv 0, 2, 3, 5, 6, 8, 9, 11 \pmod{12}$, we have $G_1 \equiv 17, 33, 47, 39, 17, 33, 47, 39 \pmod{79}$; when $t \equiv 1, 2, 3, 9, 10, 11 \pmod{12}$, we have $G_2 \equiv 56, 37, 7, 74, 74, 77 \pmod{79}$ which are all quadratic non-residues of 79. We are able to conclude without considering the values attained by G_3 modulo 79. It is seen that the relation $Z^2 = 17(V_r)^2 - 32$ does not hold for the present modulus.

As a result of the above discussion of any one modulus provided by (A) and (B), it is seen that the system of Pell's equations $U^2 - 11V^2 = 5$, $Z^2 - 17V^2 = -32$ has no common solution V_i except possibly for $i = 1$. However, when $i = 1$ we obtain $U = \pm 7$, $V = \pm 2$ and $Z = \pm 6$.

Next we consider the class of solutions provided by equation (54). Since the same characteristic number is got in this case, the results obtained for the previous class of solutions are applicable here also. Thus we have established the following:

Theorem 14. *The only positive integral solutions common to the two Pell's equations* $U^2 - 11V^2 = 5$ *and* $Z^2 - 17V^2 = -32$ *are* $U = 7$, $V = 2$ *and* $Z = 6$.

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