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AN ISOPERIMETRIC INEQUALITY FOR RIESZ CAPACITIES

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ABSTRACT. Let A be a compact set of $\mathbf{R}^n$, and let A^* be the closed ball centered at the origin with the same measure as A. We prove that, if C_α is the α-Riesz capacity with $0 < \alpha < 2$, then $C_\alpha(A) \geq C_\alpha(A^*)$. We also prove an isoperimetric inequality for the expected measure of the stable sausage generated by A. Our results also yield isoperimetric inequalities for the relativistic α-stable processes, and other Lévy processes.

1. Introduction. It is well known that, among all compact sets of equal measure, the ball has the smallest Newtonian capacity. This is one of the classical generalized isoperimetric inequalities of Pólya and Szegő [**11**]. In [**8**], Luttinger provided a new method, based on multiple integrals inequalities, to prove this and many other isoperimetric inequalities. In this paper we adapt the method of Luttinger [**8**] to obtain isoperimetric inequalities for Riesz capacities. The Riesz kernel is

$$k_\alpha(x-y) = \frac{\Gamma(n-\alpha/2)}{\Gamma(\alpha/2)\, \pi^{n/2} 2^{\alpha-1}}\ \frac{1}{|x-y|^{n-\alpha}},$$

where $n \geq 2$ and $0 < \alpha < n$. Let A be a compact set in $\mathbf{R}^n$, the α-Riesz capacity of A is defined by

$$C_\alpha(A) = \Big[\inf_\mu \iint k_\alpha(x-y)\, d\mu(x)\, d\mu(y) \Big]^{-1},$$

where the infimum is taken over all probability Borel measures supported in A. If $\alpha = 2$ and $n \geq 3$, then this is the classic Newtonian capacity. Let $|A|$ be the Lebesgue measure of A, and let A^* be the

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closed ball in $\mathbf{R}^n$ centered at the origin such that $|A^*| = |A|$. Then Pólya and Szegő's classical result is:

$$C_2(A) \geq C_2(A^*). \tag{1}$$

Naturally one might ask if (1) holds for all Riesz capacities. This problem was stated by Mattila in [**9**], where the author finds lower bounds for the Hausdorff measure of the projection of A on an m-dimensional subspace in terms of $C_m(A)$. In this paper we present a very short proof of the following result:

Theorem 1. *Let $\alpha \in (0, 2)$, and let A be a compact set of $\mathbf{R}^n$ such that $|A| > 0$. Then,*

$$C_\alpha(A) \geq C_\alpha(A^*), \tag{2}$$

where A^* is the closed ball, centered at the origin, such that $|A| = |A^*|$.

Theorem 1 was previously proved by Betsakos [**3**]. His proof relies on some polarization inequalities for transition densities of killed symmetric stable processes and a well-known relationship between Green's functions and Riesz capacities.

Luttinger obtained (1) from a probabilistic representation of the Newtonian capacity, due to Spitzer [**14**], and the following rearrangement inequality, proved by Friedberg and Luttinger [**7**].

Theorem 2. *Let $F_0, \dots, F_m : \mathbf{R}^n \to [0, 1]$, and let $H_0, \dots, H_m$ be nonnegative nonincreasing radially symmetric functions in $\mathbf{R}^n$. Then,*

$$\begin{aligned} &\int \cdots \int \left[1 - \prod_{j=0}^{m} (1 - F_j(z_j)) \right] \prod_{j=0}^{m} H_i(z_j - z_{j+1}) \, dz_0 \cdots dz_m \\ &\quad \geq \int \cdots \int \left[1 - \prod_{j=0}^{m} (1 - F_j^*(z_j)) \right] \prod_{j=0}^{m} H_i(z_j - z_{j+1}) \, dz_0 \cdots dz_m, \end{aligned}$$

where F_i^* is the symmetric decreasing rearrangement of F_i, and $z_{m+1} = z_0$.

Actually Friedberg and Luttinger proved Theorem 2 for the Steiner symmetrization of functions in $\mathbf{R}$. However, it was proved in [**4**] that the symmetric decreasing rearrangement of a function in $\mathbf{R}^n$ can be obtained as the limit of a sequence of Steiner symmetrizations with respect to different planes. It is known that such rearrangement inequalities, combined with a probabilistic representation of the heat kernel, imply the classical Raleigh-Faber-Krahn inequality and many other generalized isoperimetric inequalities for heat kernels and Green's functions of the Laplacian and fractional Laplacian; see [**1, 9, 10**].

The other key result, in Luttinger's proof of (1), is Spitzer's study of the expected volume of the Brownian sausage generated by A. The Markov processes associated to Riesz kernels are the symmetric α-stable processes. Let X_t be an n-dimensional symmetric α-stable process of order $\alpha \in (0,2)$, and let $T_A^\alpha = \inf\{t > 0 : X_t \in A\}$ be the first time X_t hits A. Define

$$E_A^\alpha(t) = \int P^x(T_A^\alpha \leq t)\, dx;$$

this quantity can be interpreted as the expected Lebesgue measure of the stable sausage $\cup_{s\leq t}[X_s + A]$.

In the case that $\alpha = 2$,

$$E_A^\alpha(t) - |A| = \int_{A^c} P^x(T_A^\alpha \leq t)\, dx$$

can be interpreted as the total heat flow, up to time t, from A to the surroundings. This was the original motivation of Spitzer to study the behavior of $E_A^\alpha(t)$. Getoor [**6**] proved that

$$\lim_{t\to\infty} \frac{E_A^\alpha(t)}{t} = C_\alpha(A), \tag{3}$$

extending Spitzer's result to all symmetric stable processes.

Not only does Theorem 2 imply isoperimetric inequalities for the expected area of the stable sausage, but this method applies, without change, to any Lévy processes whose transition probability densities are radially symmetric and nonincreasing. This class of processes includes the relativistic α-stable processes and any processes of the form B_{A_t}, where B_t is a Brownian motion and A_t is a subordinator. We will prove Theorem 1 in Section 2, and we will discuss extensions of Theorem 1 to other processes in Section 3.

Before publication we learned that Theorem 1 and our generalizations to other Lévy processes were first proved by Watanabe in [**15**], where he also settles the equality case. His argument is also based on some symmetrization techniques and an analytical representation of the capacity in terms of Dirichlet forms.

2. Proof of Theorem 1. Recall that the process X_t has right continuous sample paths and stationary independent increments. Its infinitesimal generator is

$$\big(-\Delta\big)^{\alpha/2},$$

where $0 < \alpha \le 2$ and Δ is the Laplacian in $\mathbf{R}^n$. When $\alpha = 2$ the process X_t is just an n-dimensional Brownian motion B_t running at twice the speed. If $0 < \alpha < 2$, then $X_t = B_{2\sigma_t}$, where σ_t is a stable subordinator of index $\alpha/2$ that is independent of B_t; see [**2**]. Thus,

$$p_\alpha(t,x,y) = \int_0^\infty \frac{1}{(4\pi u)^{n/2}} \exp\left[\frac{-|x-y|^2}{4u}\right] g_{\alpha/2}(t,u)\,du,$$

where $g_{\alpha/2}(t,u)$ is the transition density of σ_t. Hence, for every positive t, $p_\alpha(t,x,y) = f_t^\alpha(|x-y|)$ and the function $f_t^\alpha(r)$ is decreasing.

Let A_k be a decreasing sequence of compact sets such that the interior of A_k contains A for all k, and $\cap_{k=1}^\infty A_k = A$. By the right-continuity of the sample paths and the Markov property of stable processes, we have

$$\begin{aligned}
\int P^{z_0}\{\, T_A^\alpha \le t \,\}\, dz_0
&= \int \Big[1 - P^{z_0}\{\, T_A^\alpha > t \,\}\Big]\, dz_0 \\
&= \int \Big[1 - P^{z_0}\{X_s \in A^c, 0 \le s \le t\}\Big]\, dz_0 \\
&= \lim_{k\to\infty}\lim_{m\to\infty} \int \Big[1 - P^{z_0}\{\, X_{(jt)/m} \in A_k^c, j = 1, \dots, m \,\}\Big]\, dz_0 \\
&= \lim_{k\to\infty}\lim_{m\to\infty} \int \dots \int \Big[1 - \prod_{j=1}^m I_{A_k^c}(z_j)\Big]
\end{aligned}$$

$$\times \prod_{j=1}^{m} p_\alpha(t/m, z_j - z_{j-1})\, dz_0 \cdots dz_m$$

$$= \lim_{k\to\infty} \lim_{m\to\infty} \int \cdots \int \left[1 - \prod_{j=1}^{m} [\,1 - I_{A_k}(z_j)\,]\right]$$

$$\times \prod_{j=1}^{m} p_\alpha(t/m, z_j - z_{j-1})\, dz_0 \cdots dz_m,$$

where I_{A_k} is the indicator function of A_k. Since $f_t^\alpha(x)$ is nonincreasing and radially symmetric, we can take $H_m = 1$ and $F_0 = 0$ in Theorem 2 to obtain

$$\lim_{k\to\infty} \lim_{m\to\infty} \int \cdots \int \left[1 - \prod_{j=1}^{m} [\,1 - I_{A_k}(z_j)\,]\right]$$

$$\prod_{j=1}^{m} p_\alpha(t/m, z_j - z_{j-1})\, dz_0 \cdots dz_m$$

$$\geq \lim_{k\to\infty} \lim_{m\to\infty} \int \cdots \int \left[1 - \prod_{j=1}^{m} [\,1 - I_{A_k^*}(z_j)\,]\right]$$

$$\prod_{j=1}^{m} p_\alpha(t/m, z_j - z_{j-1})\, dz_0 \cdots dz_m.$$

Hence,

$$E_A^\alpha(t) = \int P^{z_0}\{\, T_A^\alpha \leq t\,\}\, dz_0 \geq \int P^{z_0}\{\, T_{A^*}^\alpha \leq t\,\}\, dz_0 = E_{A^*}^\alpha(t).$$

Finally we conclude from (3) that

$$C_\alpha(A) = \lim_{t\to\infty} \frac{E_A^\alpha(t)}{t} \geq \lim_{t\to\infty} \frac{E_{A^*}^\alpha(t)}{t} = C_\alpha(A^*).$$

3. Isoperimetric inequalities for radial Lévy processes. Let X_t be a Lévy process whose transition density is radially symmetric and decreasing. This class includes any process of the form B_{A_t}, where B_t

is a Brownian motion and A_t is a subordinator. An important example of such processes is the relativistic α-stable processes. The infinitesimal generator of the relativistic α-stable process is

$$[m-\Delta]^{\alpha/2}-m,$$

where $m>0$ is the mass of a relativistic particle. These processes arise in the study of relativistic Hamiltonian systems in physics; see [**2, 5, 13** and the references therein].

Let $T_A=\inf\{t>0 : X_t\in A\}$ be the first time X_t hits A, and consider

$$E_A(t)=\int P^x(T_A\le t)\,dx.$$

This quantity can be interpreted as the expected Lebesgue measure of $\cup_{s\le t}[X_t+A]$, the X_t-sausage generated by A. The following result generalizes the results of the previous section:

Theorem 3. *Let X_t be a transient Lévy processes whose transition density is radially symmetric and decreasing. Let A be a compact set in $\mathbf{R}^n$ with positive measure. If A^* is the closed ball centered at the origin with $|A|=|A^*|$, then for all $t\ge 0$*

$$E_A(t)\ge E_{A^*}(t), \tag{4}$$

and

$$C(A)\ge C(A^*), \tag{5}$$

where $C(A)$ is the capacity associated to X_t.

An examination of the proof of Theorem 1 shows that (4) follows from Theorem 2 and the fact that the transition densities of X_t are radially symmetric and decreasing. On the other hand, Port and Stone [**12**] proved that

$$\lim_{t\to\infty}\frac{E_A(t)}{t}=C(A), \tag{6}$$

thus (5) follows from (4). We include a proof of (6) for the convenience of the reader. The capacity, associated to X_t, of a compact set A is defined as

$$C(A)=\lim_{\lambda\to 0}\lambda\int E^x[\,e^{-\lambda T_A}\,]\,dx=\lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}E_A(dt).$$

On the other hand, the Markov property and the symmetry of X_t imply

$$\begin{aligned} E_A(t) - E_A(t-h) &\\ &= \int P^x(t-h < T_A \le t)\,dx \\ &= \int\int P^x(T_A > t-h, X_{t-h} \in dy)\,P^y(T_A \le h)\,dy\,dx \\ &= \int P^y(T_A > t-h)\,P^y(T_A \le h)\,dy. \end{aligned}$$

Then

$$\lim_{t\to\infty}\Big(E_A(t) - E_A(t-h)\Big) = \int P^y(T_A = \infty)\,P^y(T_A \le h)\,dy.$$

This is an additive function of h which is bounded and measurable. Hence it is a linear function of h and there exists a constant $\gamma(A)$ such that

$$\lim_{t\to\infty}\Big(E_A(t) - E_A(t-h)\Big) = h\gamma(A).$$

It follows that

$$\lim_{t\to\infty}\frac{1}{t}\,E_A(t) = \gamma(A).$$

Integration by parts implies that

$$C(A) = \lim_{\lambda\to 0}\lambda\int_0^\infty e^{-\lambda t}E_A(dt) = \gamma(A),$$

which proves (6).

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