ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 2, 2006

A PEANO-AKÔ TYPE THEOREM FOR VARIATIONAL INEQUALITIES

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ABSTRACT. We consider in this paper a Peano-Akô property of solution sets in some quasilinear elliptic variational inequalities. As consequences, variants of that property and a partial Hukuhara-Kneser theorem for inequalities are derived.

1. Introduction. This paper is about a property of solution sets in variational inequalities. We consider here a variational version of the Peano-Akô property for solutions in inequalities. Roughly speaking, the property states that under certain conditions, the solutions of an equation "fill up" the "region" between certain specific solutions (maximal and minimal solutions in our case).

In classical versions of the Peano-Akô property, cf. e.g. [**1, 12**], this is expressed by the fact that the values of $u(x_0)$ of the solutions u at any point x_0 in the domain fill up the whole interval $[u_*(x_0), u^*(x_0)]$ where u_* and u^* are the minimal and the maximal solutions of the equation.

In the case of weak solutions, those functions may not be continuous and be only defined almost everywhere. This is particularly relevant for solutions of variational inequalities, as we know, cf. [**3, 11, 26**], that those functions are not continuous in general. Hence, the pointwise interpretation above is no longer valid for such solutions.

Peano-Akô type properties are related to the connectedness of solution sets (or parts of them), which is also known as a Hukuhara-Kneser type property, which states that the solution set (of a problem) is a continuum, i.e., a compact, connected set, in an appropriate function space. Hukuhara-Kneser type theorems have been derived in [**4, 18, 28–30**, see also the references therein], for ordinary differential equations, integral equations and parabolic equations and systems, the solutions of which are smooth in most cases. We are concerned here with the elliptic variational inequalities with solutions being non-necessarily smooth.

Received by the editors on April 14, 2003, and in revised form on May 12, 2004.

In this paper, we present an extension of the above Peano-Akô property for discontinuous solutions by giving a general interpretation of the "filling up" concept. As a consequence, we obtain a variational version of the Peano-Akô property and also the classical one. This extension is proved for variational inequalities, but it seems to be new even for equations. Also, we derive a partial Hukuhara-Kneser type result for variational inequalities.

The paper is organized as follows. In the next section, after an introduction to variational inequalities, we recall some concepts and an existence result for extremal solutions of inequalities that is needed in the sequel. The main result and its corollaries are presented in Section 3.

2. Settings – Preliminary result.

2.1 Background on variational inequalities. Let X be a Banach space with dual X^* and dual pairing $\langle \cdot, \cdot \rangle$. Assume that K is a closed, convex subset of X and G is an operator from X to X^*. The general stationary variational inequality of finding $u \in K$ such that

$$\langle G(u), v-u \rangle \geq 0, \quad \forall\, v \in K, \tag{2.1}$$

can be used to formulate various problems in applied mathematics, mechanics, and other sciences. In boundary value problems for differential equations, the operator G is usually a differential operator and the set K of admissible functions represents constraints imposed on the problem. For example, in second-order elliptic problems, G could be given by

(2.2)

$$\langle G(u), v \rangle = \int_\Omega \left[\sum_{i=1}^N A_i(x, \nabla u(x)) \partial_i v(x) - F(x, u(x), \nabla u(x)) v(x) \right] dx,$$

where the mapping A given by $A(x,w) = (A_1(x,w), \dots, A_N(x,w))$, $x \in \Omega$, $w \in \mathbf{R}^N$, represents the principal operator in the differential equation and the function $F(x,u,w)$ represents the lower order term in the equation. In several obstacle problems, K is defined by the constraint $u \geq \psi$ (where ψ is the obstacle), that is, $K = \{u \in X : u \geq \psi\}$.

For another example, when $X = W^{1,2}(\Omega)$ (the usual Sobolev space) and $G(u) = -\Delta u$ or more generally $G(u) = -\Delta u - F$ (in the sense of distributions), the convex set $K = \{u \in X : u \geq 0 \text{ on } \partial\Omega\}$ corresponds to the unilateral boundary condition:

$$u \geq 0, \ \frac{\partial u}{\partial n} \geq 0, \quad \text{and} \quad u\,\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega,$$

n denotes here the outward unit normal vector on $\partial\Omega$, cf. e.g. [**8**] or [**10**]. As seen from these simple examples, a main difference between variational inequalities and equations is the association in the latter with constraints, including nonsmooth or unilateral ones. Such constraints could not generally be treated in the framework of smooth equations. Therefore, the study of variational inequalities usually requires different arguments and calculations from those used for equations.

Variational inequalities are also closely related to the calculus of variations. In fact, if u is a solution of the minimization problem

$$u \in K : g(u) = \min_{v \in K} g(v), \tag{2.3}$$

then u satisfies (2.1) where $G = g'$ (G is the Gâteaux derivative of g) in the case g is Gâteaux differentiable on X. To check this, assume that u is a solution of this minimization problem and v is any element of K. Since $u + t(v-u) \in K$ for all $t \in (0,1)$, we have

$$\frac{1}{t}\,[g(u + t(v-u)) - g(u)] \geq 0, \quad \forall\, t \in (0,1]. \tag{2.4}$$

Letting $t \to 0^+$ in this inequality and noting that

$$\langle G(u), w\rangle = \lim_{t \to 0^+} \frac{1}{t}\,[g(u+tv) - g(u)],$$

we see that u satisfies (2.1). Conversely, assume that g is convex and Gâteaux differentiable on X. Then, every solution u of (2.1), with $G = g'$, is a minimizer of (2.3). In fact, assume $u \in K$ satisfies (2.1) and let $v \in K$. From the convexity of g, one has, for any $t \in (0,1)$,

$$g(u + t(v-u)) \leq (1-t)\,g(u) + tg(v),$$

and thus

$$\frac{1}{t}\left[g(u+t(v-u))-g(u)\right] \le g(v)-g(u).$$

Letting $t \to 0^+$ in this inequality and using the assumption that u is a solution of (2.1), we obtain

$$0 \le \langle g'(u), v-u\rangle = \lim_{t\to 0^+} \frac{1}{t}\left[g(u+t(v-u))-g(u)\right] \le g(v)-g(u),$$

showing that u is a solution of (2.3). We refer to Theorem 2 of [**19**] for similar arguments for minimization problems and mildly nonlinear elliptic boundary value problems. There were also given in [**19**] finite element approximations and error analyses of the problem.

Inequality (2.1) can therefore be seen as the Euler-Lagrange equation associated with (2.3). In the particular case where $K = X$ then (2.1) is equivalent to an equation. This equivalence is no longer true in the general case.

As an example for the above discussions, let us consider an obstacle problem in classical elasticity. Assume that a homogeneous membrane, occupying a domain Ω in $\mathbf{R}^2$, is loaded by a normally distributed force H. The boundary points have prescribed displacements, for example 0. The potential energy of the deformation is given by

$$P(v) = \frac{\lambda}{2}\int_\Omega |\nabla v|^2\,dx,$$

where $v(x)$ is the (vertical) displacement at $x = (x_1, x_2) \in \Omega$ and $\lambda > 0$ is a constant depending on the elastic properties of the membrane. We assume $\lambda = 1$ for simplicity. The work done by the external force H during the actual deformation is given by $\int_\Omega Hv\,dx$. Suppose that $H = H(x, v)$ depends on both the point x and the displacement v and H is differentiable with respect to v, with some appropriate growth condition. The total energy is therefore

$$g(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega H(x,v)\,v\,dx. \tag{2.5}$$

Assume now that the deformation of the membrane is constrained by a body represented by $\{(x_1, x_2, y) \in \Omega\times\mathbf{R} : y \le \psi(x_1, x_2)\}$. The function

$\psi : \Omega \to \mathbf{R}$ represents the obstacle and is assumed to satisfy $\psi \le 0$ on $\partial\Omega$. A choice for the set of admissible displacements is therefore

$$K = \{v \in W_0^{1,2}(\Omega) : v \ge \psi \text{ in } \Omega\}. \tag{2.6}$$

At the equilibrium position u, the principle of minimum potential energy implies that u is a solution of the minimization problem (2.3) with g and K given by (2.5) and (2.6). Let us denote $h(x,v) = H(x,v)v$ and $F(x,v) = \partial h(x,v)/\partial v$. Direct calculations show that g is Gâteaux differentiable and

$$\langle g'(u), v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega F(x,u)\, v \, dx.$$

Therefore, we obtain following the variational inequality: Find $u \in K$ such that

$$\int_\Omega \nabla u \cdot \nabla(v-u)\, dx - \int_\Omega F(x,u)(v-u)\, dx \ge 0, \quad \forall\, v \in K. \tag{2.7}$$

This inequality is of the form (2.7) with $G = g'$ given above. This operator G here is a particular case (2.2) with $A_i(x,w) = w_i$, $1 \le i \le N$. The inequality (2.7) is also an example of (2.9) below with $A(x,w) = w$, for $x \in \Omega$, $w \in \mathbf{R}^2$.

As a further (and more general) example, we note that the obstacle problem in complementary form:

$$\begin{cases} -\operatorname{div}(A(x,\nabla u)) - F(x,u,\nabla u) \ge 0, \\ u \ge \psi, \\ [\operatorname{div}(A(x,\nabla u)) + F(x,u,\nabla u)](u-\psi) = 0 \text{ in } \Omega, \end{cases} \tag{2.8}$$

(with $u = 0$ on $\partial\Omega$) has (2.1) as a weak formulation with G given by (2.2) and K by (2.6), or more generally $K = \{v \in W_0^{1,p}(\Omega) : v \ge \psi \text{ in } \Omega\}$ depending on the growth of A. We refer e.g. to [**3, 11**] for the derivation, see also [**19**] and [**20**].

More detailed introductions to variational inequalities together with their existence theories and other issues, are given, for example, in [**3, 11**] or [**16**], and in [**21, 22**] for multi-valued variational inequalities. Several applications of variational inequalities are given, besides the cited references, in [**8, 9**] or [**26**]. Numerical methods for variational inequalities together with various applications are discussed e.g. in [**7, 19**] and in recent works [**23, 24**] and the references therein.

Abstract and general existence results were established for (2.1) in the case G satisfies certain coercivity and monotonicity assumptions, cf. [**16**, Theorem 8.2] or [**11**, Theorem 1.7], see also [**17, 27**]. The uniqueness of solutions usually holds when G has a strict monotonicity property, cf. e.g. [**16**, Theorem 8.3]. When G is given by (2.2), these existence and uniqueness results generally hold if $A(x, \nabla v)$ corresponds to a typical elliptic operator and $F = F(x)$ depends only on x. However, when F also depends on v or ∇v, then the coercivity and strict monotonicity of G may fail. The existence and uniqueness of solutions of (2.1)–(2.2), or (2.9) below, in this more general case have been subjects of continuing research. Some classical uniqueness (and in several cases also existence) conditions for (2.1)–(2.2) are, for example, that the operator associated with $F(x, u, \nabla u)$ is anti-monotone or Lipschitz continuous, with certain conditions on the Lipschitz and coercivity coefficients, cf. [**19, 20**] or [**27**].

Various approaches have been used to study noncoercive variational inequalities such as topological, bifurcation, variational methods, etc. As a motivation of this paper, it is a continuation of the previous works [**13, 14**], where a sub- supersolution method was proposed for noncoercive variational inequalities. In those papers, by using sub- and supersolutions for inequalities, we investigated the existence of solutions (in many cases, positive solutions) and also of maximal and minimal solutions of (2.9). We are interested here not in the existence of solutions of variational inequalities or their numerical approximations but instead in the structure of their solution sets. The class of inequalities that we study here will be described in more details in the next section, together with necessary assumptions and some preparatory results.

2.2 Sub- supersolutions in variational inequalities. We are concerned here with the following quasilinear elliptic variational inequality: Find $u \in K$ such that

$$\int_\Omega A(x, \nabla u) \cdot \nabla(v - u)\, dx \geq \int_\Omega F(x, u, \nabla u)(v - u)\, dx, \quad \forall\, v \in K. \tag{2.9}$$

Here, Ω is an open bounded subset of $\mathbf{R}^N$ with sufficiently smooth boundary, $X = W^{1,p}(\Omega)$, $X_0 = W_0^{1,p}(\Omega)$ are the usual Sobolev spaces equipped with the usual norms and K is a closed, convex subset of X_0. Also, $A : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function satisfying the following conditions:

$$\begin{aligned} &|A(x,w)| \le \nu |w|^{p-1} + \gamma(x), \\ &A(x,w)\cdot w \ge \alpha |w|^p, \quad \text{a.e.} \quad x \in \Omega, \ \text{all } w \in \mathbf{R}^N, \end{aligned} \tag{2.10}$$

where $\alpha, \nu > 0$ and $\gamma \in L^{p'}(\Omega)$ (p' is the Hölder conjugate of p), and

$$\begin{aligned} &[A(x,w_1) - A(x,w_2)]\cdot(w_1 - w_2) > 0, \quad \text{a.e.} \quad x \in \Omega, \\ &\qquad \text{all } w_1, w_2 \in \mathbf{R}^N, \quad w_1 \neq w_2. \end{aligned} \tag{2.11}$$

For an example of mappings A satisfying the above conditions, let us consider the p-Laplacian ($p \ge 2$), that is, A is given by

$$A(x,w) = |w|^{p-2}w,$$

for $x \in \Omega$, $w \in \mathbf{R}^N$. In this case (2.10) is satisfied with $\nu = \alpha = 1$ and $\gamma(x) = 0$. Also, (2.11) holds because

$$(|w_1|^{p-2}w_1 - |w_2|^{p-2}w_2)\cdot(w_1 - w_2) \ge 0, \quad \forall\, w_1, w_2 \in \mathbf{R}^N,$$

and the equality occurs only when $w_1 = w_2$. In the particular case where $p = 2$, i.e., $A(x,w) = w$, we have the classical case of the Laplacian (in the distributional sense of Sobolev space framework).

Assume that $F : \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ is a Carathéodory function with certain growth conditions to be specified later. We denote by L the functional defined by

$$\langle L(u), v\rangle = \int_\Omega A(x, \nabla u)\cdot \nabla v \, dx, \quad \forall\, u, v \in X.$$

It follows from (2.10)–(2.11) that L is coercive and strictly monotone on X. The concepts of sub- and supersolutions for the inequality (2.9) are defined in [**14**]. We recall the definitions here for the sake of completeness.

Definition 1. A function $u \in W^{1,p}(\Omega)$ is called a W-subsolution of (2.9) if

(i) $u \leq 0$ on $\partial\Omega$,

(ii) $F(\cdot, u, \nabla u) \in L^{q'}(\Omega)$, and

(iii) $\langle L(u), w - u\rangle \geq \int_\Omega F(\cdot, u)(w - u)\, dx$, for all $w \in u \wedge K$,

where $u \wedge v = \min\{u, v\}$, $u \wedge K = \{u \wedge v : v \in K\}$, also, $u \vee v = \max\{u, v\}, u \wedge K = \{u \vee v : v \in K\}$, and $1 < q < p^*$, p^* is the Sobolev conjugate of p.

We have a similar definition of W-supersolutions by reversing the inequality in (i) and replacing $\wedge$ by $\vee$ in (iii) in the above definition. A subsolution, respectively supersolution, of (2.9) is a maximum, respectively minimum, of any finite number of W-subsolutions, respectively W-supersolutions. The following result is proved in [**14**], see also [**13**], and will be used in Section 3.

Theorem 2.1. *Assume* (2.9) *has a subsolution*

$$\underline{u} = \max\{\underline{u}_1, \dots, \underline{u}_k\},$$

and a supersolution

$$\overline{u} = \min\{\overline{u}_1, \dots, \overline{u}_m\},$$

where $\underline{u}_1, \dots, \underline{u}_k$ *are* W*-subsolutions and* $\overline{u}_1, \dots, \overline{u}_m$ *are* W*-supersolutions of* (2.9)*. Suppose that* K *satisfies the following lattice conditions*:

(2.12)
$$\underline{u}_j \vee K \subset K, \overline{u}_i \wedge K \subset K, \quad \forall\, i \in \{1, \dots, m\}, \quad \forall\, j \in \{1, \dots, k\},$$

and

$$u, v \in K, d \in \mathbf{R}^+ \Longrightarrow u \wedge (v + d),\, u \vee (v - d) \in K, \tag{2.13}$$

and F *has the growth condition*

$$|F(x, u, \xi)| \leq a(x) + b|\xi|^{p/q'}, \tag{2.14}$$

for almost every $x \in \Omega$, *all* $u \in [\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\}, \max\{\overline{u}_1(x), \dots, \overline{u}_m x)\}]$, *where* $a \in L^{q'}(\Omega)$, $b \in [0, \infty)$, $q \in (1, p^*)$.

Then, there exist a minimal solution u_* *and a maximal solution* u^* *of* (2.9) *in* $W_0^{1,p}(\Omega)$ *in the sense that* u_*, u^* *are solutions of* (2.9),

$$\underline{u} \le u_* \le u^* \le \overline{u}, \tag{2.15}$$

and if $u \in W_0^{1,p}(\Omega)$ *is a solution of* (2.9) *such that* $\underline{u} \le u \le \overline{u}$, *then*

$$u_* \le u \le u^*. \tag{2.16}$$

3. Main results. In this section, we assume that the assumptions in Theorem 2.1 are satisfied and show that under certain conditions, the solution set

$$\mathcal{S} = \{u \in W_0^{1,p}(\Omega) : u \text{ is a solution of (2.9) and } \underline{u} \le u \le \overline{u}\}$$

fills up the "interval" $[u_*, u^*]$ in a certain sense. In what follows, we assume that

$$\underline{u}, \overline{u} \in L^\infty(\Omega), \tag{3.1}$$

and for almost every $x \in \Omega$ and all $\xi \in \mathbf{R}^N$,

(3.2) $F(x, u, \xi)$ is nonincreasing with respect to $u \in [\underline{u}(x), \overline{u}(x)]$.

As a consequence of the above assumptions,

$$u_*, u^* \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Let us consider the vector space

$$H = W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

with the usual intersection topology generated by the norm

$$\|u\|_H = \|u\|_{W_0^{1,p}(\Omega)} + \|u\|_{L^\infty(\Omega)}.$$

H is therefore a Banach space. From the above assumptions, $\mathcal{S} \subset H$. Let G be any (generally nonlinear) functional from $\mathcal{S}$ to $\mathbf{R}$, which is continuous with respect to $\|\cdot\|_H$. We have the following result about the "filling up" property of $\mathcal{S}$ stated above.

Theorem 3.1. *Under the assumptions in Theorem* 2.1 *and* (3.1)–(3.2)*, we have*

$$[\min\{G(u_*), G(u^*)\}, \max\{G(u_*), G(u^*)\}] \subset G(\mathcal{S}),$$

that is, for all s such that

$$\min\{G(u_*), G(u^*)\} \leq s \leq \max\{G(u_*), G(u^*)\},$$

there exists a solution $u \in \mathcal{S}$ such that $G(u) = s$.

Proof. Without loss of generality, we can assume that

$$\min\{G(u_*), G(u^*)\} = G(u_*) \leq G(u^*) = \max\{G(u_*), G(u^*)\}.$$

Assume by contradiction that there exists $s_0 \in \mathbf{R}$ such that

$$G(u_*) < s_0 < G(u^*), \tag{3.3}$$

but

$$G(u) \neq s_0, \quad \forall\, u \in \mathcal{S}. \tag{3.4}$$

Put $u_1 = u_*$, $w_1 = u^*$ and $d_1 = \|w_1 - u_1\|_{L^\infty(\Omega)}\,(< \infty)$. For $u \in \mathbf{R}$, $x \in \Omega$, let T denote the following truncating function:

$$T(x,u) = \begin{cases} \underline{u}(x) & \text{if} \quad u < \underline{u}(x) \\ u(x) & \text{if} \quad \underline{u}(x) \leq u \leq \overline{u}(x) \\ \overline{u}(x) & \text{if} \quad u > \overline{u}(x) \end{cases},$$

It is clear that $T(\cdot,u) \in W_0^{1,p}(\Omega)$, respectively $T(\cdot,u) \in H$, whenever $u \in W_0^{1,p}(\Omega)$, respectively $u \in H$. Also, T is increasing with respect to the second variable, that is

$$T(x,u_1) \leq T(x,u_2) \quad \text{if} \quad u_1 \leq u_2 \ (u_1,u_2 \in \mathbf{R}). \tag{3.5}$$

We define

$$F_0(x,u,\xi) = F(x,T(x,u),\xi) \quad \text{for } x \in \Omega,\ u \in \mathbf{R},\ \xi \in \mathbf{R}^N.$$

Then, F_0 is a Carathéodory function that also satisfies the growth condition (2.14). Let us consider the variational inequality: Find $u \in K$ such that

$$\text{(3.6)} \quad \int_\Omega A(x,\nabla u)\cdot\nabla(v-u)\,dx \geq \int_\Omega F_0(x,u,\nabla u)(v-u)\,dx, \quad \forall\, v \in K$$

We check that $u_1 + (d_1/2)$ is a W-supersolution of (3.6). In fact,

$$u_1 + \frac{d_1}{2} = \frac{d_1}{2} \geq 0 \quad \text{on} \quad \partial\Omega,$$

and since $\underline{u} \leq u_1 \leq \overline{u}$,

$$\text{(3.7)} \qquad F_0(\cdot,u_1,\nabla u_1) = F(\cdot,u_1,\nabla u_1) = F(\cdot,u_*,\nabla u_*) \in L^{q'}(\Omega).$$

Let $w \in K$. For

$$v = \left(u_1 + \frac{d_1}{2}\right) \vee w \in \left(u_1 + \frac{d_1}{2}\right) \vee K,$$

we have

$$\text{(3.8)} \quad \int_\Omega A\left[x,\nabla\left(u_1+\frac{d_1}{2}\right)\right]\cdot\nabla\left[v-\left(u_1+\frac{d_1}{2}\right)\right]dx \\ \geq \int_\Omega F_0\left[x,u_1+\frac{d_1}{2},\nabla\left(u_1+\frac{d_1}{2}\right)\right]\left[v-\left(u_1+\frac{d_1}{2}\right)\right]dx.$$

To prove this inequality, we first note that it is equivalent to the inequality

$$\int_\Omega A\left(x,\nabla u_1\right)\cdot\nabla\left[\left(v-\frac{d_1}{2}\right)-u_1\right]dx \\ \geq \int_\Omega \left[F_0\left(x,u_1+\frac{d_1}{2},\nabla u_1\right) - F_0(x,u_1,\nabla u_1)\right]\left[v-\left(u_1+\frac{d_1}{2}\right)\right]dx \\ + \int_\Omega F_0(x,u_1,\nabla u_1)\left[\left(v-\frac{d_1}{2}\right)-u_1\right]dx.$$

Now, because

$$\underline{u}(x) \le T(x, u_1(x)) \le T\left(x, u_1(x) + \frac{d_1}{2}\right) \le \overline{u}(x),$$

and $F(x, u, \xi)$ is decreasing in u on the interval $[\underline{u}(x), \overline{u}(x)]$, we have

$$\begin{aligned} F_0(x, u_1(x), \nabla u_1(x)) &= F(x, T(x, u_1(x)), \nabla u_1(x)) \\ &\ge F\left(x, T\left(x, u_1(x) + \frac{d_1}{2}\right), \nabla u_1(x)\right) \\ &= F_0\left(x, u_1 + \frac{d_1}{2}, \nabla u_1\right), \end{aligned}$$

for almost every $x \in \Omega$. On the other hand, since $v \ge u_1 + (d_1/2)$, we have $v - (u_1 + (d_1/2)) \ge 0$ almost everywhere on Ω and thus

(3.10)

$$\int_\Omega \left[F_0\left(x, u_1 + \frac{d_1}{2}, \nabla u_1\right) - F_0(x, u_1, \nabla u_1)\right]\left[v - \left(u_1 + \frac{d_1}{2}\right)\right] dx \le 0.$$

Since $u_1, w \in K$, we have

$$v - \frac{d_1}{2} = u_1 \vee \left(w - \frac{d_1}{2}\right) \in K.$$

Using (3.7) and the fact that u_1 is a solution of (2.9), one gets

$$\begin{aligned} &\int_\Omega A(x, \nabla u_1) \cdot \nabla \left[\left(v - \frac{d_1}{2}\right) - u_1\right] dx \\ &\qquad \ge \int_\Omega F(x, u_1, \nabla u_1)\left[\left(v - \frac{d_1}{2}\right) - u_1\right] dx \\ &\qquad = \int_\Omega F_0(x, u_1, \nabla u_1)\left[\left(v - \frac{d_1}{2}\right) - u_1\right] dx. \end{aligned} \tag{3.11}$$

Combining (3.10) and (3.11), we get (3.9) and thus (3.8). This shows that $u_1 + (d_1/2)$ is a W-supersolution of (3.6). By a similar proof, one can show that $w_1 - (d_1/2)$ is a W-subsolution of (3.6).

Now, since $w_1 = u^*$ is between $\underline{u}$ and $\overline{u}$, we have $F_0(\cdot, w_1, \nabla w_1) = F(\cdot, w_1, \nabla w_1)$. This implies that w_1 is also a solution, and thus a W-supersolution of (3.6). Hence,

$$\overline{p_1} = \left(u_1 + \frac{d_1}{2}\right) \wedge w_1$$

is a supersolution of (3.6). Using similar arguments, one can show that

$$\underline{p_1} = \left(w_1 - \frac{d_1}{2}\right) \vee u_1$$

is a subsolution of (3.6).

It is easy to check from the definition of d_1 that

$$w_1 - \frac{d_1}{2} \le u_1 + \frac{d_1}{2}$$

and thus $\underline{p_1} \le \overline{p_1}$. Note that (2.12) is satisfied in this case since, for every $v \in K$, we always have

$$\left(u_1 + \frac{d_1}{2}\right) \wedge v, \left(w_1 - \frac{d_1}{2}\right) \vee v, \quad u_1 \vee v, \quad w_1 \wedge v \in K.$$

It follows from the above discussion that all assumptions of Theorem 2.1 are satisfied for the inequality (3.6) and the pair of sub- supersolutions $\underline{p_1}$ and $\overline{p_1}$. According to this theorem, there exists a solution u of (3.6) such that

$$u_1 \le \underline{p_1} \le u \le \overline{p_1} \le w_1.$$

Because $\underline{u} \le u \le \overline{u}$, we have $F(\cdot, u, \nabla u) = F_0(\cdot, u, \nabla u)$ and thus u is also a solution of (2.9), i.e. $u \in \mathcal{S}$. From our assumptions, $G(u) \neq s_0$. If $G(u) > s_0$, we choose

$$u_2 = u_1 \quad \text{and} \quad w_2 = u.$$

Otherwise, we choose

$$u_2 = u \quad \text{and} \quad w_2 = w_1.$$

In both cases, u_2 and w_2 are solutions of (2.9) and

$$u_1 \le u_2 \le w_2 \le w_1,$$

i.e., $u_2, w_2 \in \mathcal{S}$ and

$$G(u_2) < s_0 < G(w_2).$$

Moreover, it is easy to check that

$$d_2 = \|w_2 - u_2\|_{L^\infty(\Omega)} \le \frac{d_1}{2}.$$

Using mathematical induction, one can construct sequences $\{u_n\}$ and $\{w_n\}$ in $\mathcal{S}$ such that

$$u_* \le u_n \le u_{n+1} \le w_{n+1} \le w_n \le u^*, \tag{3.12}$$

$$G(u_n) < s_0 < G(w_n), \tag{3.13}$$

and

$$d_n = \|w_n - u_n\|_{L^\infty(\Omega)} \le 2^{1-n} d_1. \tag{3.14}$$

Equation (3.12) implies that $\{u_n\}, \{w_n\} \subset H$ and $\{u_n\}$ is an increasing sequence, while $\{w_n\}$ is a decreasing one. Thus, it follows from (3.14) that

$$\sup_{n\in\mathbf{N}} u_n(x) = u(x) = \inf_{n\in\mathbf{N}} w_n(x),$$

and

$$\lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} w_n(x) = u(x),$$

for almost every $x \in \Omega$. Also, from (3.14), those convergences are uniform, that is,

$$u_n, w_n \to u \quad \text{in} \quad L^\infty(\Omega). \tag{3.15}$$

Let us show that $u \in W_0^1(\Omega)$ and

$$u_n, w_n \to u \quad \text{in} \quad W_0^1(\Omega). \tag{3.16}$$

Let $\|\cdot\|_0$ denote the usual norm in $W_0^1(\Omega)$: $\|u\|_0 = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$. Since $u_n \in \mathcal{S}$, by fixing $\phi \in K$, we have from (2.9) and (2.14) that

$$\begin{aligned}
&\int_\Omega A(x,\nabla u_n)\cdot\nabla u_n\,dx \\
&\le \int_\Omega A(x,\nabla u_n)\cdot\nabla\phi\,dx + \int_\Omega F(x,u_n,\nabla u_n)(\phi - u_n)\,dx \\
&\le \nu\int_\Omega |\nabla u_n|^{p-1}|\nabla\phi|\,dx + \|\gamma\|_{L^{p'}(\Omega)} + \int_\Omega [a + b|\nabla u_n|^{p/q'}](|u_n| + |\phi|)\,dx \\
&\le C\Big[\|u_n\|_0^{p-1}\|\phi\|_0 + \|a\|_{L^{q'}(\Omega)}(\|u_n\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)}) \\
&\qquad + b\,\|u_n\|_0^{p/q'}(\|u_n\|_{L^\infty(\Omega)} + \|\phi\|_{L^\infty(\Omega)})\Big] + \|\gamma\|_{L^{p'}(\Omega)} \\
&\le C\Big[\|u_n\|_0^{p-1} + \|u_n\|_0^{p/q'} + 1\Big],
\end{aligned}$$

where C is a generic constant that does not depend on n. On the other hand, it follows from (2.10) that

$$\int_\Omega A(x, \nabla u_n) \cdot \nabla u_n \, dx \geq \alpha \int_\Omega |\nabla u_n|^{p-1} \, dx.$$

Thus,

$$\alpha \|u_n\|_0^p \leq C \left[\|u_n\|_0^{p-1} + \|u_n\|_0^{p/q'} + 1 \right],$$

Because $p/q' < p$, the above estimate implies that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Therefore, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that

$$u_{n_k} \rightharpoonup \tilde{u} \quad \text{in} \quad W_0^{1,p}(\Omega),$$

and thus

$$u_{n_k} \to \tilde{u} \quad \text{in} \quad L^p(\Omega).$$

By (3.15), $\tilde{u} = u$ and $u_{n_k} \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. This holds for all weakly converging subsequences $\{u_{n_k}\}$ of $\{u_n\}$, implying that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega). \tag{3.17}$$

In particular, $u \in W_0^{1,p}(\Omega)$ and

$$u_n \longrightarrow u \quad \text{in} \quad L^p(\Omega). \tag{3.18}$$

As K is weakly closed in $W_0^{1,p}(\Omega)$, it follows from (3.17) that $u \in K$. Now, we prove that the convergence in (3.17) is in fact a strong convergence. Since $u_n \in S$, by replacing u by u_n and v by u in (2.9), we get

$$\int_\Omega A(x, \nabla u_n) \cdot (\nabla u - \nabla u_n) \, dx \geq \int_\Omega F(x, u_n, \nabla u_n)(u - u_n) \, dx. \tag{3.19}$$

Using the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, one also has $u_n - u \to 0$ in $L^q(\Omega)$. Because the sequence $\{|\nabla u_n|\}$ is bounded in $L^p(\Omega)$, the growth condition (2.14) implies that the sequence $\{F(\cdot, u_n, \nabla u_n)\}$ is bounded in $L^{q'}(\Omega)$. Since

$$\left| \int_\Omega F(x, u_n, \nabla u_n)(u - u_n) \, dx \right| \leq \|F(\cdot, u_n, \nabla u_n)\|_{L^{q'}(\Omega)} \|u - u_n\|_{L^q(\Omega)},$$

we have

$$\int_\Omega F(x,u_n,\nabla u_n)(u-u_n)\,dx \longrightarrow 0.$$

From (3.19),

$$\liminf \int_\Omega A(x,\nabla u_n)\cdot(\nabla u - \nabla u_n)\,dx \geq 0. \tag{3.20}$$

Now, since $\nabla u_n \rightharpoonup \nabla u$ in $[L^p(\Omega)]^N$,

$$\lim \int_\Omega A(x,\nabla u)\cdot(\nabla u - \nabla u_n)\,dx = 0.$$

Thus,

$$\liminf \int_\Omega [A(x,\nabla u_n) - A(x,\nabla u)]\cdot(\nabla u - \nabla u_n)\,dx \geq 0.$$

Because A is monotone, we must have

$$\lim \int_\Omega [A(x,\nabla u_n) - A(x,\nabla u)]\cdot(\nabla u - \nabla u_n)\,dx = 0.$$

This limit, together with (3.17), (3.18) and Lemma 3 of [**5**], implies that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Similar arguments show that w_n to u in $W_0^{1,p}(\Omega)$. Equation (3.16) is proved.

Equations (3.15) and (3.16) means that $u_n, w_n \to u$ in H, that is,

$$\|u_n - u\|_{L^\infty(\Omega)} + \|u_n - u\|_0 \to 0,\ \|w_n - u\|_{L^\infty(\Omega)} + \|w_n - u\|_0 \to 0.$$

By the continuity of G, we have that both $G(u_n)$ and $G(w_n)$ converge to $G(u)$ as $n \to \infty$. Equation (3.13) implies that

$$s_0 = G(u) = \lim G(u_n) = \lim G(w_n). \tag{3.21}$$

Now, let us verify that u is a solution of (2.9). As noted before, $u \in K$. For $v \in K$, we have

(3.22)

$$\int_\Omega A(x,\nabla u_n)\cdot(\nabla v - \nabla u_n)\,dx \geq \int_\Omega F(x,u_n,\nabla u_n)(v-u_n)\,dx, \quad \forall\, n.$$

By passing to a subsequence if necessary, we obtain from (3.16) that

$$u_n \to u, \nabla u_n \longrightarrow \nabla u \quad \text{a.e. in} \quad \Omega,$$

and

$$|u_n|, |\nabla u_n| \le g \quad \text{a.e. in} \quad \Omega, \quad \forall\, n,$$

for some $g \in L^p(\Omega)$. Hence, $A(x, \nabla u_n(x)) \to A(x, \nabla u(x))$ for almost every $x \in \Omega$ and $|A(\cdot, \nabla u_n)| \le \nu g^{p-1} + \gamma$ almost every in Ω, all $n \in \mathbf{N}$. The dominated convergence theorem thus implies that $A(\cdot, \nabla u_n) \to A(\cdot, \nabla u)$ in $[L^{p'}(\Omega)]^N$ and thus

$$\int_\Omega A(x, \nabla u_n) \cdot (\nabla v - \nabla u_n)\, dx \longrightarrow \int_\Omega A(x, \nabla u) \cdot (\nabla v - \nabla u)\, dx. \tag{3.23}$$

Similarly, we have $F(\cdot, u_n, \nabla u_n) \to F(\cdot, u, \nabla u)$ in $L^{q'}(\Omega)$ and

$$\int_\Omega F(\cdot, u_n, \nabla u_n)(v - u_n)\, dx \to \int_\Omega F(\cdot, u, \nabla u)(v - u)\, dx. \tag{3.24}$$

Letting $n \to \infty$ in (3.22) and using (3.23) and (3.24), we see that u is a solution of (2.9), that is $u \in \mathcal{S}$. This fact, together with (3.21), contradicts (3.4) and proves our theorem. □

Now, let us derive some corollaries of the above theorem. We consider on H the usual partial ordering:

$$u \le v \Longleftrightarrow u(x) \le v(x) \quad \text{for a.e.} \quad x \in \Omega.$$

Corollary 3.2. *Assume that* (3.1)–(3.2) *are satisfied and that* G *is increasing on* $\mathcal{S}$*, that is,*

$$u, v \in \mathcal{S} \quad \textit{and} \quad u \le v \Rightarrow G(u) \le G(v). \tag{3.25}$$

Then, $G(\mathcal{S}) = [G(u_*), G(u^*)]$.

Proof. That $G(\mathcal{S}) \subset [G(u_*), G(u^*)]$ follows from (3.25). The other inclusion follows from Theorem 3. □

With particular choices of G, we have different variants of the Peano-Akô property. For example, we have the following result.

Corollary 3.3. *Assume that* (3.1) *and* (3.2) *hold and that* $\mathcal{S} \subset C(\Omega)$. *Then, for all* $x_0 \in \Omega$, *all* $s \in [u_*(x_0), u^*(x_0)]$, *there exists* $u \in \mathcal{S}$ *such that* $u(x_0) = s$.

Proof. Let $G : \mathcal{S} \to \mathbf{R}$, $G(u) = u(x_0)$. G is well defined and continuous with respect to the topology of uniform convergence on Ω. Since $\mathcal{S} \subset C(\Omega) \cap L^\infty(\Omega)$, this topology is the same as that generated by $\|\cdot\|_{L^\infty(\Omega)}$ on $\mathcal{S}$. It follows that G is continuous on $\mathcal{S}$ with respect to the topology generated by $\|\cdot\|_H$. Our result now follows from Corollary 3.2. □

Corollary 3.4. *Assume conditions* (3.1) *and* (3.2) *are satisfied. Then, for all* $\phi \in C_0^\infty(\Omega)$, $\phi \geq 0$, *all*

$$s \in \left[\int_\Omega u_* \phi \, dx, \int_\Omega u^* \phi \, dx \right],$$

there exists a solution $u \in \mathcal{S}$ *such that*

$$\int_\Omega u\phi \, dx = s.$$

Consequently,

$$\left\{ \int_\Omega u\phi \, dx : u \in \mathcal{S} \right\} = \left[\int_\Omega u_* \phi \, dx, \int_\Omega u^* \phi \, dx \right].$$

Proof. Consider $G : H \to \mathbf{R}$, $G(u) = \int_\Omega u\phi \, dx$. It is easy to see that G is continuous and increasing, in the sense of (3.25), on H, since $\phi \geq 0$. Our claim follows from Corollary 3.2. □

Remark 3.5. (a) Corollary 3.3 is a classical Peano-Akô property. It is extended in that corollary to variational inequalities with continuous solutions.

(b) Corollary 3.4 could be seen as a variational version of the Peano-Akô property, where the pointwise property is replaced by the action of the solutions on test functions. It means that $\mathcal{S}$ fills out the interval between u_* and u^* in the distributional sense.

(c) Some other choices of G are, for example,

$$G(u) = \int_\Omega \sum_{j=1}^N \partial_j u \phi_j \, dx \quad (\phi_j \in C_0^\infty(\Omega),\ 1 \le j \le N),$$

and

$$G(u) = \int_\Omega \nabla u \cdot \nabla \phi \, dx \quad (\phi \in C_0^\infty(\Omega)).$$

Corollary 3.2, with these functionals G, shows, in the first case, that $\{\nabla u : u \in \mathcal{S}\}$ fills up the interval $[\nabla u_*, \nabla u^*]$ and, in the second case, $\{\Delta u : u \in \mathcal{S}\}$ fills up $[\Delta u_*, \Delta u^*]$ in the distributional sense.

As another consequence of the above discussion, we have the following property of $\mathcal{S}$:

Corollary 3.6. *$\mathcal{S}$ is a connected subset of H.*

Proof. Assume otherwise that there exist open sets A and B in H such that $A \cap B = \varnothing$, $\mathcal{S} \subset A \cup B$, and $A \cap \mathcal{S} \neq \varnothing$, $B \cap \mathcal{S} \neq \varnothing$. Let $G : \mathcal{S} \to \mathbf{R}$ be defined by

$$G(u) = \begin{cases} 0 & \text{if } u \in \mathcal{S} \cap A \\ 1 & \text{if } u \in \mathcal{S} \cap B. \end{cases} \tag{3.26}$$

Then G is continuous on $\mathcal{S}$. If $G(u_*) \neq G(u^*)$, then by choosing $s = 1/2$, we see from Theorem 3.1 that there exists $u \in \mathcal{S}$ such that $G(u) = 1/2$. This contradicts the definition of G in (3.26). Hence, $G(u_*) = G(u^*)$. We can assume without loss of generality that $G(u_*) = G(u^*) = 0$, i.e., u_*, u^* are both in A. Choose $u_1 \in B \cap \mathcal{S}$ and consider the set

$$(\varnothing \neq)\mathcal{S}_1 = \{u \in \mathcal{S} : u_* \le u \le u_1\}(\subset \mathcal{S}).$$

Applying Theorem 3.1 to $\mathcal{S}_1, u_*, u_1$ instead of $\mathcal{S}, u_*, u^*$, and noting that $G(u_*) = 0, G(u_1) = 1$, we see that there exists $u \in \mathcal{S}_1$ such that $G(u) = 1/2$. Again, we obtain a contradiction. □

Note that $\mathcal{S}$ is a closed and bounded subset of H. Hence, Corollary 3.6 gives us a partial Hukuhara-Kneser property of the section $\mathcal{S}$ of solutions of (2.9) between sub- and supersolutions. We refer to [**4, 18, 28**–**30**] for detailed discussions of this property for other kinds of problems. We conclude this section by noting that many convex sets K in applications satisfy the conditions in Theorem 2.1, for example,

$$K = \{u \in W_0^{1,p}(\Omega) : u \geq (\leq)\, \psi \text{ a.e. in } \Omega\}$$

in obstacle problems, or

$$K = \{u \in W_0^{1,p}(\Omega) : \Phi(\nabla u) \leq C(x) \text{ a.e. in } \Omega\}$$

in elastic-plastic torsion and sand pile problems (Φ is a convex function, and $C(x)$ is given), cf. e.g., [**2, 3, 6, 9, 25**].

Acknowledgment. The author would like to thank the referee for the valuable comments and supplementary references.

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