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THE NORM THEOREM FOR TOTALLY SINGULAR QUADRATIC FORMS

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ABSTRACT. The aim of this paper is to prove the norm theorem in the case of totally singular quadratic forms.

1. Introduction. Let F be a commutative field. In characteristic different from 2, an important part in the algebraic theory of quadratic forms is that related to function fields of quadratic forms, and a classical result in this area is the Cassels-Pfister subform theorem. This theorem is a consequence of another one due to Knebusch and known as the norm theorem [**5**, Theorem 4.2]. It asserts that for a normed irreducible polynomial $p \in F[x_1, \dots, x_n]$, an anisotropic quadratic form ϕ becomes hyperbolic over the quotient field of $F[x_1, \dots, x_n]/(p)$ if and only if p is a norm of ϕ over $F(x_1, \dots, x_n)$ the field of rational functions in n variables $x_1, \dots, x_n$ over F (normed means that the coefficient of the highest monomial occurring in p with respect to the lexicographical ordering is 1).

For quadratic forms in characteristic 2, we should distinguish between different objects: nonsingular quadratic forms, totally singular quadratic forms, and singular but not totally singular quadratic forms, cf. subsection 2.1 for definitions, and it is an interesting problem to extend some known results in characteristic $\neq 2$ to one of these objects. The theory of function fields of nonsingular quadratic forms works as in characteristic $\neq 2$. In fact, for such quadratic forms we have the norm theorem by Baeza [**2**]. This theorem and some representation results in [**1**, Satz 3.4, 3.5, Lemma 3.7] imply the Cassels-Pfister subform theorem for nonsingular quadratic forms, which was explicitly stated in [**6**, Proposition 3.4] (recall that some representation results in [**1**] have

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been also obtained by Amer [**10**, pp. 17–18]). Moreover, the author and Hoffmann obtained some progress on the theory of function fields of possibly singular quadratic forms [**3, 4, 6, 7, 9**], and the basics of a theory of totally singular quadratic forms have been established in [**3**, Section 8].

In [**8**] and [**9**] more investigations on totally singular quadratic forms have been performed, and it turns out that the theory of function fields of such quadratic forms can be done like that for quadratic forms in characteristic $\neq 2$. In fact, in [**8**] the Cassels-Pfister subform theorem has been proved after changing the notion of hyperbolicity by the so-called quasi-hyperbolicity, which means that the underlying vector space of the totally singular form is of even dimension and contains a maximal totally isotropic subspace of half dimension (totally isotropic space is just a space on which the restriction of the quadratic form is the zero form). The norm theorem was also obtained in the particular case where the normed irreducible polynomial is given by a totally singular quadratic form. The aim of this paper is to prove the general version of the norm theorem for totally singular quadratic forms:

Theorem 1.1. *Let ϕ be an anisotropic totally singular quadratic form of dimension ≥ 2 over F, and let $p \in F[x_1, \dots, x_n]$ be a normed irreducible polynomial. Let $F(p)$ be the quotient field of $F[x_1, \dots, x_n]/(p)$ and $K = F(x_1, \dots, x_n)$. Then the following are equivalent*:

(1) *ϕ is quasi-hyperbolic over $F(p)$.*

(2) *p is a norm of ϕ_K.*

Theorem 1.1 will be proved in two steps. First, we give the proof in the case of a normed irreducible polynomial in one variable. In this case, the implication (1) $\Rightarrow$ (2) will be proved by a direct argument. For the converse, we introduce a nonsingular form ϕ' which dominates generically our totally singular form ϕ and satisfies $\dim \phi' = 2 \dim \phi$ (the quadratic form ϕ' is defined at the beginning of Section 3. We refer to [**3**] for details concerning the domination relation). By using Knebusch's specialization theory, we prove in Proposition 3.3 (2) that the Witt index of $\phi'_{K(p)}$ is at least the dimension of the anisotropic part of $\phi_{F(p)}$ (here K is the field of rational functions on which ϕ' is defined), and by using Proposition 3.2 we get the quasi-hyperbolicity of $\phi_{F(p)}$. The reason of the reduction to nonsingular forms is the fact that the specialization argument that we use is not known for totally singular forms. In the second step, we will prove the theorem for a normed irreducible polynomial in more than one variable. To this end, we will use the following proposition to reduce the situation to a polynomial in one variable:

Proposition 1.2. *Let ϕ be an anisotropic totally singular quadratic form of dimension ≥ 2 over F, $f \in F[x_1, \dots, x_n]$ and $K = F(x_1, \dots, x_n)$. Let p be a normed irreducible polynomial which divides f with an odd power. If f is a norm of ϕ_K, then p is also a norm of ϕ_K.*

2. Preliminaries.

2.1 Notation and definitions. For $a, b \in F$, let denote by $[a, b]$, respectively $[a]$, the quadratic form $ax^2 + xy + by^2$, respectively the quadratic form ax^2. A quadratic form over F is called nonsingular, respectively totally singular, if it is isometric to an orthogonal sum of quadratic forms $[a, b]$, respectively $[a]$. Any quadratic form ϕ can be written as $\phi \cong R \perp S$ where R is nonsingular and S is totally singular ($\cong$ denotes the isometry of quadratic forms). The quadratic form ϕ is called singular when $\dim S > 0$.

For a nonzero quadratic form ϕ of dimension $n \geq 1$ and underlying vector space V, the homogeneous polynomial P_ϕ given by ϕ after a choice of a basis of V is reducible if and only if $\phi \cong \mathbf{H} \perp [0] \perp \cdots \perp [0]$ or $\phi \cong [a] \perp [0] \perp \cdots \perp [0]$ for some $a \in F^*$. When P_ϕ is irreducible, the function field of ϕ, denoted by $F(\phi)$, is defined as the quotient field of $F[x_1, \dots, x_n]/(P_\phi)$. We set $F(\phi) = F$ if P_ϕ is reducible or $\phi = 0$.

A quadratic form ψ is called a subform of ϕ, denoted by $\psi \subset \phi$, if there exists a quadratic form ψ' such that $\phi \cong \psi \perp \psi'$. Two quadratic forms ϕ and ψ are called similar if $\phi \cong \alpha\psi$ for some $\alpha \in F^*$.

A quadratic form ϕ of dimension $n \geq 2$ over F is called isotropic if the polynomial P_ϕ has a solution in $F^n - \{0\}$, otherwise ϕ is called anisotropic.

A scalar $\alpha \in F^*$ is called a norm of ϕ if $\phi \cong \alpha\phi$. We denote by $D_F(\phi)$, respectively $G_F(\phi)$, the set of scalars in F^* represented by ϕ, respectively the group of norms of ϕ. For a field extension K/F, let ϕ_K denote the quadratic form $\phi \otimes K$.

It was proved in [**3**, Proposition 2.4] that any quadratic form ϕ is uniquely decomposed as follows

$$\phi \cong i \times \mathbf{H} \perp j \times [0] \perp \phi_{\mathrm{an}}, \tag{1}$$

where $\mathbf{H}$ is the hyperbolic plane $[0,0]$, and ϕ_{an} is anisotropic that we call the anisotropic part of ϕ (here $n \times \psi$ denotes the orthogonal sum of n copies of a quadratic form ψ).

As in (1), the integer i, respectively j, is called the Witt index of ϕ and denoted by $i_W(\phi)$, respectively the defect index of ϕ and denoted by $i_d(\phi)$.

Clearly, the quasi-hyperbolicity of a totally singular form ϕ means that $\dim \phi$ is even and $\dim \phi = 2i_d(\phi)$.

2.2 Representation results. Recall that the anisotropic part of a totally singular form $\phi = [a_1] \perp \cdots \perp [a_n]$ is uniquely determined by the F^2-vector space inside F generated by $\{a_1, \ldots, a_n\}$. More precisely, this vector space coincides with $D_F(\phi) \cup \{0\}$, and the anisotropic part of ϕ is isometric to $[a_{i_1}] \perp \cdots \perp [a_{i_m}]$ for any F^2-basis $\{a_{i_1}, \ldots, a_{i_m}\}$ of $D_F(\phi) \cup \{0\}$. From this we easily deduce the following lemma:

Lemma 2.1 [**3**], [**9**, Lemma 2.1]. *Let $\phi = [a_1] \perp \cdots \perp [a_n]$ be a totally singular quadratic form over F.*

(1) *If $b_1, \ldots, b_m \in D_F(\phi)$ are such that the form $\psi = [b_1] \perp \cdots \perp [b_m]$ is anisotropic, then $\psi \subset \phi_{\mathrm{an}}$.*

(2) *For any field extension K/F, there exists $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that $(\phi_K)_{\mathrm{an}} \cong ([a_{i_1}] \perp \cdots \perp [a_{i_m}])_K$.*

A well-known representation theorem by Pfister, cf. [**11**, Theorem 3.2, pp. 148–149], easily extends to the case of totally singular quadratic forms. For the reader's convenience we give a proof.

Proposition 2.2. *If a nonzero polynomial $p \in F[x]$ is represented by an anisotropic totally singular form ϕ over $F(x)$, then it is represented by ϕ over the polynomial ring $F[x]$.*

Proof. Set $\phi = [a_1] \perp \cdots \perp [a_n]$. Since p is represented by $\phi_{F(x)}$, there exist polynomials $p_0 \neq 0, p_1, \dots, p_n \in F[x]$ such that

$$p = \sum_{i=1}^{n} a_i \left(\frac{p_i}{p_0}\right)^2.$$

There is nothing to prove if $\deg(p_0) = 0$. Suppose we have $\deg(p_0) > 0$. We take the sequences $(q_0, q_1, \dots, q_n)$, $(r_0, r_1, \dots, r_n)$ given by $p_i = q_i p_0 + r_i$ with $q_0 = 1$, $r_0 = 0$ and $\deg(r_i) < \deg(p_0)$ $(\deg(0) = -\infty)$. Let

$$s = p + \sum_{i=1}^{n} a_i q_i^2.$$

If $s = 0$, then the polynomial p is represented by ϕ over $F[x]$, and we are done. If not, the polynomial sp_0^2 equals $\sum_{i=1}^{n} a_i r_i^2$, and thus it has degree smaller than $2\deg(p_0)$, i.e., $\deg(s) < 0$, which is not possible. □

Remark 2.3. Proposition 2.2 remains true in the case of a polynomial in more than one variable. This was communicated to me by D.W. Hoffmann during the workshop which was held in Eilat, February 2004.

From Proposition 2.2 we deduce some corollaries:

Corollary 2.4. *If an anisotropic totally singular form ϕ over F represents a nonzero polynomial $p(x_1, \dots, x_n)$ over $F(x_1, \dots, x_n)$, and if $c = (c_1, \dots, c_n) \in F^n$ satisfies $p(c) \neq 0$ then $p(c) \in D_F(\phi)$.*

Proof. We use Proposition 2.2, and we proceed as in [**11**, Corollary 3.6, p. 150]. □

Corollary 2.5. *Let ϕ be an anisotropic totally singular form over F, and let $p \in F[x_1, \dots, x_n]$ be a norm of ϕ. Let $c_1, \dots, c_k \in F$ be such that the polynomial $q := p(c_1, \dots, c_k, x_{k+1}, \dots, x_n)$ is nonzero, $1 \leq k \leq n$. Then, q is a norm of ϕ over $F(x_{k+1}, \dots, x_n)$.*

Proof. Set $\phi = [a_1] \perp \cdots \perp [a_m]$, $K = F(x_1, \ldots, x_n)$ and $L = F(x_{k+1}, \ldots, x_n)$. For any $i \in \{1, \ldots, m\}$ we have $pa_i \in D_K(\phi_K)$. We consider p as a polynomial of $L[x_1, \ldots, x_k]$ and we apply Corollary 2.4 to get $qa_i \in D_L(\phi_L)$ for any $i \in \{1, \ldots, m\}$. Since $q\phi_L$ is anisotropic we get by Lemma 2.1 that $\phi_L \cong q\phi_L$. □

2.3 Isotropy results. A totally singular quadratic form ϕ is called a quasi-Pfister form, of degree n, if there exists an n-fold bilinear Pfister form B such that ϕ is isometric to the totally singular form given by $v \mapsto B(v, v)$. We refer to [**3, 9**] for details on quasi-Pfister forms and their neighbors.

We recall a characterization of quasi-Pfister forms using their groups of norms:

Lemma 2.6 [**3**, Proposition 8.5]. *An anisotropic totally singular form ϕ over F is isometric to a quasi-Pfister form if and only if $D_F(\phi) \subset G_F(\phi)$.*

Other characterizations of quasi-Pfister forms have been proved in [**8, 9**] by using standard splitting towers.

Lemma 2.7 [**8**, Lemma 2.4]. *Let π be an anisotropic quasi-Pfister form over F, and let $a_1, \ldots, a_n \in F^*$ be such that $a_1\pi \perp \cdots \perp a_n\pi$ is isotropic. Then $i_d(a_1\pi \perp \cdots \perp a_n\pi) \geq \dim \pi$.*

Proof. Set $\pi = [b_1] \perp \cdots \perp [b_m]$. The isotropy of $a_1\pi \perp \cdots \perp a_n\pi$ implies the existence of scalars $x_i \in D_F(a_i\pi) \cup \{0\}$, not all zero, such that $\sum_{i=1}^n x_i = 0$. Without loss of generality, we may suppose $x_i \neq 0$ for any $i \in \{1, \ldots, n\}$. Since $a_i\pi \cong x_i\pi$ (Lemma 2.6), it follows that $a_1\pi \perp \cdots \perp a_n\pi \cong b_1\phi \perp \cdots \perp b_m\phi$ where $\phi = [x_1] \perp \cdots \perp [x_n]$. Since $\sum_{i=1}^n x_i = 0$, the quadratic form ϕ is isotropic. Hence the claim. □

Lemma 2.8. *Let ϕ be a totally singular form over F, and let L/F be an extension which is purely transcendental or separable. If ϕ_L is isotropic, respectively quasi-hyperbolic, then ϕ is isotropic, respectively quasi-hyperbolic.*

Proof. (1) Suppose that L/F is purely transcendental.

(i) It is well known that if ϕ_L is isotropic, then ϕ is also isotropic.

(ii) If ϕ_L is quasi-hyperbolic, then any subform of ϕ of dimension $\geq (\dim\phi/2)+1$ is isotropic over L, and thus it is also isotropic over F. Hence $\dim\phi_{\rm an} \leq (\dim\phi/2)$. Moreover, by Lemma 2.1 there exists $\phi' \subset \phi$ of dimension $\dim\phi/2$ such that $(\phi_L)_{\rm an} \cong \phi'_L$. In particular, ϕ' is anisotropic over F, and again by Lemma 2.1 $\phi' \subset \phi_{\rm an}$. Hence $\dim\phi_{\rm an} \geq (\dim\phi/2)$, and thus $\dim\phi_{\rm an} = (\dim\phi/2)$.

(2) (i) Suppose that L is separable over F and ϕ_L is isotropic. Set $\phi = [a_1] \perp \cdots \perp [a_m]$, and let $\{e_1, \ldots, e_n\}$ be an F-basis of L. Let $v = (\sum_{j=1}^n \lambda_j^1 e_j, \ldots, \sum_{j=1}^n \lambda_j^m e_j) \in L^m - \{0\}$ be such that $\phi(v) = 0$. Then $\sum_{j=1}^n \left(\sum_{i=1}^m a_i(\lambda_j^i)^2\right) e_j^2 = 0$. Since L/F is separable, $\{e_1^2, \ldots, e_n^2\}$ is also an F-basis of L. Then $\sum_{i=1}^m a_i(\lambda_j^i)^2 = 0$ for any $j \in \{1, \ldots, n\}$, and thus ϕ is isotropic since $v \neq 0$.

(ii) For the quasi-hyperbolicity we use the same argument as in (1)(ii). □

As a corollary we get that only the case of inseparable irreducible polynomials will be considered in this paper:

Corollary 2.9. *Let ϕ be a totally singular form of dimension ≥ 2 over F, and let $f \in F[x_1, \ldots, x_n]$ be a norm of ϕ. Let p be an irreducible polynomial which divides f with an odd power. Then*:

(1) *$\phi_{F(p)}$ is isotropic.*

(2) *If ϕ is anisotropic, then p is inseparable in the sense that $\partial p/\partial x_i = 0$ for any $i \in \{1, \ldots, n\}$.*

Proof. Set $\phi = [a_1] \perp \cdots \perp [a_m]$, $A = F[x_1, \ldots, x_n]$ and $K = F(x_1, \ldots, x_n)$. Since $K^2 \subset G_K(\phi_K)$ we may suppose that p^2 does not divide f.

(1) Without loss of generality, we may suppose that ϕ is anisotropic and $1 \in D_F(\phi)$. Hence, $f \in D_K(\phi_K)$ and there exist polynomials

$r \neq 0, q_1, \ldots, q_m \in A$ such that

$$r^2 f = \sum_{i=1}^{m} a_i q_i^2. \tag{2}$$

Since p^2 does not divide f, we may suppose in (2) that p does not divide all the polynomials $q_1, \ldots, q_m$. To get the claim it suffices to extend (2) to the field $F(p)$.

(2) For any $i \in \{1, \ldots, n\}$ the polynomial p considered as an element of $L_i := F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)[x_i]$ stays irreducible. Since $A/(p)$ and $L_i/(p)$ possess the same quotient field, the claim follows from statement (1) and Lemma 2.8. □

We give results concerning the isotropy of totally singular forms over purely inseparable extensions of type $F(\sqrt[2^m]{d})$:

Lemma 2.10. *Let $d \in F^*$ be such that the polynomial $x^{2^m} + d$ is irreducible over F, and let ϕ be an anisotropic totally singular form of dimension ≥ 2 over F. Then ϕ becomes isotropic over $F(\sqrt[2^m]{d})$ if and only if $[1] \perp [d]$ is similar to a subform of ϕ.*

Proof. Set $k = 2^m - 1$, $l = 2^{m-1} - 1$, $\alpha = \sqrt[2^m]{d}$ and $\phi = [a_1] \perp \cdots \perp [a_n]$. The condition of the lemma is clearly sufficient. Conversely, suppose that $\phi_{F(\alpha)}$ is isotropic and let $v = \left(\sum_{j=0}^{k} l_j^1 \alpha^j, \ldots, \sum_{j=0}^{k} l_j^n \alpha^j\right) \in F(\alpha)^n - \{0\}$ be such that $\phi(v) = 0$. The condition $\phi(v) = 0$ implies

$$\sum_{i=1}^{n} a_i \left(\sum_{j=0}^{l} l_j^i \alpha^j \right)^2 = \sum_{i=1}^{n} a_i \left(\sum_{j=l+1}^{k} l_j^i \alpha^j \right)^2.$$

Hence

$$\begin{aligned} \sum_{j=0}^{l} \left(\sum_{i=1}^{n} a_i (l_j^i)^2 \right) \alpha^{2j} &= \alpha^{2^m} \sum_{j=l+1}^{k} \left(\sum_{i=1}^{n} a_i (l_j^i)^2 \right) \alpha^{2j-2^m} \\ &= \alpha^{2^m} \sum_{j=0}^{l} \left(\sum_{i=1}^{n} a_i (l_{j+l+1}^i)^2 \right) \alpha^{2j}. \end{aligned} \tag{3}$$

Set $v_j = (l_j^1, \dots, l_j^n)$ for $j \in \{0, \dots, k\}$. By comparing the two sides in (3), and using the fact that $\alpha^{2^m} = d$ and $\{1, \alpha, \dots, \alpha^k\}$ is an F-basis of L, we get $\phi(v_j) = d\phi(v_{j+l+1})$ for any $j \in \{0, \dots, l\}$. Since $v \neq 0$, there exists $j_0 \in \{0, \dots, k\}$ such that $v_{j_0} \neq 0$, for example $j_0 = 0$. Hence $\phi(v_0) = d\phi(v_{l+1}) \neq 0$ because ϕ is anisotropic. The form $[\phi(v_0)] \perp [\phi(v_{l+1})]$ is anisotropic since the polynomial $x^{2^m} + d$ is irreducible. By Lemma 2.1 $[\phi(v_0)] \perp [\phi(v_{l+1})]$ is a subform of ϕ, i.e., $\phi(v_{l+1})([1] \perp [d])$ is a subform of ϕ. □

Proposition 2.11. *Let $d \in F^*$ be such that the polynomial $x^{2^m} + d$ is irreducible over F, and let ϕ be an anisotropic totally singular form of dimension ≥ 2 over F. If ϕ becomes isotropic over $L = F(\sqrt[2^m]{d})$, then there exists ρ a totally singular form of dimension $i_d(\phi_L)$ such that $\rho \perp d\rho \subset \phi$.*

Proof. We proceed by induction on the dimension of ϕ. By Lemma 2.10 the result is true in dimension 2. Suppose that the result is true for any anisotropic totally singular form of dimension $< \dim \phi$. Set $n = i_d(\phi_L)$. Since ϕ is isotropic over L, it follows from Lemma 2.10

$$\phi \cong \phi' \perp \alpha([1] \perp [d]), \tag{4}$$

for some $\alpha \in F^*$ and a totally singular form ϕ' of dimension $\dim \phi - 2$. After extending (4) to L, we get

$$\phi_L \cong [0] \perp ([\alpha] \perp \phi')_L.$$

Then $i_d\left(([\alpha] \perp \phi')_L\right) = n - 1$. Since the form $[\alpha] \perp \phi'$ is anisotropic of dimension $< \dim \phi$, we get by induction totally singular forms ρ' and ϕ'' such that $\dim \rho' = n - 1$ and

$$[\alpha] \perp \phi' \cong \rho' \perp d\rho' \perp \phi''. \tag{5}$$

We add in each side of (5) the form $\alpha([1] \perp [d])$ and we use the isometry $[\alpha] \perp [\alpha] \cong [0] \perp [\alpha]$ to get

$$[0] \perp \phi \cong ([\alpha] \perp \rho') \perp d([\alpha] \perp \rho') \perp \phi''. \tag{6}$$

The form $\psi := [\alpha] \perp \rho' \perp d([\alpha] \perp \rho')$ is anisotropic, otherwise we would get by Lemma 2.7 $i_d(\psi) \geq 2$, and by (6) ϕ would be isotropic. It follows from Lemma 2.1 that ψ is a subform of $([0] \perp \phi)_{\mathrm{an}}$, i.e., $\rho \perp d\rho$ is a subform of ϕ where $\rho = [\alpha] \perp \rho'$. Hence the claim. $\square$

We get the following corollaries:

Corollary 2.12. *With the same notation and hypotheses as in Proposition* 2.11 *we have*:

(1) $i_d(\phi_L) \leq [\dim \phi/2]$ *where* $[n]$ *denotes the integer part of* n.

(2) *If* ϕ_L *becomes quasi-hyperbolic, then*:

(i) *there exists* ρ *a totally singular form such that* $\phi \cong \rho \perp d\rho$.

(ii) $x^{2^m} + d \in G_{F(x)}(\phi_{F(x)})$.

Proof. (1) and (2)(i) are immediate consequences of Proposition 2.11, and for (2)(ii) we use (i) and the fact that $x^{2^m} + d$ is a norm of $[1] \perp [d]$. $\square$

Corollary 2.13. *Let* ϕ *be an anisotropic totally singular form of dimension* ≥ 2 *over* F. *If* $p \in F[x]$ *is an irreducible polynomial such that* $\phi_{F(p)}$ *is isotropic, then* $i_d(\phi_{F(p)}) \leq [\dim \phi/2]$.

Proof. By Lemma 2.8, the polynomial p is inseparable. Then there exists a separable irreducible polynomial $q \in F[x]$ such that $p = q(x^{2^m})$ for some integer $m \geq 1$. Let α be a root of q in the algebraic closure of F, and let $\beta = \sqrt[2^m]{\alpha}$. Since $S = F(\alpha)$ is separable over F, the form ϕ_S is anisotropic. It follows from Corollary 2.12 that $i_d(\phi_{F(p)}) \leq [\dim \phi/2]$ since $F(p)$ is isomorphic to $S(\beta)$. $\square$

3. Proof of Theorem 1.1 in the case $n = 1$. Throughout this section, we fix the following

$$(\star) \qquad \begin{cases} t_1, \ldots, t_n \text{ are variables over } F \quad (n \geq 2) \\ K = F(t_1, \ldots, t_n) \\ a_1, \ldots, a_n \in F^* \\ \phi = [a_1] \perp \cdots \perp [a_n] \\ \phi' = [a_1, a_1^{-1}t_1^{-1}] \perp \cdots \perp [a_n, a_n^{-1}t_n^{-1}]. \end{cases}$$

3.1 Preparatory results.

Lemma 3.1. *Let ψ be a quadratic form over F, and $b_1, \dots, b_n \in F^*$ such that $\psi \perp [b_1] \perp \cdots \perp [b_n]$ is anisotropic. Then $\psi \perp [b_1, b_1^{-1}t_1^{-1}] \perp \cdots \perp [b_n, b_n^{-1}t_n^{-1}]$ is also anisotropic.*

Proof. By induction, it suffices to prove the lemma for $n = 1$. Suppose that $\psi \perp [b_1, b_1^{-1}t_1^{-1}]$ is isotropic over $F(t_1)$, and let $(p_1, \dots, p_m, r, s) \in F(t_1)^{m+2} - \{0\}$ be such that

$$\psi(p_1, \dots, p_m) + b_1 r^2 + rs + b_1^{-1}t_1^{-1}s^2 = 0. \tag{7}$$

We may suppose that $p_1, \dots, p_m, r, s$ are polynomials not all divisible by t_1. After multiplying (7) by t_1 and specializing t_1 to 0, we deduce that $s = s't_1$ for some $s' \in F[t_1]$. We substitute in (7) and we specialize t_1 to 0 to get

$$\psi(p_1(0), \dots, p_m(0)) + b_1 r(0)^2 = 0.$$

Since $\psi \perp [b_1]$ is anisotropic, t_1 divides $p_1, \dots, p_m, r$, which is not possible since t_1 also divides s. $\square$

It is well known and easy to prove that for $a, b, c, d \in F$ and $\alpha \in F^*$, we have the following

$$\begin{aligned} [a, b] \perp [c, d] &\cong [a + c, b] \perp [c, b + d] \\ \alpha[a, b] &\cong [\alpha a, \alpha^{-1}b]. \end{aligned}$$

We will refer to these isometries as "standard relations."

Proposition 3.2. *As in ($\star$) we have $i_d(\phi) \geq i_W(\phi')$.*

Proof. We proceed by induction on $\dim \phi$. If ϕ is anisotropic, then ϕ' is also anisotropic by Lemma 3.1, and thus the proposition is true. Suppose that ϕ is isotropic. Let $(x_1, \dots, x_n) \in F^n - \{0\}$ be such that $\sum_{i=1}^n a_i x_i^2 = 0$. Without loss of generality, we may suppose $x_1 \neq 0$. Hence, a_1 is represented by $\psi := [a_2] \perp \cdots \perp [a_n]$. Let $\psi' = [a_2, a_2^{-1}t_2^{-1}] \perp \cdots \perp [a_n, a_n^{-1}t_n^{-1}]$. By using standard relations we get the isometry

$$\phi' \cong [0, \beta_1] \perp [\alpha_2, \beta_2] \perp \cdots \perp [\alpha_n, \beta_n], \tag{8}$$

where

$$\beta_1 = x_1^{-2} a_1^{-1} t_1^{-1},$$

and for $i \geq 2$,

$$\alpha_i = \begin{cases} a_i & \text{if } x_i = 0 \\ x_i^2 a_i & \text{if } x_i \neq 0 \end{cases}$$

$$\beta_i = \begin{cases} a_i^{-1} t_i^{-1} & \text{if } x_i = 0 \\ \beta_1 + x_i^{-2} a_i^{-1} t_i^{-1} & \text{if } x_i \neq 0. \end{cases}$$

Hence

$$[\alpha_2, \beta_2] \perp \cdots \perp [\alpha_n, \beta_n] \cong (i_W(\phi') - 1) \times \mathbf{H} \perp \rho, \tag{9}$$

for some anisotropic quadratic form ρ over K. Let $\lambda : K \to F(t_2, \dots, t_n)^\infty$ be the $F(t_2, \dots, t_n)$-place satisfying $t_1 \mapsto \infty$. The quadratic forms $[\alpha_2, \beta_2] \perp \cdots \perp [\alpha_n, \beta_n]$ and $(i_W(\phi') - 1) \times \mathbf{H}$ are of good reduction with respect to λ, cf. [**5**] for the notion of good reduction and related details. Hence, ρ also possesses a good reduction with respect to λ [**5**, Proposition 2.2]. We specialize in (9), and we use standard relations with [**5**, Proposition 2.2] to get

$$\psi' \cong (i_W(\phi') - 1) \times \mathbf{H} \perp \lambda_*(\rho), \tag{10}$$

where $\lambda_*(\rho)$ is the specialization of ρ with respect to λ. By induction we get $i_d(\psi) \geq i_W(\psi') \geq i_W(\phi') - 1$. Since a_1 is represented by ψ, we get the isometry $\phi \cong [0] \perp \psi$. Hence $i_d(\phi) = i_d(\psi) + 1 \geq i_W(\phi')$. □

Proposition 3.3. *We keep the same notation as in* ($\star$)*, and we suppose that* ϕ *is anisotropic. Let* $f \in G_{F(x)}(\phi_{F(x)})$ *and* $p \in F[x]$ *be an irreducible polynomial which divides* f *with an odd power. Let* $\lambda : K(x) \to K(p)^\infty$ *be the* K*-place associated to* p*, and* $\varepsilon = \dim(\phi_{F(p)})_{\mathrm{an}}$*. Then we have the following*:

(1) *For any* $a_{k_1}, \dots, a_{k_\varepsilon} \in \{a_1, \dots, a_n\}$ *such that* $(\phi_{F(p)})_{\mathrm{an}} \cong ([a_{k_1}] \perp \cdots \perp [a_{k_\varepsilon}])_{F(p)}$*, there exist indices* $i_1, \dots, i_\varepsilon \in \{1, \dots, n\}$ *such that*

$$\phi' \cong [f a_{k_1}, \mu_{k_1}] \perp \cdots \perp [f a_{k_\varepsilon}, \mu_{k_\varepsilon}] \perp \left(\perp^n_{\substack{i=1 \\ i \neq i_1, \dots, i_\varepsilon}} [a_i, \nu_i] \right),$$

where $\mu_{k_1}, \dots, \mu_{k_\varepsilon}$ *and* ν_i $(i \neq i_1, \dots, i_\varepsilon)$ *belong to* A, *the ring of* λ.

(2) $i_W(\phi'_{K(p)}) \geq \varepsilon$.

(3) $\phi_{F(p)}$ *is quasi-hyperbolic.*

Proof. Since $F(x)^2 \subset G_{F(x)}(\phi_{F(x)})$, we may suppose that p^2 does not divide f. Let $f' \in F[x]$ be such that $f = pf'$.

(1) By Corollary 2.9 $\phi_{F(p)}$ is isotropic, and by Lemma 2.1 (2) there exist $a_{k_1}, \dots, a_{k_\varepsilon} \in \{a_1, \dots, a_n\}$ such that $(\phi_{F(p)})_{\mathrm{an}} \cong ([a_{k_1}] \perp \cdots \perp [a_{k_\varepsilon}])_{F(p)}$. We may suppose, possibly after reindexing, that $k_i = i$ for $i = 1, \dots, \varepsilon$.

Claim. *There exist indices* $i_1, \dots, i_\varepsilon \in \{1, \dots, n\}$ *such that for any* $l \in \{1, \dots, \varepsilon\}$ *we have the relation*

$$(\mathbf{R}_l) \qquad fa_l = \left(\sum_{i=1}^{l-1} (\beta_i^l)^2 fa_i \right) + (\alpha_{i_l}^l)^2 a_{i_l} + \left(\sum_{\substack{j=1 \\ j \neq i_1, \dots, i_l}}^{n} (\alpha_j^l)^2 a_j \right),$$

with the following conditions:

- $\beta_1^l, \dots, \beta_{l-1}^l \in A$.
- $\alpha_j^l \in A$ *for any* $j \neq i_1, \dots, i_l$.
- $\alpha_{i_l}^l$ *is a unit of* A.

Before we prove the claim we use it to justify the statement. Without loss of generality, we may assume in $(\mathbf{R}_1)$ that $\alpha_j^1 \neq 0$ for any $j \in \{1, \dots, n\} - \{i_1\}$. By using standard relations and $(\mathbf{R}_1)$ we get

$$(11) \qquad \phi' \cong [fa_1, \mu_1] \perp \left(\perp_{\substack{j=1 \\ j \neq i_1}}^{n} [a_j, \nu_j] \right),$$

where

$$\begin{cases} \mu_1 = (\alpha_{i_1}^1)^{-2} a_{i_1}^{-1} t_{i_1}^{-1} \\ \nu_j = a_j^{-1} t_j^{-1} + (\alpha_{i_1}^1)^{-2} (\alpha_j^1)^2 a_{i_1}^{-1} t_{i_1}^{-1} \quad \text{if } j \neq i_1. \end{cases}$$

Note that the elements μ_1, ν_j, $j \neq i_1$, belong to A since $\alpha_{i_1}^1$ is a unit. If $\varepsilon = 1$, then the proof is complete. If not, suppose we have for some

$l < \varepsilon$ the isometry

$$\phi' \cong [fa_1, \mu'_1] \perp \cdots \perp [fa_l, \mu'_l] \perp \left(\perp^n_{\substack{j=1 \\ j \neq i_1, \dots, i_l}} [a_j, \nu'_j] \right), \tag{12}$$

with $\mu'_1, \dots, \mu'_l, \nu'_j \in A$, $j \neq i_1, \dots, i_l$. Without loss of generality, we may suppose in $(\mathbf{R}_{l+1})$ that the elements $\beta_1^{l+1}, \dots, \beta_l^{l+1}$ and α_j^{l+1} are all nonzero, $j \neq i_1, \dots, i_{l+1}$. Set $\beta = \alpha_{i_{l+1}}^{l+1}$. By using standard relations, the relation (12) and $(\mathbf{R}_{l+1})$, we get the following

$$\phi' \cong [fa_1, \mu_1] \perp \cdots \perp [fa_{l+1}, \mu_{l+1}] \perp \left(\perp^n_{\substack{j=1 \\ j \neq i_1, \dots, i_{l+1}}} [a_j, \nu_j] \right), \tag{13}$$

where

$$\begin{cases} \mu_j = \mu'_j + (\beta_j^{l+1})^2 \beta^{-2} \nu'_{i_{l+1}} & \text{if } 1 \leq j \leq l \\ \mu_{l+1} = \beta^{-2} \nu'_{i_{l+1}} \\ \nu_j = \nu'_j + (\alpha_j^{l+1})^2 \beta^{-2} \nu'_{i_{l+1}} & \text{if } j \neq i_1, \dots, i_{l+1}. \end{cases}$$

The elements $\mu_1, \dots, \mu_{l+1}, \nu_j$, $j \neq i_1, \dots, i_{l+1}$, belong to A since β is a unit. Now it is clear that if we continue the process we get the statement.

(2) By statement (1) we have

$$\phi' \cong [fa_{k_1}, \mu_{k_1}] \perp \cdots \perp [fa_{k_\varepsilon}, \mu_{k_\varepsilon}] \perp \left(\perp^n_{\substack{i=1 \\ i \neq i_1, \dots, i_\varepsilon}} [a_i, \nu_i] \right), \tag{14}$$

for some $\mu_{k_1}, \dots, \mu_{k_\varepsilon}, \nu_i \in A$, $i \neq i_1, \dots, i_\varepsilon$. On the one hand, the form ϕ' is of good reduction with respect to λ, and its specialization $\lambda_*(\phi')$ equals $\phi'_{K(p)}$. On the other hand, the form $[fa_{k_1}, \mu_{k_1}] \perp \cdots \perp [fa_{k_\varepsilon}, \mu_{k_\varepsilon}]$ is also of good reduction with respect to λ, and its specialization equals $\varepsilon \times \mathbf{H}$ since $fa_{k_1}, \dots, fa_{k_\varepsilon}$ belong to the maximal ideal of A. Hence, after taking the specialization in (14) we get $i_W(\phi'_{K(p)}) \geq \varepsilon$.

(3) By statement (2) we have $i_W(\phi'_{K(p)}) \geq \varepsilon$. We apply Proposition 3.2 over the field $F(p)$ to get $i_d(\phi_{F(p)}) \geq \varepsilon$, i.e., $i_d(\phi_{F(p)}) \geq (\dim \phi/2)$. By Corollary 2.13 $i_d(\phi_{F(p)}) \leq (\dim \phi/2)$. Hence, $\phi_{F(p)}$ is quasi-hyperbolic.

Let us now prove the claim. Since $fa_1 \in D_{F(x)}(\phi_{F(x)})$, there exist by Proposition 2.2 polynomials $\alpha_1^1, \dots, \alpha_n^1$ such that

$$fa_1 = \sum_{i=1}^{n} (\alpha_i^1)^2 a_i. \tag{15}$$

Since p^2 does not divide f, it follows from (15) that there exists $i_1 \in \{1, \dots, n\}$ such that p does not divide $\alpha_{i_1}^1$, and thus $\alpha_{i_1}^1$ is a unit for A. If $\epsilon = 1$ then the proof is complete. If not, suppose we have indices $i_1, \dots, i_l$ such that $l < \varepsilon$ and the relations $(\mathbf{R}_1), \dots, (\mathbf{R}_l)$ hold. Since $fa_{l+1} \in D_{F(x)}(\phi_{F(x)})$, we get by Proposition 2.2 polynomials $\gamma_1^{l+1}, \dots, \gamma_n^{l+1}$ such that

$$fa_{l+1} = \sum_{i=1}^{n} (\gamma_i^{l+1})^2 a_i. \tag{16}$$

For any $m \in \{1, \dots, l\}$ we multiply $(\mathbf{R}_m)$ by $(\alpha_{i_m}^m)^{-2}$, and we use successive substitutions from $(\mathbf{R}_1), \dots, (\mathbf{R}_l)$ to get easily from (16) the following

$$fa_{l+1} = \left(\sum_{i=1}^{l} (\beta_i^{l+1})^2 fa_i \right) + \left(\sum_{\substack{j=1 \\ j \neq i_1, \dots, i_l}}^{n} (\alpha_j^{l+1})^2 a_j \right), \tag{17}$$

where $\beta_1^{l+1}, \dots, \beta_l^{l+1}, \alpha_j^{l+1} \in A$, $j \neq i_1, \dots, i_l$. Moreover, we claim that there exists $i_{l+1} \in \{1, \dots, n\} - \{i_1, \dots, i_l\}$ such that $\alpha_{i_{l+1}}^{l+1}$ is a unit for A. If not, we get easily from (17) that the element

$$f' a_{l+1} + \sum_{i=1}^{l} (\beta_i^{l+1})^2 f' a_i$$

belongs to the maximal ideal of A, and after taking its image by λ we deduce that $[a_1] \perp \cdots \perp [a_{l+1}]$ is isotropic over $F(p)$ (because p does not divide f'), and thus $(\phi_{F(p)})_{\mathrm{an}}$ is also isotropic since $l < \varepsilon$ and $(\phi_{F(p)})_{\mathrm{an}} \cong ([a_1] \perp \cdots \perp [a_\varepsilon])_{F(p)}$, a contradiction. Now it is clear that if we continue the process we get the claim. □

3.2 Proof of Theorem 1.1 in the case $n = 1$**.** By Proposition 3.3 (3) the condition $p \in G_{F(x)}(\phi_{F(x)})$ implies the quasi-hyperbolicity of $\phi_{F(p)}$. For the converse, suppose that $\phi_{F(p)}$ is quasi-hyperbolic. As in the proof of Corollary 2.13 $p = q(x^{2^m})$ for some $q \in F[x]$ an irreducible separable polynomial and $m \geq 1$. Let $\alpha_1, \ldots, \alpha_r$ be the roots of q in the algebraic closure of F. For any $i \in \{1, \ldots, r\}$, let $\beta_i = \sqrt[2^m]{\alpha_i}$ and $S_i = F(\alpha_i)$ which is a separable extension of F. The quadratic form ϕ_{S_i} is then anisotropic. Moreover, the polynomial $x^{2^m} + \alpha_i$ is irreducible over S_i. Since $\phi_{F(p)}$ is quasi-hyperbolic and $F(p)$ is isomorphic to $S_i(\beta_i)$, it follows from Corollary 2.12 that $x^{2^m} + \alpha_i$ is a norm of $\phi_{S_i(x)}$. Since p is normed, we have $p = \prod_{i=1}^{r}(x^{2^m} + \alpha_i)$. Hence over the field $S = F(\alpha_1, \ldots, \alpha_r)(x)$, the polynomial p is a norm of ϕ_S. In particular, the form $[a_i] \perp p\phi$ is isotropic over S for any $i \in \{1, \ldots, n\}$. Since $S/F(x)$ is separable, it follows from Lemma 2.8 that $pa_i \in D_{F(x)}(\phi_{F(x)})$ for any $i \in \{1, \ldots, n\}$. Lemma 2.1 (1) implies that $p \in G_{F(x)}(\phi_{F(x)})$. □

4. Proof of Proposition 1.2. We may suppose that p^2 does not divide f. We proceed by induction on n.

(1) The case $n = 1$. By Proposition 3.3 (3) $\phi_{F(p)}$ is quasi-hyperbolic, and by Theorem 1.1 in the case of a polynomial in one variable we conclude that $p \in G_{F(x)}(\phi_{F(x)})$.

(2) The case $n > 1$. We will follow some steps of the induction argument given by Knebusch in [**5**, p. 296, (ii) $\Rightarrow$ (i)]. Set $x' = (x_2, \ldots, x_n)$, $x = (x_1, x')$, $L = F(x_2, \ldots, x_n)$, r the degree of p considered as a polynomial of $L[x_1]$ and $\zeta \in F[x_2, \ldots, x_n]$ the highest coefficient of $p \in L[x_1]$.

(i) Suppose that F is infinite:

• If $r = 0$. We write $f = p(x')g(x) \in F[x_1, \ldots, x_n]$. Since p^2 does not divide f, then p does not divide all coefficients of $g \in L[x_1]$. Since F is infinite, there exists $c \in F$ such that $p(x')$ does not divide $g(c, x')$ in $F[x']$. By Corollary 2.5 $p(x')g(c, x')$ is a norm of ϕ_L. By induction we deduce that $p(x')$ is a norm of ϕ_L, and thus it is also a norm of ϕ_K.

• If $r > 0$. Let $p' = \zeta^{-1}p$ which is a normed polynomial in $L[x_1]$. It is easy to verify that ${p'}^2$ does not divide f in $L[x_1]$. By case (1), the polynomial p' is a norm of $\phi_{L(x_1)}$, and thus $p\phi_{L(x_1)} \cong \zeta\phi_{L(x_1)}$. We claim that ζ is a norm of ϕ_L. In fact, take h any normed irreducible divisor of ζ in $F[x_2, \dots, x_n]$ with odd power. Since p is irreducible, the polynomial h does not divide all coefficients of $p \in L[x_1]$. Since F is infinite, there exists $c \in F$ such that h does not divide $p(c, x')$. By Corollary 2.5 we have the isometry $p(c, x')\phi_L \cong \zeta\phi_L$. Hence $\zeta p(c, x')$ is a norm of ϕ_L, and by induction hypothesis h is a norm of ϕ_L. Since ζ is normed and any normed irreducible factor of it is a norm of ϕ_L, we deduce that ζ is a norm of ϕ_L, and thus p is a norm of $\phi_{L(x_1)}$.

(ii) Suppose that F is finite: As used in [**5**, p. 297] we change F by $F(t)$ for some variable t over F. Hence, over $F(t)$ we are in conditions of (2)(i), and thus p is a norm of $\phi_{F(t)}$. The claim follows from Corollary 2.5 by substituting 0 in t.

5. General proof of Theorem 1.1. Let $L = F(x_2, \dots, x_n)$ (read $L = F$ if $n = 1$), and let ζ be the highest coefficient of p considered as a polynomial of $L[x_1]$.

If p is a norm of $\phi_{L(x_1)}$, then we get by Proposition 3.3 (3) that ϕ is quasi-hyperbolic over $L(p) = F(p)$. Conversely, suppose that ϕ is quasi-hyperbolic over $F(p)$. By Theorem 1.1 in the case of a polynomial in one variable, the polynomial $\zeta^{-1}p \in L[x_1]$ is a norm of $\phi_{L(x_1)}$. In particular, ζp is a norm of $\phi_{L(x_1)}$. By Proposition 1.2, p is a norm of ϕ.

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