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COMPACTOIDNESS

IWO LABUDA

Dedicated to the memory of Paweł Szeptycki,
a mathematician and an artist

ABSTRACT. Although there is no doubt, today, what is the proper definition of compactness for a *subset* of a topological space X, the corresponding definition for a *family of subsets* of X is no longer evident. Two answers, arguably, are provided via the notion of *compactoidness*. The latter notion is the leitmotif of the topical survey below.

0. Introduction. The notion of compactoidness, in the form of a 'total net', can be traced back at least to 1969, see Pettis [**19**]. There, the total nets (of sets) were defined, a few facts (including a generalized version of Tikhonov product theorem) about them were proven and several interesting applications were indicated. Without any deeper analysis, essentially the same notion also made an appearance in the 1970 papers by Topsøe (as a 'compact net' [**22**]) and Wilker (as a 'compact filter' [**26**]), as well as in a 1976 paper by Kats ('compact filter' [**11**]).

It seems that it was Vaughan who first realized the importance of Pettis' contribution. He discusses the net versus filter approach in [**24**], the Proceedings of a Conference in Memphis, and gives there (some of the proofs appeared later in [**25**]) the basic characterization: *in a regular space a filter is compactoid if and only if it aims at its adherence which is compact.*

It looks as if not too many mathematicians read proceedings of conferences … . Compactoid filters, i.e., total filters of Pettis were then rediscovered again by Penot, and by Dolecki and Lechicki, see [**7, 18**]. In both cases, characteristically, the (re)discovery was motivated

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by applications, especially abundant in the paper by Penot. This is not at all surprising, taking into account the central role of compactness in analysis. However, applications will rarely be mentioned in this survey: we focus on the internal structure of the notion under scrutiny.

Most of the subsequent development is devoted to research in *convergence spaces* and the results are quite scattered throughout the literature; let us mention [**3**–**5, 15**–**17**]. Although the just-quoted paper [**5**] by Dolecki, Greco and Lechicki, where filters compactoid *relative to a set* were defined, could be taken as our starting point here, we restrict attention to topological spaces only.

On the other hand, we try to organize the existing material very carefully and with attention to detail. A measure of success attained in this direction, we think, is a general feeling of easiness with which most of the results are achieved. Not without a price, though: the number of steps that lead to important theorems may seem large.

We start with the general scheme of compactoidness defined by an arbitrary class of filters or overcovers, say $\mathbf{D}$. This approach, found in a more primitive form already in [**5**], compare Lemmas 3.5 and 3.6 there, gives a unification.

Once a type, determined by $\mathbf{D}$, of compactoidness of a family of sets $\mathcal{B}$ relative to another family of sets (say $\mathcal{A}$) is defined, we can consider $\mathbf{D}$-*selfcompactoidenss* of $\mathcal{B}$ whenever $\mathcal{B} = \mathcal{A}$. $\mathbf{D}$-selfcompactoidness is a type of compactness property for a family $\mathcal{B}$ and in many papers, [**6, 9, 20, 23**], $\mathbf{D}$-selfcompactoid families are indeed called $\mathbf{D}$-compact. However, in order to recover the classical notion of compactness, a slightly weaker notion, that of a family $\mathcal{B}$ being *nearly* $\mathbf{D}$-*compactoid* relative to another family $\mathcal{A}$, is more appropriate. $\mathcal{B}$ is called $\mathbf{D}$-*compact* if it is *nearly* $\mathbf{D}$-*selfcompactoid*. The latter notion, together with $\mathbf{D}$-selfcompactoidness, are the notions sought for in the Abstract above. A short description of the paper follows.

The results which can be treated in full generality, that is, with respect to more or less arbitrary classes of filters or overcovers, are discussed first. Some of them may be of independent interest but mainly this is a unified preparation for what follows.

In Sections 3 and 4, we concentrate on two fundamental cases: that of $\aleph$-*compactoidness* (which can be defined by arbitrary overcovers of cardinality less than $\aleph$) and that of $\aleph$-*midcompactoidness* (which can be defined by closed overcovers of cardinality less than $\aleph$). Thus, for instance, a nonvoid family $\mathcal{B}$ of nonvoid subsets of a topological space X is *midcompactoid* if it is $\aleph$-midcompactoid for every $\aleph$ or, which is the same, whenever for every closed overcover of X there exists a set $B \in \mathcal{B}$ with a finite subcover.

In Section 5, versions of the Alexander subbase theorem are given. Corollary 5.6 generalizes the classical statement to the case of a filter base of sets. Corollary 5.4 is its analog in the case of *midcompactoidness* and seems to be new even in the classical framework.

1. Compactoidness. Let X be a topological space and $P \subset X$. We denote $\overline{P}$ the closure of P, P° the interior of P and $P^c = X \setminus P$. Let $\mathcal{P}$ be a family of subsets of X. Its *family of closures* $\overline{\mathcal{P}}$ is $\{\overline{P} : P \in \mathcal{P}\}$, its *family of interiors* $\mathcal{P}^\circ$ is $\{P^\circ : P \in \mathcal{P}\}$ and its *family of complements* $\mathcal{P}^c$ is $\{P^c : P \in \mathcal{P}\}$. Let $\mathbf{P}$ be a class of families $\mathcal{P}$ in X. Its *class of closures* $\overline{\mathbf{P}}$ is $\{\overline{\mathcal{P}} : \mathcal{P} \in \mathbf{P}\}$, its *class of interiors* $\mathbf{P}^\circ$ is $\{\mathcal{P}^\circ : \mathcal{P} \in \mathbf{P}\}$ and its *class of complements* $\mathbf{P}^c$ is $\{\mathcal{P}^c : \mathcal{P} \in \mathbf{P}\}$. We will never regard the families of sets as subsets in the hyperspace, so our notation will not lead to any confusion. As an attentive reader must have already noticed, we reserve script for *families* of sets, and bold for *classes*, that is, sets of families of sets.

Let $\mathcal{H}$ be another family of subsets of X.

We write $\mathcal{P}\#\mathcal{H}$ and say that $\mathcal{P}$ *meshes* with $\mathcal{H}$ if $P \cap H \neq \varnothing$ for each $P \in \mathcal{P}$ and each $H \in \mathcal{H}$.

We write $\mathcal{P} \geq \mathcal{H}$ and say that $\mathcal{P}$ is *finer* than $\mathcal{H}$ if for each $H \in \mathcal{H}$ there exists $P \in \mathcal{P}$ such that $P \subset H$. Note that if $\mathcal{P}$ is a filter base finer than $\mathcal{H}$, then $\mathcal{P}\#\mathcal{H}$.

$\mathcal{P}$ is a *refinement* of $\mathcal{H}$ if for each $P \in \mathcal{P}$ there exists $H \in \mathcal{H}$ such that $P \subset H$. Note that $\mathcal{P}$ is a refinement of $\mathcal{H}$ if and only if $\mathcal{P}^c \leq \mathcal{H}^c$.

We reserve $\mathcal{B}$ to denote a fixed, but otherwise arbitrary, *nonvoid family of nonvoid subsets of* X. In many situations, however, it will be necessary to assume additionally that $\mathcal{B}$ is downward directed by inclusion, i.e., that $\mathcal{B}$ is a *filter base*. This stronger assumption will always be mentioned explicitly.

The *adherence* of $\mathcal{B}$, $\operatorname{adh}\mathcal{B}$, is defined by the formula

$$\operatorname{adh}\mathcal{B} = \bigcap \overline{\mathcal{B}} = \bigcap \{\overline{B} : B \in \mathcal{B}\};$$

the term *cluster set* of $\mathcal{B}$ is also used, e.g., by Bourbaki himself in the English translation of [**1**].

The *limit set* of $\mathcal{B}$, $\lim\mathcal{B}$, is the set of points to which $\mathcal{B}$ converges, that is,

$$\lim\mathcal{B} = \{x \in X : \mathcal{B} \geq \mathcal{N}(x)\}$$

where $\mathcal{N}(x)$ denotes the filter of neighborhoods of x. We say that x is the limit of $\mathcal{B}$, and write $\lim\mathcal{B} = x$, only if unicity is guaranteed, i.e., if $\lim\mathcal{B} = \{x\}$. However, in this paper Hausdorffness is rarely needed and, accordingly, the notions of *compactness* and *regularity* (each point x has a base of closed neighborhoods) *will not* presuppose it, compare Kelley [**12**].

Let $A \subset X$. As in [**5**], a family $\mathcal{P}$ of subsets of X is an *overcover* of A, if

$$A \subset \bigcup \{P^\circ : P \in \mathcal{P}\}.$$

Let $\mathbf{P}$ be a class of overcovers of A. Suppose $\mathcal{B}$ has the following property: for each $\mathcal{P} \in \mathbf{P}$, there exist $B \in \mathcal{B}$ and a finite subcover $\mathcal{P}_0 \subset \mathcal{P}$ of B, i.e., we postulate the existence of $\mathcal{P}_0 = \{P_1, P_2, \dots, P_n\}$ such that $P_i \in \mathcal{P}$ and $B \subset \bigcup_{i=1}^n P_i$.

Note that in the just stated property of $\mathcal{B}$, we could, in a sense for free, also postulate the family $\mathcal{P}$ to be an *ideal* of sets, i.e., to be stable by subsets and by finite unions. Indeed, if it were not, we could enlarge it into one and the existence of a finite subcover of B chosen from the ideal guarantees the existence of a subcover chosen from the original overcover.

$\mathcal{Q}$ is an ideal if and only if the family of its complements $\{F = Q^c : Q \in \mathcal{Q}\}$ is a *filter*. Thus, we expect some sort of duality between statements involving covers and those involving filters. The following is due to Dolecki, [**2**, Theorem 2.1]; it will be convenient to refer to it as *cover-filter duality* (CF-duality).

Let $\mathbf{P}$ be a class of overcovers of A, and denote by $\mathbf{P}_\star$ the class of all (possibly degenerate) filters generated by the elements of $\mathbf{P}^c$.

CF-duality. The following conditions are equivalent for $\mathcal{B}$.

($\mathfrak{F}_{\#}$) For each $\mathcal{F} \in \mathbf{P}_{\star}$ such that $\mathcal{F}\#\mathcal{B}$, $A \cap \operatorname{adh} \mathcal{F} \neq \varnothing$, that is, $\mathcal{F}$ has a cluster point in A.

($\mathfrak{C}_1$) For each $\mathcal{P} \in \mathbf{P}$ overcovering A, there exist $B \in \mathcal{B}$ and a finite subfamily $\mathcal{P}_0 \subset \mathcal{P}$ such that $B \subset \bigcup \mathcal{P}_0$.

We say that $\mathcal{B}$ is $\mathbf{P}$-*compactoid, respectively* $\mathbf{P}_{\star}$-*compactoid*, relative to A if the condition ($\mathfrak{C}_1$), respectively ($\mathfrak{F}_{\#}$), is satisfied. To streamline the language, we adopt a few conventions. When we say that $\mathcal{B}$ is $\mathbf{D}$-compactoid relative to A without indicating whether cover- or filter-compactoidness is dealt with, we mean that the class $\mathbf{D}$ can be a class of either filters or overcovers and, in the first case, 'compactoid' refers to filter-compactoidness; in the second case, 'compactoid' refers to cover-compactoidness. The corresponding statement is then meant to apply to both types of compactoidness. If $\mathcal{B}$ is $\mathbf{D}$-compactoid relative to A and $\mathbf{D}$ is the class of *all* overcovers or filters, then '$\mathbf{D}$' is dropped and we say that $\mathcal{B}$ is *compactoid relative to* A. Further, if $\mathcal{B}$ is $\mathbf{D}$-compactoid relative to X, then 'relative to X' is dropped. Thus, for instance, $\mathcal{B}$ is compactoid means that for every overcover $\mathcal{P}$ of X there exists $B \in \mathcal{B}$ and a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ with $B \subset \cup \mathcal{P}_0$ or that the equivalent condition involving filters is satisfied.

We note that the set A appearing in the definition *cannot be empty.* This is, of course, unless the class $\mathbf{P}_{\star}$ in the condition ($\mathfrak{F}_{\#}$) is empty which would mean that there is no compactoidness to speak of in the first place.

We also consider $\mathbf{D}$-compactoidness relative to a family of subsets of X. Namely, if $\mathcal{A}$ is such a family, then $\mathcal{B}$ is said to be $\mathbf{D}$-*compactoid relative to* $\mathcal{A}$ if $\mathcal{B}$ is $\mathbf{D}$-compactoid relative to A for each $A \in \mathcal{A}$. If $\mathcal{A} = \mathcal{B}$, then $\mathcal{B}$ is said to be $\mathbf{D}$-*selfcompactoid*.

Naturally, a nonempty subset $E \subset X$ is called $\mathbf{D}$-*compactoid relative to* A if the family $\{E\}$ consisting of one set E is so. Most important special cases occur when $A = X, A = \overline{E}$ or $A = E$. In the first case E is $\mathbf{D}$-*compactoid* (because 'relative to X' was dropped), in the second case E is $\mathbf{D}$-*compactoid relative to its adherence* and, finally, in the third case E is $\mathbf{D}$-*selfcompactoid.* In the overcover case, the latter means that for each overcover $\mathcal{P} \in \mathbf{P}$ of A, there exists a finite subfamily $\mathcal{P}_0 \subset \mathcal{P}$ such that $A \subset \cup \mathcal{P}_0$. In the filter case, it means that for each filter $\mathcal{F} \in \mathbf{P}_{\star}$ meshing with A, $\mathcal{F}$ has a cluster point in A.

Remark. In the original terminology of [**5**], $\mathcal{B}$ was compactoid 'with respect to A.' In more recent papers there is a tendency to replace 'with respect to A' by 'in A.' This, strictly speaking, is abusive because not every cluster point has to be in A. This is one reason for which we try 'relative to A,' a term shorter than the original 'with respect to A.' Another reason is the symmetry with the following definition.

A subset $E \subset X$ is called *countably compact relative to* A if every sequence (x_n) in E has a cluster point in A (and 'countably compact relative to X' is shortened to *relatively countably compact*, [**14**]).

1.1 Theorem. *Let $\mathcal{B}$ be $\mathbf{D}$-compactoid relative to A. If $A \subseteq \cap\mathcal{B}$, then both $\mathcal{B}$ and A are $\mathbf{D}$-selfcompactoid.*

Proof. We give a proof in the case $\mathbf{D}$ is a class of filters. As $A \subset B$ for each B in $\mathcal{B}$, it is trivial that $\mathcal{B}$ is $\mathbf{D}$-selfcompactoid. Let $\mathcal{D} \in \mathbf{D}$ be a filter meshing with A. As $A \subset \cap\mathcal{B}$, $\mathcal{D}$ meshes with $\mathcal{B}$. By condition $(\mathfrak{F}_\#)$, $\mathcal{D}$ has a cluster point in A. $\square$

Remarks 1. A predecessor to the above statement, for compactoid filters, is essentially proven already in [**18**, Proposition 14] although the statement of the proposition there is unnecessarily restricted to the case of filters having a base consisting of closed sets (which are called *regular* filters in [**18**] and [**5**]).

2. In the investigations concerning the active boundary of upper semi-continuous set valued maps, see, e.g., [**14**], cluster sets which are strict subsets of the adherence are used in an essential way. This, together with Section 1.3 below, motivates the present form of the theorem involving $A \subset \cap\mathcal{B}$.

We write $\mathcal{B} \rightsquigarrow A$ and say that $\mathcal{B}$ *aims at* A if $\mathcal{B}$ is finer than the filter of neighborhoods of A, that is, if for each neighborhood V of A there exists $B \in \mathcal{B}$ such that $B \subset V$. The following proposition, for compactoid filters, can be found, e.g., in [**18**, Proposition 13(c)]. However, it seems that corresponding statements for nets of values of an upper semi-continuous set-valued map have been known much earlier, see, e.g., [**21**, Lemma].

1.2 Proposition. *Let* $\mathbf{P}$ *be a class of open overcovers of* A*, and let* A *be* $\mathbf{P}$*-selfcompactoid. If* $\mathcal{B}$ *aims at* A*, then* $\mathcal{B}$ *is* $\mathbf{P}$*-compactoid relative to* A*.*

Proof. Let $\mathcal{P} \in \mathbf{P}$. By assumption on A, there exists a finite subcover $\mathcal{P}_0$ of $\mathcal{P}$ such that $A \subset \cup\mathcal{P}_0$. As $\mathcal{P}$ is an open cover, $\cup\mathcal{P}_0$ is a neighborhood of A and, as $\mathcal{B} \rightsquigarrow A$, there is a $B \in \mathcal{B}$ such that $B \subset \cup\mathcal{P}_0$. $\square$

Remark. Since $\mathcal{P} \in \mathbf{P}$ are assumed to be *open*, our overcovers here are just open covers. Then, as easily seen, $\mathbf{P}$-selfcompactoidness amounts to $\mathbf{P}$-compactness, to be defined below.

1.3 Theorem. *Let* $\mathbf{C}$ *be a class of closed overcovers of* A*, and let* $\mathcal{B}$ *be* $\mathbf{C}$*-compactoid relative to* A*. Then* $\overline{\mathcal{B}}$ *and* $\operatorname{adh}\mathcal{B}$ *are* $\mathbf{C}$*-compactoid relative to* A*. If* $A \subseteq \operatorname{adh}\mathcal{B}$*, then* $\overline{\mathcal{B}}$ *is* $\mathbf{C}$*-compactoid relative to its intersection and* A *is* $\mathbf{C}$*-selfcompactoid.*

Proof. Let $\mathcal{P} \in \mathbf{C}$. By condition $(\mathfrak{C}_1)$, there exists $B \in \mathcal{B}$ and a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ such that $B \subset \cup\mathcal{P}_0$. As $\cup\mathcal{P}_0$ is closed, $\overline{B} \subset \cup\mathcal{P}_0$. This shows that $\operatorname{adh}\mathcal{B}$ and $\overline{\mathcal{B}}$ are $\mathbf{C}$-compactoid relative to A. Now, as A is contained in the intersection of $\overline{\mathcal{B}}$, Theorem 1.1 applied with $\mathcal{B} = \overline{\mathcal{B}}$ implies the theorem. $\square$

Remark. Of course, the most important case of Theorem 1.1, respective 1.2, is $A = \cap\mathcal{B}$, respectively $A = \cap\overline{\mathcal{B}}$. But see Remark 2 above.

1.4 Proposition. *Let* $\mathcal{A}_1$*,* $\mathcal{A}_2$ *be families of subsets of* X *such that the* $\mathbf{P}$*-overcovers of* $\mathcal{A}_2$ *refine the* $\mathbf{P}$*-overcovers of* $\mathcal{A}_1$ *in the sense that for each* $A_2 \in \mathcal{A}_2$ *and* $\mathcal{P} \in \mathbf{P}$ *overcovering* A_2 *there exists* $A_1 \in \mathcal{A}_1$ *overcovered by* $\mathcal{P}$*. If* $\mathcal{B}$ *is* $\mathbf{P}$*-compactoid relative to* $\mathcal{A}_1$*, then it is so relative to* $\mathcal{A}_2$*.*

Proof. Let $A_2 \in \mathcal{A}_2$, and let $\mathcal{P} \in \mathbf{P}$ be an overcover of A_2. By assumption, $\mathcal{P}$ is an overcover of A_1 for some $A_1 \in \mathcal{A}_1$. As $\mathcal{B}$ is **P**-compactoid relative to $\mathcal{A}_1$, there is a finite subfamily $\mathcal{P}_0$ of $\mathcal{P}$ and $B \in \mathcal{B}$ such that $B \subset \cup \mathcal{P}_0$. This shows that $\mathcal{B}$ is **P**-compactoid relative to $\mathcal{A}_2$. $\square$

Remark. Let $\mathcal{O}$ be the topology of X and $\mathcal{A}$ a family of subsets of X. Denote $\mathcal{O}(A) = \{O \in \mathcal{O} : A \subset O\}$ and $\mathcal{O}(\mathcal{A}) = \cup \{\mathcal{O}(A) : A \in \mathcal{A}\}$. We note that the condition on $\mathcal{A}_1$ and $\mathcal{A}_2$ is satisfied if and only if $\mathcal{O}(\mathcal{A}_2)$ is a refinement of $\mathcal{O}(\mathcal{A}_1)$.

Overcovers provide a scheme which permits the unified approach to various notions of compactoidness. However, in topology the traditional overcovers are *open covers*. It will, therefore, be of importance to identify the arising notions in terms of open covers. The following two lemmas will be helpful.

1.5 Lemma. *Let* $\mathbf{P}$ *be a class of overcovers of a set* $A \subset X$ *such that* $\mathbf{P}^\circ \subset \mathbf{P}$. *The following are equivalent.*

(i) $\mathcal{B}$ *is* $\mathbf{P}$*-compactoid relative to* A.

(ii) $\mathcal{B}$ *is* $\mathbf{P}^\circ$*-compactoid relative to* A.

1.6 Lemma. *Let* $\mathbf{P}$ *be a class of closed overcovers such that* $\overline{\mathbf{P}^\circ} \subset \mathbf{P}$. *Then the following are equivalent.*

(i) $\mathcal{B}$ *is* $\mathbf{P}$*-compactoid relative to* A.

(ii) $\mathcal{B}$ *is* $\overline{\mathbf{P}^\circ}$*-compactoid relative to* A.

2. Near compactoidness. Let $\mathcal{B}$ and $\mathcal{A}$ be two families of subsets of X, and let $\mathbf{Q}$ be a class of filters (on X). We say that a filter base $\mathcal{B}$ is *nearly* $\mathbf{Q}$*-compactoid relative to* $\mathcal{A}$, if

$$(\mathfrak{F}_>) \qquad \mathcal{Q} \in \mathbf{Q},\ \mathcal{Q} \geq \mathcal{B} \implies \operatorname{adh} \mathcal{Q} \# \mathcal{A}.$$

Remark. Formally, we could state the above definition, as in the case of **Q**-compactoidness, for a family $\mathcal{B}$. However, it may be observed that, whenever a filter finer than $\mathcal{B}$ exists, $\mathcal{B}$ is forced to be a filter subbase.

$\mathcal{B}$ is said to be **Q**-*compact*, if it is it nearly **Q**-selfcompactoid, that is, if

$$\mathcal{Q} \in \mathbf{Q},\ \mathcal{Q} \geq \mathcal{B} \implies \operatorname{adh} \mathcal{Q} \# \mathcal{B}.$$

As just noted, **Q**-compactness is defined for filter bases only. **Q**-selfcompactoidness, which can be written

$$\mathcal{Q} \in \mathbf{Q},\ \mathcal{Q} \# \mathcal{B} \implies \operatorname{adh} \mathcal{Q} \# \mathcal{B},$$

makes sense for any family $\mathcal{B}$. The difference between the two is most visible when $\mathcal{B}$ equals $\{E\}$, where $\varnothing \neq E \subset X$. **Q**-compactness of E means, as expected, that any **Q**-filter on E has a cluster point in E. On the other hand, **Q**-selfcompactoidness of E involves, by its very definition, **Q**-filters that *mesh* with E. These filters live, so to say, in the surrounding space X and therefore **Q**-selfcompactoidness is a more stringent requirement on E which, however, is not *intrinsic*. We are, of course, interested when the two notions coincide on filter bases or, at least, on subsets of X.

A filter base $\mathcal{B}$ is said to be **Q**-refinable if, for each $\mathcal{Q} \in \mathbf{Q}$ meshing with $\mathcal{B}$, there exists $\mathcal{F} \in \mathbf{Q}$ which is finer than both $\mathcal{Q}$ and $\mathcal{B}$. It is clear that, for such filter bases near **Q**-compactoidness and **Q**-compactoidness coincide. Thus

2.1 Proposition. *Let $\mathcal{B}$ be a* **Q***-refinable filter base. Then $\mathcal{B}$ is* **Q***-selfcompactoid if, and only if, it is* **Q***-compact.*

An obvious consequence is, that $\mathcal{B}$ is selfcompactoid if and only if it is compact. This, because any filter base is refinable with respect to the class of all filters. Some other examples will appear below, but we will not delve too much into that in this article. Even in the case of subsets of X, although for any class **D** of *open* overcovers the coincidence trivially holds, the case of general **D** is non-obvious.

Let $\mathbf{D} = \mathbf{D}(X)$ be a class of filters on X. If $E \subset X$, then $\mathbf{D}(E)$ is the class $\mathbf{D}$ defined on the (sub)space E. We do not exclude the case when $\mathbf{D}(E)$ could be the empty class. Let (P) denote a property that a filter base may or may not have. A filter base having the property (P) is also said to be a (P)-base (the same way a filter belonging to $\mathbf{D}$ is called a $\mathbf{D}$-filter).

$\mathbf{D}$ is called *locally determined*, by a property (P), if a filter $\mathcal{D}$ belongs to the class provided it admits a base having property (P). Such a class $\mathbf{D}$ is, moreover, *hereditary*, if

(1) If $E \subset X$ and $\mathcal{D} \in \mathbf{D}(X)$ such that $\mathcal{D} \# E$, then $\mathcal{D} \cap E \in \mathbf{D}(E)$;

(2) If $\mathcal{H}$ is a (P)-base on E, then there exists a (P)-base $\mathcal{H}'$ on X such that $\mathcal{H}' \cap E \geq \mathcal{H}$.

2.2 Proposition. *Suppose* $\mathbf{D}$ *is a class of filters on* X *which is locally determined and hereditary. Let* $E \subset X$. *Then* E *is* $\mathbf{D}$*-selfcompactoid if and only if* E *is* $\mathbf{D}$*-compact.*

Proof. 'If'. Let $\mathcal{D} \in \mathbf{D}$ be meshing with E. By (1), $\mathcal{F} := \mathcal{D} \cap E$ is a filter on E belonging to $\mathbf{D}(E)$. Hence $\mathcal{F}$ has a cluster point, say x, in E. As is easily seen, x is also a cluster point of $\mathcal{D}$.

'Only if'. Let $\mathcal{D}$ be a filter in $\mathbf{D}(E)$. It is generated by a base $\mathcal{H}$ having property (P). By (2), there exists a base $\mathcal{H}'$ on X such that the filter it generates, call it $\mathcal{D}'$, is in $\mathbf{D}(X)$. As E is $\mathbf{D}$-selfcompactoid, the filter $\mathcal{D}'$ has a cluster point, say x, in E. Noting that $\mathcal{D}'$ induces on E a filter which is finer than $\mathcal{D}$, we see that x is a cluster point of $\mathcal{D}$. $\square$

3. $\aleph$-compactoidness. Let $\aleph$ be a cardinal number. We denote by $\mathbf{P}(\aleph)$ the class of all overcovers such that $\operatorname{card}\{P : P \in \mathcal{P}\} < \aleph$ for each $\mathcal{P} \in \mathbf{P}(\aleph)$ and by $\mathbf{F}(\aleph)$ the family of all filters $\mathcal{F}$ such that each $\mathcal{F}$ can be generated by a base of cardinality less than $\aleph$.

CF-duality. *Let* $A \subset X$. *The following conditions are equivalent.*

$(F_{\#}(\aleph))$ *If* $\mathcal{F} \in F(\aleph)$ *and* $\mathcal{F} \# \mathcal{B}$, *then* $\mathcal{F}$ *has a cluster point in* A $(\operatorname{adh} \mathcal{F} \cap A \neq \varnothing)$.

($C_1(\aleph)$) *If $\mathcal{P} \in P(\aleph)$ overcovers A, then there exist a finite subfamily $\mathcal{P}_0 \subset \mathcal{P}$ and $B \in \mathcal{B}$ such that $B \subset \cup \mathcal{P}_0$.*

($C_2(\aleph)$) *For each open cover $\mathcal{V}$ of A of cardinality less than $\aleph$ there exist a finite subfamily $\mathcal{V}_0 \subset \mathcal{V}$ and $B \in \mathcal{B}$ such that $B \subset \cup \mathcal{V}_0$.*

Indeed, $\mathbf{F}(\aleph) = \mathbf{P}_\star(\aleph)$. The equivalence $C_1(\aleph) \Leftrightarrow C_2(\aleph)$ follows from Lemma 1.5.

• $\mathcal{B}$ is said to be $\aleph$-*compactoid relative to* A if it satisfies one (hence all) of the above conditions.

It is customary, and we will do that, to refer to $\aleph_0$-compactoidness as *finite* compactoidness, and to $\aleph_1$-compactoidness as the *countable* one. Also, one denotes by $\mathbf{P}_1$ the class $\mathbf{P}(\aleph_0)$, by $\mathbf{P}_\omega$ the class $\mathbf{P}(\aleph_1)$. Again, it is customary to denote by $\mathbf{F}_1$ the family $\mathbf{F}(\aleph_0)$ of principal filters and use $\mathbf{F}_\omega$ instead of $\mathbf{F}(\aleph_1)$.

If $\aleph$ is dropped in the above, we obtain the conditions $(F_\#)$, (C_1) and (C_2) characterizing compactoidness relative to A. It is clear that $\mathcal{B}$ is compactoid if it is $\aleph$-compactoid for every $\aleph$.

3.1 Proposition. *If $\mathcal{B}$ is finitely compactoid relative to A and A is $\aleph$-compact, then $\mathcal{B}$ is $\aleph$-compactoid relative to A.*

Proof. Indeed, $\mathcal{B}$ is finitely compactoid relative to A if and only if $\mathcal{B}$ aims at A. The condition $(C_2(\aleph))$ allows the use of open covers. Apply Proposition 1.2. □

Remark. Our terminology is chosen in such a way that 'full properties,' e.g., being compactoid, follow from the corresponding '$\aleph$-properties,' being $\aleph$-compactoid, by dropping the $\aleph$. This is consistent with the fact that an object has the 'full property' if it has the '$\aleph$-property' for every $\aleph$, and also with the fact that, if an object is fixed, then the 'full property' is nothing else than the '$\aleph$-property' for a sufficiently large $\aleph$. An obvious consequence is that a statement, like Proposition 3.1, which is valid for every $\aleph$, implies automatically the 'full' statement. For instance, in the case of Proposition 3.1, we have

If $\mathcal{B}$ is finitely compactoid relative to A and A is compact, then $\mathcal{B}$ is compactoid relative to A.

For this reason, below, the 'full' statements which are obtained 'by dropping the $\aleph$' are treated as self-evident and will not be formulated.

The conditions $C_2(\aleph)$, 1.1 and 3.1 combine into

3.2 Theorem. *$\mathcal{B}$ is $\aleph$-compactoid relative to its intersection if and only if $\mathcal{B}$ aims at $\cap\mathcal{B}$, which is $\aleph$-compact.*

Following [**17**], $\mathcal{B}$ is *closed* if $\operatorname{adh}\mathcal{B} = \cap\mathcal{B}$. In the case of a filter base $\mathcal{B}$, the 'only if' part of the 'full' statement in the next corollary is [**17**, Theorem 2.5].

3.3 Corollary. *Suppose $\mathcal{B}$ is closed. Then $\mathcal{B}$ is $\aleph$-compactoid relative to its intersection if and only if $\mathcal{B}$ aims at its adherence which is $\aleph$-compact.*

4. $\aleph$-midcompactoidness. Denote by $\mathbf{C}(\aleph)$ the class of all *closed* overcovers such that $\operatorname{card}\{C : C \in \mathcal{C}\} < \aleph$ for each $\mathcal{C} \in \mathbf{C}(\aleph)$, and by $\mathbf{G}(\aleph)$ the family of all filters $\mathcal{G} \in \mathbf{C}_\star(\aleph)$, that is, of those filters which admit a base of cardinality less than $\aleph$ consisting of open sets.

CF-duality. *Let $A \subset X$. The following conditions are equivalent.*

($F^\circ_\#(\aleph)$) *Each $\mathcal{G} \in G(\aleph)$ meshing with $\mathcal{B}$ has a cluster point in A.*

($\overline{C}_1(\aleph)$) *For each $\mathcal{C} \in C(\aleph)$ overcovering A, there exist $B \in \mathcal{B}$ and a finite subfamily $\mathcal{C}_0 \subset \mathcal{C}$ such that $B \subset \cup\mathcal{C}_0$.*

($\overline{C}_2(\aleph)$) *For each open cover $\mathcal{V}$ of A of cardinality less than $\aleph$, there exist $B \in \mathcal{B}$ and a finite subfamily $\{V_1, V_2, \dots, V_n\} \subset \mathcal{V}$ such that $B \subset \cup_{i=1}^n \overline{V}_i$.*

The equivalence of the last two conditions follows from Lemma 1.6.

• $\mathcal{B}$ is said to be $\aleph$-*midcompactoid relative to* A if it satisfies one (hence all) of the above conditions.

So $\mathcal{B}$ *is midcompactoid relative to* A if $\mathcal{B}$ is $\mathbf{C}$-compactoid relative to A, where $\mathbf{C}$ is the class of all closed overcovers of A. As usual, $\mathcal{B}$ is midcompactoid relative to A if it is $\aleph$-midcompactoid relative to A for every $\aleph$.

Keeping our conventions intact, we refer to $\aleph_0$-midcompactoidness as *finite* midcompactoidness, and to $\aleph_1$-midcompactoidness as the *countable* one. Also, one denotes by $\mathbf{C}_1$ the class $\mathbf{C}(\aleph_0)$, and by $\mathbf{C}_\omega$ the class $\mathbf{C}(\aleph_1)$. The notation $\mathbf{G}_1$ and $\mathbf{G}_\omega$ should be clear.

In the present situation, Lemma 1.6 can be given a more precise form. Recall that an *open domain*, or a regular open set, is the interior of a closed set or, equivalently, a set $C \subset X$ such that $C = (\overline{C})^\circ$. Similarly, a *closed domain*, or a regular closed set, is the closure of an open set or a set $C \subset X$ such that $C = \overline{C^\circ}$, see, e.g., [**13**].

4.1 Proposition. *The following conditions are equivalent.*

(i) *For each* $\mathcal{C} \in \mathbf{C}(\aleph)$ *overcovering* A *there exist* $B \in \mathcal{B}$ *and a finite subfamily* $\mathcal{C}_0$ *of* $\mathcal{C}$ *such that* $B \subset \cup\, \mathcal{C}_0$.

(ii) *For each cover* $\mathcal{D}$ *of* A *by open domains with* $\operatorname{card}\mathcal{D} < \aleph$ *there exist* $B \in \mathcal{B}$ *and a finite subfamily* $\mathcal{D}_0$ *of* $\mathcal{D}$ *such that* $B \subset \cup\{\overline{D} : D \in \mathcal{D}_0\}$.

(iii) *For each open cover* $\mathcal{O}$ *of* A *with* $\operatorname{card}\mathcal{O} < \aleph$ *there exist* $B \in \mathcal{B}$ *and a finite subfamily* $\mathcal{O}_0$ *such that* $B \subset \cup\{\overline{O} :\ O \in \mathcal{O}_0\}$.

(iv) *For each overcover* $\mathcal{C}$ *of* A *by closed domains with* $\operatorname{card}\mathcal{C} < \aleph$, *there exist* $B \in \mathcal{B}$ *and a finite subfamily* $\mathcal{C}_0$ *such that* $B \subset \cup\, \mathcal{C}_0$.

Proof. (i) implies (iv) is trivial.

(iv) implies (iii). Let $\mathcal{O}$ be an open cover of A. Then $\overline{\mathcal{O}}$ is an overcover of A by closed domains. Hence by (iv) there exist $B \in \mathcal{B}$ and $\overline{O_1}, \overline{O_2}, \ldots, \overline{O_n}$ such that $B \subset \cup_{i=1}^n \overline{O_i}$. This shows (iii).

(iii) implies (ii) is trivial.

(ii) implies (i). Let $\mathcal{C}$ be a closed overcover of A. Then $\mathcal{C}^\circ$ is an overcover of A by open domains. Hence by (ii) there exists $B \in \mathcal{B}$ and its finite cover $\{\overline{C_1^\circ}, \overline{C_2^\circ}, \dots, \overline{C_n^\circ}\}$. Noting that, for each $i = 1, 2, \dots, n$, $\overline{C_i^\circ} \subset C_i$, we get $B \subset \cup_{i=1}^n C_i$ which shows (i).

The next proposition is a special case of 1.3.

4.2 Proposition. *If $\mathcal{B}$ is $\aleph$-midcompactoid relative to A, then its adherence* $\operatorname{adh}\mathcal{B}$ *is $\aleph$-midcompactoid relative to A.*

According to our terminology, a filter base $\mathcal{B}$ must be called $\aleph$-*midcompact*, if it is nearly $\aleph$-selfmidcompactoid. Let us identify what $\aleph$-midcompactness of a set means. If $\aleph$ is sufficiently large with respect to the cardinality of E, $\mathbf{C}(\aleph) = \mathbf{C}$. Hence, if X is a Hausdorff space, the condition means that E is a H-*closed* subspace of X in the terminology of Engelking [**10**] or *absolutely closed* in that of Bourbaki [**1**]. The countable condition, obtained when $\aleph = \aleph_1$, means that E is a *pseudocompact* subspace of X if like, e.g., Engelking in [**10**], one assumes X to be a Tychonoff space (Engelking and other authors assume X to be a Tychonoff space in order to get the equivalence with the known definition of pseudocompactness in terms of continuous functions defined on E). Alternatively, if one assumes X to be merely Hausdorff, one could call E a *countably* H-*closed* subspace.

4.3 Proposition. *A subset E of X is $\aleph$-selfmidcompactoid if and only if it is $\aleph$-midcompact.*

Proof. We will use Proposition 2.2. The class $\mathbf{D} = \mathbf{G}(\aleph)$ is locally determined by the property:

(P) $\mathcal{G} \in \mathbf{G}(\aleph)$ *provided it admits an open base of cardinality less than* $\aleph$.

The only fact that needs a proof is the 'extension' condition (2) stated before Proposition 2.2. Let $\mathcal{H}$ be a (P)-base on E. For each $H \in \mathcal{H}$ (which is open relative to E) choose an open set H' in X such that $H' \cap E = H$. Let $\mathcal{H}' = \{H' : H \in \mathcal{H}\}$; it is a subbase of a filter on X whose base is the family $\mathcal{H}''$ of finite intersections of elements of $\mathcal{H}'$. Clearly, $\operatorname{card}\mathcal{H}'' < \aleph$. Moreover $\mathcal{H}''$ induces $\mathcal{H}$ on E.

Theorem 1.3 and Proposition 4.3 in the present context give

4.4 Theorem. *If $\mathcal{B}$ is $\aleph$-midcompactoid relative to A, then so is $\overline{\mathcal{B}}$. If $A \subseteq \operatorname{adh} \mathcal{B}$, then $\overline{\mathcal{B}}$ is $\aleph$-midcompactoid relative to its intersection and A is $\aleph$-midcompact.*

The following proposition should be compared with Proposition 2.1.

4.5 Proposition. *If $\mathcal{B}$ is finitely midcompactoid relative to A which is $\aleph$-compact, then $\overline{\mathcal{B}}$ is $\aleph$-midcompactoid relative to A.*

Proof. Let $\mathcal{P}$ be a closed overcover of A with $\operatorname{card} \mathcal{P} < \aleph$. As A is $\aleph$-compact, there exists $\mathcal{P}_0 = \{P_1, P_2, \dots, P_n\}$ such that the interiors of P_i's cover A. That is, $A \subset \cup \mathcal{P}_0$ and $\cup \mathcal{P}_0$ is a closed overcover of A. As $\mathcal{B}$ is finitely midcompactoid relative to A, there is $B \in \mathcal{B}$ such that $B \subset \cup \mathcal{P}_0$. Hence $\overline{B} \subset \cup \mathcal{P}_0$. $\square$

Remark. We sometimes write $\mathcal{B} \rightsquigarrow_c A$ and say that $\mathcal{B}$ *clings to* A if $\mathcal{B}$ is finer than the filter generated by the *closed* neighborhoods of A, that is, if for each open set O containing A there exists $B \in \mathcal{B}$ such that $B \subset \overline{O}$. Noting that $\mathcal{B}$ is finitely midcompactoid means also that $\mathcal{B}$ clings to A, we could state the above proposition in a way similar to Proposition 1.2, that is, assuming that $\mathcal{B}$ clings to A.

For our purposes, it will be convenient to call X of weight $\aleph$ if it admits a base of open sets whose cardinality is less than $\aleph$.

4.6 Theorem. *Let X be a regular space of weight $\aleph$. $\mathcal{B}$ is $\aleph$-midcompactoid relative to A (if and) only if $\overline{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}$ is compactoid relative to A.*

Proof. Let $\mathcal{B}$ be $\aleph$-midcompactoid relative to A. As X is of weight $\aleph$, any open cover of A contains a subcover of cardinality less than $\aleph$. Let therefore $\mathcal{C}$ be an open cover of A which, without loss of generality, can be assumed to be of cardinality less than $\aleph$, and $\mathcal{O}$ a base of the topology of X, also of cardinality less than $\aleph$. Consider $\mathcal{O}_C = \{O \in \mathcal{O} : \overline{O} \subset C\}$

for each $C \in \mathcal{C}$. Further, define (a refinement of $\mathcal{C}$) $\mathcal{V} = \cup \{\mathcal{O}_C : C \in \mathcal{C}\}$. It is clear that $\mathcal{V}$ is an open cover of A and $\operatorname{card} \mathcal{V} < \aleph$. Applying the condition $\overline{C}_2(\aleph)$, we find $B \in \mathcal{B}$ and $\{\overline{O_1}, \overline{O_2}, \dots, \overline{O_n}\}$ whose union contains B. Let C_i be an element of $\mathcal{C}$ for which $\overline{O_i} \subset C_i$, $i = 1, 2, \dots, n$. Then $\overline{B} \subset \cup_{i=1}^n \overline{O_i} \subset \cup_{i=1}^n C_i$ which ends the proof. ◻

4.7 Corollary. *Let X be regular of weight $\aleph$. The following are equivalent.*

(i) *$\mathcal{B}$ is $\aleph$-midcompactoid relative to its adherence.*

(ii) *$\mathcal{B}$ aims at* $\operatorname{adh} \mathcal{B}$ *which is $\aleph$-compact.*

(iii) *$\overline{\mathcal{B}}$ is compactoid relative to its intersection.*

4.8 Corollary. *Let X be regular. The following are equivalent.*

(i) *$\mathcal{B}$ is midcompactoid relative to its adherence.*

(ii) *$\mathcal{B}$ is compactoid.*

(iii) *$\mathcal{B}$ aims at* $\operatorname{adh} \mathcal{B}$ *which is compact.*

(iv) *$\overline{\mathcal{B}}$ is compactoid.*

(v) *$\overline{\mathcal{B}}$ is compactoid relative to its intersection.*

Remark. As mentioned in the introduction, (ii) $\Leftrightarrow$ (iii) is due to Vaughan [**24, 25**], compare also [**7, 18**]. (ii) $\Leftrightarrow$ (iv) is the corollary in [**24**], compare also [**4**].

5. Alexander subbase theorem. Let $X = (X, \mathcal{O})$ be a topological space. A *filter in* $\mathcal{O}$ is a nonempty subfamily $\mathcal{G} \subset \mathcal{O}$ which does not contain the empty set, is stable under finite intersections and such that if $G \in \mathcal{G}$ and $G \subset H \in \mathcal{O}$, then $H \in \mathcal{G}$. A filter which is a maximal element with respect to inclusion in the family of filters in $\mathcal{O}$ is called an *ultrafilter in* $\mathcal{O}$. Note that if $\mathcal{O} = 2^X$, i.e., the topology is discrete, we may drop 'in $\mathcal{O}$' recovering the usual notion of an (ultra)filter in X.

A filter base $\mathcal{B}$ is said to be *midrefinable* if, for every filter $\mathcal{G}$ in $\mathcal{O}$ meshing with $\mathcal{B}$, there exists a filter $\mathcal{F}$ in $\mathcal{O}$ which is finer than both $\mathcal{G}$ and $\mathcal{B}$. It is clear that any filter base $\mathcal{B}$ which itself consists of open sets is midrefinable.

Here are a few simple facts.

(1) *If $\mathcal{G}$ is a filter in $\mathcal{O}$, then there exists an ultrafilter $\mathcal{U}$ in $\mathcal{O}$ finer than $\mathcal{G}$.*

(2) *Let $\mathcal{U}$ be an ultrafilter in $\mathcal{O}$ and let $\mathcal{B}$ be a midrefinable filter base. If $\mathcal{U}\#\mathcal{B}$, then $\mathcal{U}$ is finer than $\mathcal{B}$.*

(3) *Let $\mathcal{U}$ be an ultrafilter in $\mathcal{O}$. If $x_0 \in \operatorname{adh}\mathcal{U}$, then $x_0 \in lim\,\mathcal{U}$.*

We consider the following conditions.

(U) *Each ultrafilter $\mathcal{F}$ finer than $\mathcal{B}$ has a limit point in A* $(\lim \mathcal{F} \cap A \neq \varnothing)$.

(U°) *Each ultrafilter in $\mathcal{O}$ finer than $\mathcal{B}$ has a limit point in A.*

5.1 Proposition. *Suppose $\mathcal{B}$ is a midrefinable filter base. $\mathcal{B}$ is midcompactoid relative to A if (and only if) (U°) is satisfied.*

Proof. Let $\mathcal{F}$ be a filter base in $\mathcal{O}$ meshing with $\mathcal{B}$. Then, as $\mathcal{B}$ is midrefinable, there exists a filter $\mathcal{F}'$ in $\mathcal{O}$ which is finer than both $\mathcal{F}$ and $\mathcal{B}$. Let $\mathcal{F}''$ be an ultrafilter in $\mathcal{O}$ finer than $\mathcal{F}'$. Then $\varnothing \neq A \cap \lim \mathcal{F}'' \subset \operatorname{adh}\mathcal{F}' \cap A \subset \operatorname{adh}\mathcal{F} \cap A$. This shows $(U^\circ) \Rightarrow (F^\circ_\#)$. $\square$

Applying the above with $\mathcal{O} = 2^X$ and noting that midrefinability of $\mathcal{B}$ is then automatic, we have

5.2 Corollary. *Suppose $\mathcal{B}$ is a filter base. $\mathcal{B}$ is compactoid relative to A if and only if (U) is satisfied.*

Let $\mathcal{S}$ be a subbase of (the topology of) X, and denote by $\mathbf{S}$ the class of all covers of A consisting of subbasic sets. A family $\mathcal{B}$ is said to be $\mathbf{S}$-*midcompactoid relative to* A if for each cover $\mathcal{G} \in \mathbf{S}$ there exist sets $G_1, G_2, \ldots, G_n$ in $\mathcal{G}$ and $B \in \mathcal{B}$ such that $B \subset \cup_{i=1}^n \overline{G}_i$.

5.3 Theorem. *Let $\mathcal{B}$ be* **S**-*midcompactoid relative to A. Then each ultrafilter in $\mathcal{O}$ meshing with $\mathcal{B}$ has a limit point in A.*

Proof. Let $\mathcal{U}$ be an ultrafilter in $\mathcal{O}$, meshing with $\mathcal{B}$, which does not have a limit point in A.

For each $x \in A$, there exists $G(x)$ in $\mathcal{S}$ such that $x \in G(x)$ and $G(x)$ does not mesh with $\mathcal{U}$. Otherwise, by (2), $G(x)$ would belong to $\mathcal{U}$ and, as finite intersections of such sets form a fundamental system of the neighborhoods of x, the point x would be a limit point of $\mathcal{U}$. Applying the assumption of the theorem to the cover $\{G(x) : x \in A\}$, we find $x_1, x_2, \ldots, x_k$ and $B \in \mathcal{B}$ such that $B \subset \cup_{i=1}^{k} \overline{G}(x_i)$.

As $G(x_i)$'s do not mesh with $\mathcal{U}$, it follows that $G = \cup_{i=1}^{n} G(x_i)$ does not mesh with $\mathcal{U}$ either. Then, U being open, $U \cap \overline{G} = \varnothing$ for some $U \in \mathcal{U}$. This contradicts the fact that $\overline{G} \supset B \in \mathcal{B}$ and $\mathcal{U} \# \mathcal{B}$. □

It is not clear to the present author what exactly are the families $\mathcal{B}$ for which the conclusion of the above theorem amounts to their midcompactoidness relative to A. However, we have

5.4. Corollary. *Let $\mathcal{B}$ be a midrefinable filter base. If $\mathcal{B}$ is* **S**-*midcompactoid relative to A, then it is midcompactoid relative to A. If $\mathcal{B}$ is* **S**-*selfmidcompactoid, then it is selfmidcompactoid.*

Proof. Indeed, by Theorem 5.3 the condition (U°) is satisfied. Apply Proposition 5.1. □

By a similar, and even slightly simpler, argument, we get also

5.5 Theorem. *Let $\mathcal{B}$ be* **S**-*compactoid relative to A. Every ultrafilter meshing with $\mathcal{B}$ has a limit point in A.*

5.6. Corollary. *Let $\mathcal{B}$ be a filter base. If $\mathcal{B}$ is* **S**-*compactoid relative to A, then it is compactoid relative to A. If $\mathcal{B}$ is* **S**-*selfcompactoid, then it is selfcompactoid.*

Concluding, let us mention that historically the investigations of upper semi-continuity were providing both a motivation as well as a field of applications for the notion of compactoidness. This research is not an exception and so some of the results obtained above found applications in the author's paper [**14**] devoted to a study of the active boundary of upper semi-continuous set valued maps.

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