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L^p ESTIMATES FOR ROUGH SINGULAR INTEGRALS ASSOCIATED TO SOME HYPERSURFACES

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ABSTRACT. We consider a class of rough singular integral operators that are associated to a Calderón-Zygmund type kernel K and a hypersurface given by the graph $\{(y, \phi(\psi(y))) : y \in \mathbf{R}^n\}$. Here $\phi(t)$ is an increasing convex C^2 function on $[0, \infty)$, and ψ is a smooth convex function on $\mathbf{R}^n$, which is homogeneous of degree 1 and of finite type. Also, $K|_{S^{n-1}}$ satisfies a cancelation condition and some other hypotheses but may fail to be smooth. We obtain L^p estimates for these operators assuming that the maximal function related to the function $\phi(t)$ is bounded on L^p.

1. Introduction. Let S^{n-1} be the unit sphere in $\mathbf{R}^n$, $n \geq 2$, with the induced Lebesgue measure $d\sigma$. Throughout this paper, we shall assume that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$, which satisfies the condition $\Omega \in L^1(S^{n-1})$ and the cancelation condition

$$\int_{S^{n-1}} \Omega(y)\, d\sigma(y) = 0.$$

Let $K(y) = \Omega(y)/|y|^n$ for $y \in \mathbf{R}^n \backslash \{0\}$. For $d \geq n$ and a suitable mapping $\Phi : \mathbf{R}^n \to \mathbf{R}^d$, define the singular integral operator $T_{K,\Phi}$ by

$$T_{K,\Phi} f(x) = p.v. \int_{R^n} f(x - \Phi(y)) K(y)\, dy$$

for $x \in \mathbf{R}^d$.

When $n = d$, $\Omega \in C^\infty(S^{n-1})$ and $\Phi(y) \equiv y$ for $y \in \mathbf{R}^n$, $T_{K,\Phi}$ reduces to a classical Calderón-Zygmund singular integral operator T_K, given by

$$T_K f(x) \equiv T_{K,\Phi} f(x) = p.v. \int_{\mathbf{R}^n} f(x - y) K(y)\, dy$$

for $x \in \mathbf{R}^n$, and hence it is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.

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In [3], Calderón and Zygmund used the method of rotations to establish the L^p-boundedness of T_K for $1 < p < \infty$ under the weaker hypothesis $\Omega \in L\log^+ L(S^{n-1})$, that is,

$$\int_{S^{n-1}} |\Omega(y)| \log^+ |\Omega(y)|\, d\sigma(y) < \infty.$$

They also showed that the condition $\Omega \in L\log^+ L(S^{n-1})$ is sharp in the sense that if $\Omega \in L(\log^+ L)^{1-\varepsilon}(S^{n-1})$ for some $\varepsilon > 0$, T_K cannot be bounded on $L^p(\mathbf{R}^n)$ for any p.

When $\Omega \in L^q(S^{n-1})$ for some $q > 1$, Duoandikoetxea and Rubio de Francia obtained the L^p estimates for T_K by using a method involving some Fourier transform estimates instead of the method of rotations. See [3] and [10].

Ricci and Weiss [20], and independently Connett [9], obtained the improved result that if $\Omega \in H^1(S^{n-1})$, then T_K is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. Now let us recall that, for $1 < q \le \infty$, the inclusion relations

$$\begin{aligned} C^\infty(S^{n-1}) \subset L^\infty(S^{n-1}) \subset L^q(S^{n-1}) &\subset L\log^+ L(S^{n-1}) \\ &\subset H^1(S^{n-1}) \subset L^1(S^{n-1}) \end{aligned}$$

hold and that all the inclusions are proper. (For a definition of the Hardy space $H^1(S^{n-1})$, see Section 2.)

From now on we will restrict our attention to the case of hypersurfaces. That is, we take $d = n+1$. First let us consider the surface of revolution $\Phi(y) = (y, \phi(|y|))$ for $y \in \mathbf{R}^n$. For a given Ω, let

$$(1.1)\quad T_{K,\phi} f(x,t) \equiv T_{K,\Phi} f(x,t) = p.v. \int_{R^n} f(x-y, t-\phi(|y|)) K(y)\, dy$$

for $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. This operator $T_{K,\phi}$ is called the singular integral operator along the surface of revolution $\{(y, \phi(|y|)) : y \in \mathbf{R}^n\}$. Just like in the case of T_K, L^p estimates for $T_{K,\phi}$ have been obtained for Ω belonging to spaces ranging from $C^\infty(S^{n-1})$ to $H^1(S^{n-1})$. For $\Omega \in C^\infty(S^{n-1})$, Kim, Wainger, Wright and Ziesler in [14] proved the L^p boundedness of singular integrals along certain surfaces of revolution.

Theorem A [**14**]. *Suppose that* $\Omega \in C^\infty(S^{n-1})$ *and* $\phi : [0,\infty) \to \mathbf{R}$ *is* C^2*, convex and increasing. Then* $T_{K,\phi}$ *in* (1.1) *extends to a bounded operator on* $L^p(\mathbf{R}^{n+1})$ *for* $1 < p < \infty$.

Chen and Fan improved the above result to the situation $\Omega in L^q(S^{n-1})$ or some $q > 1$, assuming that ϕ satisfies weaker hypotheses than those of Theorem A. In fact, they proved the L^p estimate of $T_{K,\phi}$ when Ω belongs to the Block space $B_r^\beta(S^{n-1})$, $\beta > 0$ and $r > 1$, introduced in [**17**], which contains $L^q(S^{n-1})$ for $q > 1$. However, we would like to point out that by a result of Keitoku and Sato in [**13**], it follows that the statement that $T_{K,\phi}$ is bounded on L^p for every $1 < p < \infty$ for $\Omega \in L^q(S^{n-1})$ for some $q > 1$ is equivalent to the statement that $T_{K,\phi}$ is bounded on L^p for every $1 < p < \infty$ for $\Omega \in B_r^\beta(S^{n-1})$, see [**15**].

Let $\phi : [0,\infty) \to \mathbf{R}$ be continuously differentiable on $(0,\infty)$, which satisfies

$$|\phi(t) - \phi(0)| \le ct^\alpha, \tag{1.2}$$

for some $\alpha > 0$ and sufficiently small t, where c is independent of t.

Theorem B [**6**]. *Suppose* $\Omega \in L^q(S^{n-1})$ *for some* $q > 1$ *and* ϕ *satisfies the condition* (1.2)*. If the maximal operator* V_ϕ *given by*

$$V_\phi g(t) = \sup_{k\in Z} \left| \int_{2^k}^{2^{k+1}} g(t-\phi(r))\,\frac{dr}{r} \right| \tag{1.3}$$

is bounded on $L^p(\mathbf{R})$ *for all* $1 < p < \infty$*, then* $T_{K,\phi}$ *in* (1.1) *is bounded in* $L^p(\mathbf{R}^{n+1})$ *for all* $1 < p < \infty$.

In [**1**], Al-Salman and Pan extended this result to the case $\Omega \in L\log^+ L(S^{n-1})$ also by the Fourier transform estimate method in [**10**], using Theorem B.

Theorem C [**1**]. *Suppose* $\Omega \in L\log^+ L(S^{n-1})$ *and* ϕ *satisfies the condition* (1.2)*. Let* $T_{K,\phi}$ *be given by* (1.1)*. Let* $m \in \mathbf{N}$ *and* $V_\phi^{(m)}$ *be the maximal function on* $\mathbf{R}$ *given by*

$$V_\phi^{(m)} g(t) = \sup_{k\in Z} \left| \int_{2^{mk}}^{2^{m(k+1)}} g(t-\phi(r))\,\frac{dr}{r} \right| \tag{1.4}$$

for $t \in \mathbf{R}$. If there exists a constant C_p independent of m such that

$$\|V_\phi^{(m)} g\|_{L^p(R)} \le C_p\, m \|g\|_{L^p(R)} \tag{1.5}$$

for all $1 < p < \infty$, then $T_{K,\phi}$ is bounded on $L^p(\mathbf{R}^{n+1})$ for all $1 < p < \infty$.

Lu, Pan and Yang [**16**] also obtained the following extension to the case $\Omega \in H^1(S^{n-1})$ with a stronger assumption on ϕ, again using Theorem B. See [**8**] and [**7**] for the definition of the Hardy space $H^1(S^{n-1})$ on the sphere.

Theorem D [**16**]. *Suppose that $\Omega \in H^1(S^{n-1})$ and ϕ satisfies condition* (1.2). *Let us define M_ϕ the maximal operator along $\{(r, \phi(r)) : r \in \mathbf{R}^+\}$ by*

$$M_\phi g(x,t) = \sup_{k \in Z} \left| \int_{2^k}^{2^{k+1}} g(x - r, t - \phi(r))\, \frac{dr}{r} \right| \tag{1.6}$$

for $x \in \mathbf{R}$ and $t \in \mathbf{R}$. Then $T_{K,\phi}$ given by (1.1) *is bounded on $L^p(\mathbf{R}^{n+1})$ for $1 < p < \infty$, provided that M_ϕ is a bounded operator in $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$.*

We note that the L^p boundedness of V_ϕ in (1.3) is equivalent to the estimate (1.5) for $V_\phi^{(m)}$ in (1.4) and that if the maximal operator M_ϕ in (1.6) is bounded in $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$, then the operator V_ϕ is bounded in $L^p(\mathbf{R})$ for all $1 < p < \infty$. Chen and Fan showed that if $\phi : [0, \infty) \to \mathbf{R}$ is C^2, convex and increasing, then V_ϕ is a bounded operator in $L^p(\mathbf{R})$ for all $1 < p < \infty$. For more details on $T_{K,\phi}$, we refer the readers to [**18**], where Pan gave a survey of some recent results concerning this topic and also stated some open problems.

So far we have briefly recalled some well-known results for the classical Calderón-Zygmund singular integral operator T_K and the related operator $T_{K,\phi}$ associated to the surface of revolution. Now it is natural to ask the following question whether the results on $T_{K,\phi}$ may be extended to more general surfaces than just the surfaces of revolution:

Question. Let ϕ satisfy (1.2), and let ψ be a smooth convex function on $\mathbf{R}^n$, which is homogeneous of degree 1 and of finite type with $\psi(0) = \nabla\psi(0) = 0$. Let $\Phi(y) = (y, \phi(\psi(y)))$ for $y \in \mathbf{R}^n$. For a suitable $\Omega \in L^1(S^{n-1})$, define

$$T_{K,\phi,\psi} f(x,t) \equiv T_{K,\Phi} f(x,t) = p.v. \int_{R^n} f(x-y, t-\phi(\psi(y)))K(y)\, dy.$$

Then, does $T_{K,\phi,\psi}$ extend to a bounded operator on $L^p(\mathbf{R}^{n+1})$ for $1 < p < \infty$, if Ω satisfies the hypotheses in Theorems A, B, C and D, respectively?

The first step in answering this question was taken by Wainger, Wright and Ziesler [**24**]. Namely, they obtained the L^p estimates for $T_{K,\phi,\psi}$, when $\Omega \in C^\infty(S^{n-1})$. They also determined precisely for which convex functions ψ of finite type, $T_{K,\phi,\psi}$ is bounded on L^2 for all C^1 functions ϕ with $\phi(0) = 0$. In this context they introduced the linear subspaces $E_l = \{v \in \mathbf{R}^n : \psi(v) = O(s^{l+1}) \text{ for small } s > 0\}$. Clearly, $E_{l+1} \subset E_l \subset \cdots \subset E_1 = \mathbf{R}^n$. Let l_0 be the smallest l such that $E_l \neq \mathbf{R}^n$. They showed that the L^2-boundedness holds for all such ϕ if and only if $\operatorname{codim} E_{l_0} \geq 2$. Specifically for the question stated above, they showed the following result when $E_{l_0} = \{0\}$, which is satisfied by a homogeneous function ψ. Let us denote $\phi_{l_0}(r) = \phi(r^{l_0})$ for any $r \in \mathbf{R}^+$.

Theorem E [**24**]. *Suppose that $\Omega \in C^\infty(S^{n-1})$, $E_{l_0} = \{0\}$ and that ϕ_{l_0} is C^1 and convex. Let ψ be a smooth convex function of finite type with $\psi(0) = \nabla\psi(0) = 0$. Then*

$$\|T_{K,\phi,\psi} f\|_{L^p(R^{n+1})} \leq C_p \|f\|_{L^p(R^{n+1})}, \quad 1 < p < \infty.$$

However, this leaves the question open for rough Ω, that is, $\Omega \notin C^\infty(S^{n-1})$. In this paper, we answer this question in the affirmative when ψ is a certain homogeneous function. Our approach is mainly the one which originated in the work of Duoandikoetxea and Rubio de Francia [**10**] and was further developed by Fan and Pan [**12**]. But in our case some difficulty arises, because the rotation-invariance of a surface of revolution is not available. We will try to adapt the approach of [**12**] by making the level surface $H = \{x \in \mathbf{R}^n : \ \psi(x) = 1\}$ play the role of the sphere and using a sort of polar coordinates adapted to H. Here the difficulty is that the cancelation hypothesis $\int_{S^{n-1}} \Omega(y)\, d\sigma(y) = 0$ is made for the sphere and not on H. This difficulty may be resolved as follows.

In order to obtain a similar cancelation condition on H, we replace the Lebesgue measure by the weighted measure which is related to the polar coordinates with respect to ψ and use an idea in [**24**]. Also, when $\Omega \in H^1(S^{n-1})$, we cannot apply the oscillatory integral estimate of Fan and Pan [**12**], which was used to obtain Theorem D. So we replace their oscillatory integral estimate by an estimate better adapted to our hypersurface and use these estimates to prove an extension of Theorem D.

Following [**16**], we define a more general singular integral operator T, which is associated to the hypersurfaces of the form $\{(y, \phi(\psi(y))) : y \in \mathbf{R}^n\}$, as follows:

$$Tf(x,t) = p.v. \int_{R^n} f(x-y, t-\phi(\psi(y)))\, b(\psi(y)) K(y)\, dy \tag{1.7}$$

for a bounded function b on $[0,\infty)$ and answer the question about T.

Our main results may be stated as follows. Suppose that ϕ satisfies the condition (1.2) and that ψ is a homogeneous function of degree one in $\mathbf{R}^n$. Let H be the hypersurface defined by $\{y \in \mathbf{R}^n : \psi(y) = 1\}$ with the induced Lebesgue measure $d\sigma_H$. For a fixed point $\omega \in H$, let u_ω be the outward unit normal to the surface H at ω, and let T_ω be the affine tangent plane to H at ω. Also, following [**2**] we define the "ball" $\tilde{B}(\omega, s)$ by $\tilde{B}(\omega, s) = \{y \in H : \operatorname{dist}(y, T_\omega) < s\}$ for $s > 0$.

We will now introduce a definition concerning the function ψ, which we will use instead of the usual condition that ψ is a smooth convex function of finite type. The advantage of this definition is that it allows some non-smooth and non-finite type examples.

Definition 1.1. Let ψ be a C^2 convex function on $\mathbf{R}^n$ with $\psi(0) = 0$ such that the associated surface $H = \{x : \ \psi(x) = 1\}$ is compact. We will say that ψ is a-*convex type* if there is a constant $a > 0$ such that, for some small $\delta > 0$,

$$\sup_{\omega\in H}\int_0^{\delta}\sigma_H(\tilde{B}(\omega,s))\,\frac{ds}{s^{1+a}}\ <\infty. \tag{1.8}$$

We remark that if ψ is a smooth convex function of finite type, then ψ is *a-convex type* for some $a > 0$. For $\Omega \in L^q(S^{n-1})$, we obtain the following extension of Theorem B by using the method of Fourier transform estimate in [**6**] and an observation of Duoandikoetxea and Rubio de Francia in [**10**]. In Section 4, we shall establish the following

Theorem 1. *Let $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Suppose that ϕ satisfies* (1.2) *and that ψ is a*-convex type *for some $a > 0$, which is a homogeneous function of degree* 1*. Then T is a bounded operator in $L^p(\mathbf{R}^{n+1})$ for all $1 < p < \infty$, provided that the maximal operator V_ϕ, in* (1.3) *is bounded on $L^p(\mathbf{R})$ for all $1 < p < \infty$.*

By using Theorem 1 and some methods in [**1**], we obtain the following extension of Theorem C. This is established in Section 4.

Theorem 2. *Let $\Omega \in L\log^+ L(S^{n-1})$. Suppose that ϕ satisfies* (1.2) *and that ψ is a homogeneous function of degree* 1*, which is a-convex type for some $a > 0$. Then T is a bounded operator on $L^p(\mathbf{R}^{n+1})$ for all $1 < p < \infty$, provided that the maximal operator $V_\phi^{(m)}$ in* (1.4) *satisfies* (1.5)*.*

Finally, we obtain the following extension of Theorem D, with the condition $\Omega \in H(S^{n-1})$, by using Theorem 1 and some modified oscillatory integral estimates for a polynomial phase of degree 1. In Section 3, we shall establish the following

Theorem 3. *Suppose $\Omega \in H^1(S^{n-1})$. Let ϕ satisfy* (1.2) *and ψ be a smooth convex function of finite type with $\psi(0) = \nabla\psi(0) = 0$, which is a homogeneous function of degree* 1*. If the maximal operator M_ϕ defined by* (1.6) *is bounded on $L^p(\mathbf{R}^2)$ for all $1 < p < \infty$, then T is bounded on $L^p(\mathbf{R}^{n+1})$ for all $1 < p < \infty$.*

This paper is organized as follows. In Section 2, we shall state some important known lemmas which are useful for obtaining our results. Since the proofs of Theorems 1 and 2 have some similarity to that of Theorem 3, we shall first prove Theorem 3 in Section 3 in some detail and only sketch the proof of Theorems 1 and 2 in Section 4. Some oscillatory integral estimates that are important for our proofs are shown in Section 5.

Throughout the paper, $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$. We say that $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2. Some lemmas. We begin by recalling the definition of the Hardy space $H^1(S^{n-1})$ on S^{n-1}. Let $P_{r,y}$ be the Poisson kernel on S^{n-1} defined by

$$P_{r,y}(x) = \frac{1-r^2}{|x-ry|^2}$$

where $r \in [0,1)$ and x, $y \in S^{n-1}$. For $f \in L^1(S^{n-1})$, we define

$$P^+f(x) = \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} P_{r,y}(y) f(y)\, d\sigma(y) \right|$$

where $x \in S^{n-1}$. The Hardy space $H^1(S^{n-1})$ is given by

$$H^1(S^{n-1}) = \{ f \in L^1(S^{n-1}) \ : \ \|P^+f\|_{L^1(S^{n-1})} < \infty \}$$

and $\|f\|_{H^1(S^{n-1})} = \|P^+f\|_{L^1(S^{n-1})}$. See [**12, 8**] and [**7**] for the details. There are two types of H^1 atoms on the unit sphere.

Definition 2.1. A function $a(\cdot)$ on S^{n-1} is a regular atom if there exist $\zeta \in S^{n-1}$ and $\rho \in (0,2]$ such that

(i) $\operatorname{supp}(a) \subset S^{n-1} \cap B(\zeta,\rho)$ where $B(\zeta,\rho) = \{y \in \mathbf{R}^n : \ |y-\zeta| < \rho\}$

(ii) $\|a\|_{L^\infty(S^{n-1})} \leq \rho^{-n+1}$

(iii) $\int_{S^{n-1}} a(y)\, d\sigma(y) = 0$.

Definition 2.2. A function $a(\cdot)$ on S^{n-1} is an exceptional atom if $a(\cdot) \in L^\infty(S^{n-1})$ and $\|a\|_{L^\infty(S^{n-1})} \leq 1$.

And we shall use the following characterization of $H^1(S^{n-1})$ in [**7**] or [**8**].

Lemma 2.3. *For any $f \in H^1(S^{n-1})$, there are complex numbers c_j and atoms (regular and exceptional) a_j such that*

$$f = \sum_j c_j\, a_j$$

converges in $H^1(S^{n-1})$ norm and $\|f\|_{H^1(S^{n-1})} \simeq \sum_j |c_j|$.

The following lemmas are the main tools for proving Theorem 3.

Lemma 2.4. *Let $l, n \in \mathbf{N}$ and $\{\sigma_{j,k} : j = 1, \dots, l$ and $k \in \mathbf{Z}\}$ be a family of measures on $\mathbf{R}^n$ with $\sigma_{0,k} = 0$ for every $k \in \mathbf{Z}$. Let $\alpha_{j,1}$, $\alpha_{j,2} > 0$, $\eta \in \mathbf{R}\backslash\{1\}$, $\{n_j : j = 1, \dots, l\} \subset \mathbf{N}$ and $L_j : \mathbf{R}^n \to \mathbf{R}^{n_j}$ be linear transformations for $j = 1, \dots, l$. Suppose*

(i) $\|\sigma_{j_k}\| \le 1$ *for* $k \in \mathbf{Z}$ *and* $j = 1, \dots, l$

(ii) $|\widehat{\sigma}_{j,k}(\xi)| \le C(\eta^k |L_j \xi|)^{-\alpha_{j,2}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$ *and* $j = 1, \dots, l$

(iii) $|\widehat{\sigma}_{j,k}(\xi) - \widehat{\sigma}_{j-1,k}(\xi)| \le C(\eta^k |L_j \xi|)^{\alpha_{j,1}}$ *for* $\xi \in \mathbf{R}^m$, $k \in \mathbf{Z}$ *and* $j = 1, \dots, l$

(iv) *For some $q > 1$, there exists $A_q > 0$ such that*

$$\| \sup_{k \in Z} | \, |\sigma_{j,k}| * f | \, \|_{L^q(R^n)} \le A_q \|f\|_{L^q(R^n)}$$

for all $f \in L^q(\mathbf{R}^n)$ and $j = 1, \dots, l$.

Then for every $p \in ((2q/q+1), (2q/q-1))$, there exists a constant C_p such that

$$\Big\| \sum_{k \in Z} \sigma_{l,k} * f \Big\|_{L^p(R^n)} \le C_p \|f\|_{L^p(R^n)}$$

and

$$\Big\| \Big(\sum_{k \in Z} |\sigma_{l,k} * f|^2 \Big)^{1/2} \Big\|_{L^p(R^n)} \le C_p \|f\|_{L^p(R^n)}$$

hold for every $f \in L^p(\mathbf{R}^n)$. The constant C_p is independent of the linear transformations $\{L_j\}_{j=1}^{l}$.

Lemma 2.5 [**11**]. *Let $n, m \in \mathbf{N}$, $\eta \in \mathbf{R}^+\backslash\{1\}$, $\delta_1, \delta_2 > 0$ and $L : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Suppose that $\{\sigma_k\}_{k\in Z}$ is a sequence of measures on $\mathbf{R}^m$ satisfying*

(i) $\|\sigma_k\| \leq 1$ *for* $k \in \mathbf{Z}$

(ii) $|\widehat{\sigma}_k(\xi)| \leq C[\min\{(\eta^k|L\xi|)^{\delta_1}, (\eta^k|L\xi|)^{-\delta_2}\}]$ *for* $\xi \in \mathbf{R}^n$ *and* $k \in \mathbf{Z}$

(iii) *For some $q > 1$, there exists $A_q > 0$ such that for all $f \in L^q(R^m)$*

$$\|\sigma^* f\|_{L^q(R^m)} = \|\sup_{k\in Z} | \, |\sigma_k| * f| \, \|_{L^q(R^m)} \leq A_q \|f\|_{L^q(R^m)}$$

Then for every $p \in ((2q/q+1), (2q/q-1))$ there exists a constant $C_p = C(p, n, m, \eta, \delta_1, \delta_2)$ such that

$$\left\| \sum_{k\in Z} \sigma_k * f \right\|_{L^p(R^m)} \leq C_p \|f\|_{L^p(R^m)}$$

and

$$\left\| \left(\sum_{k\in Z} |\sigma_k * f|^2 \right)^{1/2} \right\|_{L^p(R^m)} \leq C_p \|f\|_{L^p(R^m)}$$

hold for any $f \in L^p(\mathbf{R}^m)$. The constant C_p is independent of the linear transformation L.

In order to handle the truncation in the phase space, we need the following useful lemma in [**12**].

Lemma 2.6. *For $s \leq d$, let $P : \mathbf{R}^s \to \mathbf{R}^s$ and $Q : \mathbf{R}^d \to \mathbf{R}^d$ be two nonsingular linear transformations and $\phi \in S(\mathbf{R}^s)$. Let $x \in \mathbf{R}^d$ and $r > 0$. Define J and $X_r = X_r(\phi, Q, P)$ by*

$$(Jf)(x) = f(Q^t(P^t \otimes id_{R^{d-s}})(x))$$

and

$$X_r f(x) = J^{-1}((|\Phi_r| \otimes \delta_{R^{d-s}}) * Jf)(x)$$

where/ Q^t *is the transpose of* Q, $\delta_{R^{d-s}}$ *is the Dirac delta on* $\mathbf{R}^{d-s}$ *and* $\Phi \in S(\mathbf{R}^s)$ *satisfies* $\widehat{\Phi} = \phi$. *Let* $X = X\ (\phi, Q, P)$ *be given by*

$$Xf(x) = \sup_{r>0} |X_r f(x)|;$$

then for $1 < p \leq \infty$, *there exists a constant* $C_p = C(p, \phi, s, d)$ *such that*

$$\|Xf\|_p \leq C_p \|f\|_p$$

for $f \in L^p(\mathbf{R}^d)$. *The constant* C_p *is independent of the linear transformations* Q *and* P.

By introducing polar coordinates with respect to ψ, we get that for some continuous function h,

$$\begin{aligned} Tf(x,t) &= \sum_k \int_{I_k} \int_H f(x - r\omega, t - \phi(r)) K(r\omega) r^{n-1} h(\omega)\, d\sigma_H(\omega) b(r)\, dr \\ &= \sum_k \int_{I_k} \int_H f(x - r\omega, t - \phi(r)) K(\omega) h(\omega)\, d\sigma_H(\omega) b(r)\, \frac{dr}{r} \\ &\equiv \sum_k \sigma_k * f(x,t) \end{aligned}$$

where $x \in \mathbf{R}^n$, $t \in \mathbf{R}^n$, $I_k = [2^k, 2^{k+1}]$ and $d\sigma_H$ is the measure on H induced by Lebesgue measure on $\mathbf{R}^n$. And

$$\widehat{\sigma_k}(\xi, \tau) = \int_{I_k} \int_H \exp\{-i\,[r\xi \cdot \omega + \tau\phi(r)]\} K(\omega) h(\omega)\, d\sigma_H(\omega)\, b(r)\, \frac{dr}{r}.$$

To prove our result, we need the mean value zero property on H of K with the weighted measure $hd\sigma_H$, which is similar to the mean value zero property on S^{n-1} of Ω with $d\sigma$.

Lemma 2.7 [**24**]. *Suppose that* ψ *is a* C^2 *convex function and* $\int_{S^{n-1}} \Omega(y)\, d\sigma(y) = 0$. *Then, for some continuous function* h,

$$\int_H K(\omega) h(\omega)\, d\sigma_H(\omega) = 0.$$

We shall denote $d\mu = h d\sigma_H$, $L^p(H) = L^p(H, d\mu)$. Then we note that $\|K\|_{L^p(H)} \simeq \|\Omega\|_{L^p(S^{n-1})}$. By using the preceding lemmas, we shall prove Theorem 3.

3. Proof of Theorem 3. Since $\Omega \in H^1(S^{n-1})$ and $\int \Omega \, d\sigma = 0$, Lemma 2.3 and Lemma 2.7 imply that there are regular atoms $a_s(.)$ on S^{n-1} and $\{c_s\} \subset \mathbf{C}$ such that

$$K(y) = \frac{\Omega(y)}{|y|^n} = \sum_s \frac{c_s a_s(y)}{|y|^n} \equiv \sum_s c_s K_s(y) \quad \text{for} \quad y \in \mathbf{R}^n$$

where a_s is homogeneous of degree 0, $\int_H K_s(\omega)\, d\mu(\omega) = 0$, $\operatorname{supp} a_s \cap H \subset \{y \in H\,; |y - \eta_s| \leq \rho_s\}$ for some $\eta_s \in H$ and $\rho_s > 0$ and $\|a_s\|_{L^\infty(H)} \leq \rho_s^{-n+1}$. And

$$Tf(x,t) \equiv \sum_s c_s T_s f(x,t) \equiv \sum_s \sum_k c_s \sigma_k^s * f(x,t)$$

where

$$T_s f(x,t) = p.v. \int_{R^n} f(x-y, t - \phi(\psi(y)))\, b(\psi(y)) K_s(y)\, dy,$$

and

$$\widehat{\sigma_k^s}(\xi,\tau) = \int_{I_k} \int_H \exp\{-i\,[r\xi \cdot \omega + \tau\phi(r)]\} K_s(\omega)\, d\mu(\omega)\, b(r)\, \frac{dr}{r}.$$

If there is a constant C_p that is independent of s such that $\|T_s f\|_p \leq C_p \|f\|_p$ for any $p > 1$, then $\|Tf\|_p \leq \sum_s |c_s|\, \|T_s f\|_p \leq C_p \sum_s |c_s|\, \|f\|_p \simeq C_p \|\Omega\|_{H_1(S^{n-1})} \|f\|_p$ for all $p > 1$. So, it is enough to show that $\|T_s f\|_p \leq C_p \|f\|_p$ for any $p > 1$ where C_p is independent of s. For $1/4 \leq \rho_s$, $\|T_s f\|_p \leq C_p \|f\|_p$ for all $p > 1$ is induced from K_s is in L^∞ and Theorem 1 which shall be proven in Section 4. So we assume that $0 < \rho_s < 1/4$ for all s.

Since H is of finite type, we may parametrize H in the neighborhood of η_s, $B(\eta_s, \rho_s)$ as

$$\eta_s + (\tilde{z}, g_s(\tilde{z})) \quad \text{for} \quad \tilde{z} \in B(0,\rho_s) \equiv \{\tilde{\omega} \in \mathbf{R}^{n-1}; |\tilde{\omega}| < \rho_s\}$$

where $g_s(0) = \nabla g_s(0) = 0$ and for some integer $a \geq 2$,

$$\sum_{|\alpha|=a} |\partial^\alpha g_s/\partial \tilde{z}^\alpha| \neq 0.$$

Let a_1 be the smallest integer such that $\sum_{|\alpha|=a_1} |\partial^\alpha g_s/\partial \tilde{z}^\alpha| \neq 0$. By induction, we can obtain that $2 \leq a_1 < a_2 < \cdots < a_j < cdots < a_{l-1} \leq 4a_1(n-1) < a_l$ such that $\sum_{|\alpha|=a_k} |\partial^\alpha g_s/\partial \tilde{z}^\alpha| \neq 0$ for all $k = 2, \ldots, l$. And we have that, for $j \leq l$ and $\tilde{z} \in B(0, \rho_s)$,

$$\left| g_s(\tilde{z}) - \sum_{|\alpha|=a_1}^{a_j} \frac{1}{\alpha!} \frac{\partial^\alpha g_s(0)}{\partial \tilde{z}^\alpha} \tilde{z}^\alpha \right| \leq C_j \ {\rho_s}^{a_{j+1}}$$

where C_j is independent of s. For a simple notation, let us denote $\widetilde{K_s}(\tilde{z}) = K_s(\eta_s + (\tilde{z}, g_s(\tilde{z})))$ and $d\tilde{\mu}(\tilde{z}) = d\mu(\eta_s + (\tilde{z}, g_s(\tilde{z})))$ for $\tilde{z} \in B(0, \rho_s)$.

From now on, we will fix $s, z \in \mathbf{Z}$ and make a family of measures $\{\sigma_{K_s,j,k}\}_{j=0,1,\ldots,l}$ on $\mathbf{R}^{n+1}$ to prove Theorem 3 by following Lemma 2.4. For $\xi = (\xi_1, \ldots \xi_n) = (\tilde{\xi}, \xi_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ and $\tau \in \mathbf{R}$, define a family of measures $\{\widehat{\sigma_{K_s,j,k}}\}_{j=0,1,\ldots,l}$ on $\mathbf{R}^{n+1}$ by

$$\begin{aligned}
\widehat{\sigma_{K_s,l,k}}(\xi,\tau) &= \widehat{\sigma_k^s}(\xi,\tau) \\
&= \int_{I_k} \int_H \exp\{-i\,[r\xi \cdot y + \tau\phi(r)]\} K_s(y)\, d\mu(y)\, b(r)\, \frac{dr}{r} \\
&= \int_{I_k} \int_{|\tilde{z}|<\rho_s} \exp\{-i\,[r\xi \cdot \eta_s + r\tilde{\xi} \cdot \tilde{z} + r\xi_n g_s(\tilde{z}) + \tau\phi(r)]\} \\
&\qquad\qquad \times \widetilde{K_s}(\tilde{z}) d\tilde{\mu}(\tilde{z})\, b(r) \frac{dr}{r}.
\end{aligned}$$

For $j = 2, \ldots, l-1$,

$$\begin{aligned}
\widehat{\sigma_{K_s,j,k}}(\xi,\tau) = \int_{I_k} \int_{|\tilde{z}|<\rho_s} &\exp\Big\{ - i \Big[r\xi \cdot \eta_s + r\tilde{\xi} \cdot \tilde{z} \\
&+ r\xi_n \sum_{|\alpha|=a_1}^{a_j} \frac{1}{\alpha!} \frac{\partial^\alpha g_s(0)}{\partial \tilde{z}^\alpha} \tilde{z}^\alpha + \tau\phi(r) \Big] \Big\} \widetilde{K_s}(\tilde{z}) d\tilde{\mu}(\tilde{z})\, b(r)\, \frac{dr}{r}
\end{aligned}$$

$$\widehat{\sigma_{K_s,1,k}}(\xi,\tau) = \int_{I_k}\int_{|\tilde z|<\rho_s} \exp\{-i\,[r\xi\cdot\eta_s + r\tilde\xi\cdot\tilde z + \tau\phi(r)]\} \times \widetilde{K_s}(\tilde z)d\tilde\mu(\tilde z)\, b(r)\,\frac{dr}{r}$$

$$\widehat{\sigma_{K_s,0,k}}(\xi,\tau) = \int_{I_k}\int_{|\tilde z|<\rho_s} \exp\{-i\,[r\xi\cdot\eta_s + \tau\phi(r)]\}\widetilde{K_s}(\tilde z)d\tilde\mu(\tilde z)\, b(r)\,\frac{dr}{r}.$$

To apply Lemma 2.4, we will check the condition (i)–(iv) for $\{\sigma_{K_s,j,k};\ j=0,1,\dots,l$ and $k\in\mathbf{Z}\}$.

Condition (i). By Lemma 2.7, we have that for all $s,k\in\mathbf{Z}$ and $j=1,\dots,l$, $\widehat{\sigma_{K_s,0,k}}=0$ and $\|\sigma_{K_s,j,k}\|\le 1$.

Condition (ii). Since H is of finite type, we have that

$$|\widehat{\sigma_{K_s,l,k}}(\xi,\tau)| \le C(2^k{\rho_s}^{4a_1(n-1)}|\xi_n|)^{-1/4}.$$

For $j=1,\dots,l$, we get that

$$|\widehat{\sigma_{K_s,j,k}}(\xi,\tau)| \le C\left(2^k|\xi_n|{\rho_s}^{a_j}\sum_{|\alpha|=a_j}\frac{1}{\alpha!}\left|\frac{\partial^\alpha g_s(0)}{\partial\tilde z^\alpha}\right|\right)^{-1/4j}, \quad j=2,\dots,l-1,$$

and

$$|\widehat{\sigma_{K_s,1,k}}(\xi,\tau)| \le C(2^k\rho_s|(\xi_1,\dots,\xi_n)|)^{-1/4}$$

by using the estimates on the oscillatory integral related to the hypersurface H, which is proven in Section 5.

Proposition 3.1. *Let $F_m(\tilde z)=\sum_{|\alpha|\le m} b_\alpha\tilde z^\alpha$ for $\tilde z\in\mathbf{R}^{n-1}$. Suppose $\|\widetilde{K_s}\|_{L^\infty(H)}\le\rho^{-\alpha+1}$ and* supp $\widetilde{K_s}\subset B(0,\rho)=\{\tilde y\in\mathbf{R}^{n-1};|\tilde y|<\rho\}$. *Then there exists a constant C such that*

$$\int_{2^k}^{2^{k+1}}\left|\int_{|\tilde z|<\rho}\exp\{-i\,rF_m(\tilde z)\}\tilde K_s(\tilde z)\,d\tilde\mu(\tilde z)\right|\frac{dr}{r} \le C\left(2^k\rho^m\sum_{|\alpha|=m}|b_\alpha|\right)^{-1/4m}.$$

The constant C may depend on m and the dimension n, but it is independent of k, ρ and $\{b_\alpha\}_{|\alpha|\le m}$.

Now let us denote

$$\begin{aligned} G_1(\xi,\tau) &= \rho_s\tilde{\xi} = \rho_s(\xi_1,\dots,\xi_{n-1}), \quad \pi_1 = \frac{1}{4} \\ G_j(\xi,\tau) &= \rho_s^{a_j}\xi_n \sum_{|\alpha|=a_j} \frac{1}{\alpha!}\left|\frac{\partial^\alpha g_s(0)}{\partial \tilde{z}^\alpha}\right|, \quad \pi_j = \frac{1}{4j}\,, \quad 2\le j\le l-1 \\ G_l(\xi,\tau) &= {\rho_s}^{4a_1(n-1)}\xi_n, \quad \pi_l = \frac{1}{4}. \end{aligned}$$

Then we have that, for $1\le j\le N$, $k\in \mathbf{Z}$, $\xi\in\mathbf{R}^n$ and $\tau\in\mathbf{R}$,

$$|\widehat{\sigma_{K_s,j,k}}(\xi,\tau)| \le C\,[2^k|G_j(\xi,\tau)|]^{-\pi_j} \tag{3.9}$$

where the constant C is independent of j and s.

Condition (iii). If $j=l$, Lemma 2.7 implies that

$$\begin{aligned} &|\widehat{\sigma_{K_s,l,k}} - \sigma\widehat{_{K_s,l-1,k}}(\xi,\tau)| \\ &\le \int_{I_k}\Big|\int_{|\tilde{z}|<\rho_s} 1-\exp\Big\{-i\,r\xi_n\Big[g_s(\tilde{z}) - \sum_{|\alpha|=a_1}^{a_{l-1}}\Big(\frac{1}{\alpha!}\,\frac{\partial^\alpha g_s(0)}{\partial\tilde{z}^\alpha}\,\tilde{z}^\alpha\Big)\Big]\Big\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times \tilde{K_s}(\tilde{z})\,d\tilde{\mu}(\tilde{z})|\,b(r)\,\frac{dr}{r} \\ &\le C\,2^k\,|\xi_n|{\rho_s}^{a_l} \le C\,2^k\,|\xi_n|{\rho_s}^{4a_1(n-1)} = C\,|2^k G_l(\xi,\tau)|. \end{aligned}$$

For $j=2,\dots,l-1$,

$$\begin{aligned} &|\widehat{\sigma_{K_s,j,k}} - \sigma\widehat{_{K_s,j-1,k}}(\xi,\tau)| \\ &\le \int_{I_k}\int_{|\tilde{z}|<\rho_s}\Big|1-\exp\Big\{-i\,r\xi_n\sum_{|\alpha|=a_j}\Big(\frac{1}{\alpha!}\,\frac{\partial^\alpha g_s(0)}{\partial\tilde{z}^\alpha}\,\tilde{z}^\alpha\Big)\Big\}\Big| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \times |\tilde{K_s}(\tilde{z})|\,d\tilde{\mu}(\tilde{z})\,b(r)\,\frac{dr}{r} \\ &\le C\,2^k|\xi_n|{\rho_s}^{a_j}\sum_{|\alpha|=a_j}\frac{1}{\alpha!}\left|\frac{\partial^\alpha g_s(0)}{\partial\tilde{z}^\alpha}\right| = C\,|2^k G_j(\xi,\tau)|. \end{aligned}$$

And

$$
\begin{aligned}
|\widehat{\sigma_{K_s,1,k}} - \widehat{\sigma_{K_s,0,k}}(\xi,\tau)| \\
&\leq \int_{I_k}\int_{|\tilde{z}|<\rho_s} |1-\exp\{-i\, r\tilde{\xi}\cdot\tilde{z}\}|\,|\tilde{K}_s(\tilde{z})|\,d\tilde{\mu}(\tilde{z})\,b(r)\,\frac{dr}{r} \\
&\leq C\,2^k|\tilde{\xi}|\rho_s = C\,|2^k G_1(\xi,\tau)|.
\end{aligned}
$$

So we have that, for $j = 1, \dots, l$,

$$
|\widehat{\sigma_{K_s,j,k}} - \sigma\widehat{_{K_s,j-1},_k}(\xi,\tau)| \leq C\,|2^k G_j(\xi,\tau)| \tag{3.10}
$$

Condition (iv). To verify the estimate (iv) in Lemma 2.4, it suffices to establish the L^p boundedness of the operators $\sigma^*_{|K_s|,j}$ defined by

$$
\sigma^*_{|K_s|,j} f(x,t) = \sup_{k\in Z} |(\sigma_{|K_s|,j,k} * f)(x,t)| \quad \text{for} \quad j = 0,\dots,l.
$$

When $j = 0$, the assumption of M_{ϕ_l} and the change of variable derive that

$$
\begin{aligned}
\|\sigma^*_{|K_s|,0} f\|_p^p &= \int_{R^n}\int_R \left|\sup_k \int_{I_k} f(x - r\eta_s, t-\phi(r))\,\frac{dr}{r}\right|^p dt\,dx \\
&\leq C_p \|(i\,d_{n-1}\otimes M_\phi) f\|_p^p \leq C_p\|f\|_p^p.
\end{aligned}
$$

First, we consider the case $j = 1$,

$$
\sigma^*_{|K_s|,1} f(x,t) = \sup_{k\in Z} |(\sigma_{|K_s|,1,k} * f)(x,t)|.
$$

Choose $\zeta \in C_0^\infty$ such that $\zeta(t) \equiv 1$ for $|t| \leq 1/2$ and $\zeta(t) \equiv 0$ for $|t| \geq 1$. For $k \in \mathbf{Z}$, we denote another measure ν_k by

$$
\widehat{\nu_k}\,(\xi,\tau) = \zeta(2^k G_1(\xi,\tau))\widehat{\sigma_{|K_s|,0,k}}\,(\xi,\tau)
$$

for $\xi \in \mathbf{R}^n$ and $\tau \in \mathbf{R}$. Let $\tau_k = \sigma_{|K_s|,1,k} - \nu_k$.

Then by (3.10), (3.9) and $|\widehat{\sigma_{|K_s|,0,k}}| \leq c$, we have

$$
\begin{aligned}
|\widehat{\tau_k}\,(\xi,\tau)| &\leq |\widehat{\sigma_{|K_s|,1,k}}\,(\xi,\tau) - \widehat{\sigma_{|K_s|,0,k}}\,(\xi,\tau)| \\
&\quad + |1-\zeta(2^k G_1(\xi,\tau))|\,|\widehat{\sigma_{|K_s|,0,k}}\,(\xi,\tau)| \\
&\lesssim |2^k G_1(\xi,\tau)|,
\end{aligned}
$$

and

$$|\widehat{\tau_k}(\xi,\tau)| \lesssim |\widehat{\sigma_{|K_s|,1,k}}(\xi,\tau)| + |\zeta(2^k\rho_s(\xi_1,\dots,\xi_{n-1}))| \lesssim |2^k G_1(\xi,\tau)|^{-\pi_1}$$

So, we have that

$$|\widehat{\tau_k}(\xi,\tau)| \lesssim \min\left\{|2^k G_1(\xi,\tau)|, |2^k G_1(\xi,\tau)|^{-\pi_1}\right\}. \tag{3.11}$$

Define that

$$\begin{aligned} \tau^*(f)(x,t) &= \sup_k |(|\tau_k| * f)(x,t)| \\ \nu^*(f)(x,t) &= \sup_k |(|\nu_k| * f)(x,t)| \quad \text{and} \\ g_\tau(f)(x,t) &= \left\{\sum_k [(|\tau_k| * f)(x,t)]^2\right\}^{1/2}. \end{aligned}$$

Then

$$\sigma^*_{|K_s|,1}(f)(x,t) \lesssim g_\tau(f)(x,t) + \nu^*(f)(x,t) \tag{3.12}$$

$$\begin{aligned} \tau^*(f)(x,t) &\lesssim \sigma^*_{|K_s|,1} f(x,t) + \nu^*(f)(x,t) \\ &\lesssim g_\tau(f)(x,t) + 2\nu^*(f)(x,t) \end{aligned} \tag{3.13}$$

By the $L^p(\mathbf{R}^2)$ boundedness of M_ϕ, Lemma 2.6 and its remark, we have

$$\begin{aligned} \|\nu^*(f)\|_{L^p(R^{n+1})} &\lesssim \|\sigma^*_{|K_s|,0} f\|_{L^p(R^{n+1})} \\ &\lesssim \|(i\,d_{n-1} \otimes M_\phi) f\|_{L^p(R^{n+1})} \\ &\lesssim \|f\|_{L^p(R^{n+1})} \end{aligned}$$

Also from (3.11),

$$\|g_\tau(f)\|_{L^2(R^{n+1})} \lesssim \|f\|_{L^2(R^{n+1})}.$$

Thus (3.13) implies that

$$\|\tau^*(f)\|_{L^2(R^{n+1})} \lesssim \|f\|_{L^2(R^{n+1})}.$$

By invoking Lemma 2.5, we obtain

$$\|g_\tau(f)\|_{L^p(R^{n+1})} \lesssim \|f\|_{L^p(R^{n+1})} \quad \text{for} \quad 4/3 < p < 4.$$

Thus by (3.13) again, we obtain

$$\|\tau^*(f)\|_{L^p(R^{n+1})} \lesssim \|f\|_{L^p(R^{n+1})} \quad \text{for} \quad 4/3 < p < 4. \tag{3.14}$$

By using (3.13), (3.14), and a bootstrap argument, we obtain

$$\|g_\tau(f)\|_{L^p(R^{n+1})} \lesssim \|f\|_{L^p(R^{n+1})} \quad \text{for any} \quad 1 < p < \infty.$$

Now from (3.12),

$$\|\sigma^*_{|K_s|,1}(f)\|_{L^p(R^{n+1})} \lesssim \|f\|_{L^p(R^{n+1})} \quad \text{for any} \quad 1 < p < \infty.$$

By the same process as in the case $j = 1$, we get that

$$\|\sigma^*_{|K_s|,j}(f)\|_{L^p(R^{n+1})} \leq C_p \|f\|_{L^p(R^{n+1})} \tag{3.15}$$

for $1 \leq j \leq N$, where C_p is independent of s. Now by (3.9), (3.10), (3.15) and Lemma 2.4, we have

$$\|T_s f\|_{L^p(R^{n+1})} = \left\| \sum_{k \in Z} \sigma_{K_s,l,k} * f \right\|_{L^p(R^{n+1})} \leq C_p \|f\|_{L^p(R^{n+1})}$$
$$\text{for} \quad 1 < p < \infty$$

where the constant C is independent of s. This completes the proof of Theorem 3. □

4. Proof of Theorems 1 and 2.

Proof of Theorem 1. *The case* $\Omega \in L^q(S^{n-1})$. Given a finite measure μ in $\mathbf{R}^{n+1}$, define another measure μ^0 in $\mathbf{R}^n$ as follows: $\mu^0(E) = \mu(E \times \mathbf{R})$ for every Borel subset E of $\mathbf{R}^n$ in terms of Fourier transforms; this means $\widehat{\mu^0}(\xi) = \widehat{\mu}(\xi, 0)$ for all $\xi \in \mathbf{R}^n$. To obtain Theorem 1, we need the following lemmas in [**10**].

Lemma 4.1 [**10**]. *Suppose that the probability measures* $\{\nu_k\}_{k=-\infty}^{\infty}$ *in* $\mathbf{R}^{n+1}$ *satisfy that*

(i) $|\widehat{\nu}_k(\xi,s)-\widehat{\nu}_k(0,s)| \le C|2^k\xi|^{\alpha}$, $|\widehat{\nu}_k(\xi,s)| \le C|2^k\xi|^{-\alpha}$ *for some* $\alpha>0$ *and*

(ii) $M^0 g(t)=\sup_k |\nu_k^0 * g(t)|$ *is a bounded operator in* $L^p(\mathbf{R})$ *for all* $p>1$.

Then $Mf(x,t)=\sup_k|\nu_k * f(x,t)|$ *is also bounded in* $L^p(\mathbf{R}^{n+1})$ *for any* $p>1$.

Lemma 4.2 [**10**]. *Suppose that the measures* $\{\sigma_k\}_{k=-\infty}^{\infty}$ *satisfy that*

(i) $\|\sigma_k\|\le 1$, $\widehat{\sigma}_k(0,s)=0$ *for all* $s\in\mathbf{R}$,

(ii) $|\widehat{\sigma}_k(\xi,s)|\le C\min\{|2^k\xi|,|2^k\xi|^{-1}\}^{\alpha}$ *for some* $\alpha>0$ *and*

(iii) $\sigma_0^* g=\sup_k ||\sigma_k^0| * g|$ *is bounded in* $L^q(\mathbf{R})$ *and*

(iv) $\sigma^* f=\sup_k \||\sigma_k| * f|$ *is bounded in* $L^q(\mathbf{R}^{n+1})$, *for some* $1<q<\infty$.

Then the operators

$$Tf(x,t)=\sum_{k=-\infty}^{\infty}\sigma_k * f(x,t) \quad \textit{and} \quad T^{**}f(x,t)=\sup_k\left|\sum_{j\ge k}\sigma_j * f(x,t)\right|$$

are bounded in $L^p(\mathbf{R}^{n+1})$ *for any* $p\in((2q/q+1),(2q/q-1))$.

Suppose that $\Omega\in L^q(S^{n-1})$ for some $q>1$ and V_ϕ is a bounded operator in $L^p(\mathbf{R})$ for any $1<p<\infty$. Let ϕ satisfy (1.2) and ψ be a *a-convex type* for some $a>0$, which is a homogeneous function of degree 1. We have that

$$\begin{aligned} Tf(x,t) &= \sum_k \sigma_k * f(x,t) \\ &= \sum_k \int_{I_k}\int_H f(x-r\omega,t-\phi(r))K(\omega)\,d\mu(\omega)\,gb(r)\,\frac{dr}{r}. \end{aligned}$$

Now let us begin to prove Theorem 1 by applying Lemma 4.2 to $\{\sigma_k\}_{k\in Z}$. By the cancelation property of K in H (Lemma 2.7), we get that $\widehat{\sigma_k}(0,s)=0$ and

$$\begin{aligned}|\widehat{\sigma_k}(\xi,s)| &\le \int_{I^k}\int_H |e^{-ir\xi\cdot\omega}-1|\,|K(\omega)|\,d\mu(\omega)\,\frac{dr}{r}\\ &\lesssim |2^k\xi|\,\|K\|_{L^1(H)}\lesssim |2^k\xi|.\end{aligned}$$

It follows from the method in [**10**, p. 553] that

$$\begin{aligned}|\widehat{\sigma_k}(\xi,s)| &\le \int_{I^k}\left|\int_H e^{-ir\xi\cdot\omega}K(\omega)d\mu(\omega)\right|\,\frac{dr}{r}\\ &\lesssim \left[\int_{I^k}\left|\int_H e^{-ir\xi\cdot\omega}K(\omega)d\mu(\omega)\right|^2\,\frac{dr}{r}\right]^{1/2}\\ &= \left[\iint\limits_{H\times H}\int_{I^k} e^{-ir\xi\cdot(\omega-\theta)}\,\frac{dr}{r}\,K(\omega)K(\theta)\,d\mu(\omega)d\mu(\theta)\right]^{1/2}.\end{aligned}$$

Let $I(\xi,\omega-\theta)=\int_{I^k}e^{-ir\xi\cdot(\omega-\theta)}\,dr/r$. By Van der Corput's lemma, we know that

$$|I(\xi,\omega-\theta)|\le C\min\{1,|2^k\xi\cdot(\omega-\theta)|^{-1}\}.$$

For any $0<\alpha<1$,

$$|I(\xi,\omega-\theta)|\lesssim|2^k\xi|^{-\alpha}|\xi'\cdot(\omega-\theta)|^{-\alpha},$$

where $\xi'=\xi/|\xi|$. Then we get by Hölder inequality

$$\begin{aligned}&|\widehat{\sigma_k}(\xi,s)|\\ &\quad\le (2^k|\xi|)^{-\alpha/2}\left[\iint\limits_{H\times H}|\xi'\cdot(\omega-\theta)|^{-\alpha}K(\omega)K(\theta)\,d\mu(\omega)\,d\mu(\theta)\right]^{1/2}\\ &\quad\le (2^k|\xi|)^{-\alpha/2}\left[\iint\limits_{H\times H}|\xi'\cdot(\omega-\theta)|^{-\alpha q'}\,d\mu(\omega)\,d\mu(\theta)\right]^{1/2q'}\|K\|_{L^q(H)}.\end{aligned}$$

Let us consider the following integral

$$\iint\limits_{H\times H}|(\theta-\omega)\cdot u|^{-a}\,d\sigma_H(\theta)\,d\sigma_H(\omega)$$

for some $a > 0$ and for all unit vectors u, where $d\sigma_H$ is the Lebesque measure on H. Recall the definitions. For a fixed $\omega \in H$, let u_ω be the outward unit normal to the surface H at ω, and let T_ω be the affine tangent plane to H at ω. Define $\widetilde{B}(\omega, s) = \{y \in H : \operatorname{dist}(y, T_\omega) < s\}$ for any $s > 0$. We investigate the following integral

$$\begin{aligned}\int_H |(\theta-\omega)\cdot u|^{-a}\, d\sigma_H(\theta) &= \int_0^\infty \sigma_H\{\theta \in H : |(\theta-\omega)\cdot u|^{-a} > t\}\, dt \\ &\simeq \int_0^\infty \sigma_H\{\theta \in H : |(\theta-\omega)\cdot u| < s\}\, \frac{ds}{s^{1+a}}.\end{aligned}$$

Here it suffices that we consider the last integral near 0. Since ψ is convex, we have the following inequality, for a sufficiently small δ,

$$\begin{aligned}&\int_0^\delta \sigma_H\{\theta \in H : |(\theta-\omega)\cdot u| < s\}\, \frac{ds}{s^{1+a}} \\ &\qquad \le \int_0^\delta \sigma_H\{\theta \in H : |(\theta-\omega)\cdot u_\omega| < s\}\, \frac{ds}{s^{1+a}} \\ &\qquad \simeq \int_0^\delta \sigma_H(\widetilde{B}(\omega,s))\, \frac{ds}{s^{1+a}} \\ &\qquad \le \sup_{\omega\in H} \int_0^\delta \sigma_H(\widetilde{B}(\omega,s))\, \frac{ds}{s^{1+a}} \lesssim 1\end{aligned}$$

since ψ satisfies the condition (1.8). So we have proven the uniform estimate for any unit vector u

$$\iint_{H\times H} |(\theta-\omega)\cdot u|^{-a}\, d\sigma_H(\theta)\, d\sigma_H(\omega) \lesssim 1.$$

Since h is continuous and H is compact, we have

$$\begin{aligned}&\iint_{H\times H} |(\theta-\omega)\cdot \xi'|^{-\alpha q'}\, d\mu(\omega)\, d\mu(\theta) \\ &= \iint_{H\times H} |(\theta-\omega)\cdot \xi'|^{-\alpha q'} h(\theta)\, d\sigma_H(\theta) h(\omega)\, d\sigma_H(\omega) \\ &\le \|h\|_\infty^2 \iint_{H\times H} |(\theta-\omega)\cdot \xi'|^{-\alpha q'}\, d\sigma_H(\theta)\, d\sigma_H(\omega) \le C \quad \text{for} \quad \alpha q' = a.\end{aligned}$$

So

$$|\widehat{\sigma_k}(\xi,s)| \le C\,|2^k\xi|^{-\alpha/2} \tag{4.16}$$

for $\Omega \in L^q(S^{n-1})$ where $q > 1$ and $\alpha q' = a$. This proves the condition (ii) in Lemma 4.2. Then we can obtain the conditions (i)–(ii) of $\{|\sigma_k|\}_{k\in Z}$ in Lemma 4.1 and the conditions (iii)–(iv) of $\{\sigma_k\}_{k\in Z}$ in Lemma 4.2 by similar methods with those of Section 3. Thus Lemma 4.2 completes the proof of Theorem 1. □

Proof of Theorem 2. *The case* $\Omega \in L\log^+ L(S^{n-1})$. For $\Omega \in L\log^+ L(S^{n-1})$, we begin with an appropriate decomposition of Ω which is described in [**1**]. Let $A_m = \{y \in \mathbf{R}^n : 2^m < |\Omega(y)| \le 2^{m+1}\}$ for $m \in \mathbf{N}$ and $A(\Omega) = \{m \in \mathbf{N} : \sigma(A_m) > 2^{-4m}\}$ where σ is the normalized Lebesque measure on S^{n-1}. For each $m \in A(\Omega)$, let

$$a_m = \|\Omega\|^{-1}_{L^1(A_m\cap S^{n-1})}\left[\Omega\chi_{A_m} - \int_{A_m}\Omega\,d\sigma\right].$$

Then the following hold for all m in $A(\Omega)$:

(i) $\int_{S^{n-1}} a_m\,d\sigma = 0$;

(ii) $\|a_m\|_{L^1(S^{n-1})} \le 2$;

(iii) $\|a_m\|_{L^2(S^{n-1})} \le 2^{2m+2}$.

In addition, we have the following decomposition

$$\Omega = \Omega_0 + \sum_{m\in A(\Omega)} \|\Omega\|_{L^1(A_m\cap S^{n-1})} a_m$$

where $\Omega_0 \in L^2(S^{n-1})$ and satisfies

$$\int_{S^{n-1}} \Omega_0\,d\sigma = 0.$$

Clearly, Ω_0 and a_m are homogeneous of degree 0 for all $m \in A(\Omega)$. This induces the following decomposition of T,

$$T = T_0 + \sum_{m\in A(\Omega)} \|\Omega\|_{L^1(A_m\cap S^{n-1})} T_m$$

where

$$T_0 f(x,t) = p.v. \int_{R^n} f(x-y, t-\phi(\psi(y)))\, \frac{\Omega_0(y)}{|y|^n}\, dy,$$

and

$$T_m f(x,t) = p.v. \int_{R^n} f(x-y, t-\phi(\psi(y)))\, \frac{a_m(y)}{|y|^n}\, dy,$$

for any $m \in A(\Omega)$ and $(x,t) \in \mathbf{R}^n \times \mathbf{R}$. See [**1**] for the details. Since $\Omega_0 \in L^2(S^{n-1})$, we know that $\|T_0 f\|_p \le C_p \|f\|_p$ for all $1 < p < \infty$ by Theorem 1. If we can show that $\|T_m f\|_p \le C_p\, m\, \|f\|_p$ for each $m \in A(\Omega)$ and all $1 < p < \infty$, then

$$\begin{aligned}\|Tf\|_p &\le C_p \Big[1 + \sum_{m \in A(\Omega)} m\, \|\Omega\|_{L^1(A_m \cap S^{n-1})}\Big]\, \|f\|_p \\ &\le C_p\, [1 + \|\Omega\|_{L \log^+ L(S^{n-1})}]\, \|f\|_p.\end{aligned}$$

To prove Theorem 2, it suffices to show that $\|T_m f\|_p \le C_p\, m\, \|f\|_p$ for each $m \in A(\Omega)$ and all $1 < p < \infty$. Let us fix $m \in A(\Omega)$. We shall then proceed to further decompose T_m as $T_m = \sum_{k \in Z} T_{m,k}$, that is,

$$\begin{aligned}T_m f(x,t) &= \sum_{k \in Z} \int_{2^{mk}}^{2^{m(k+1)}} \int_H f(x - r\omega, t - \phi(r^l))\, \frac{a_m(y)}{|y|^n}\, d\mu(\omega)\, \frac{dr}{r} \\ &\equiv \sum_{k \in Z} \sigma_k^m * f(x,t) \equiv \sum_{k \in Z} T_{m,k} f(x,t).\end{aligned}$$

By invoking Lemma 2.7, we know that a_m satisfies the cancelation property on H with $d\mu$, i.e.,

$$\int_H \frac{a_m(y)}{|y|^n}\, d\mu(y) = 0. \tag{4.17}$$

From now on, we shall follow the proof of Theorem 1. And so we need the following lemma in [**1**] modified from Lemma 2.5.

Lemma 4.3. *Let $s, d \in \mathbf{N}$, $\eta > 2$, $\delta_1, \delta_2 > 0$, $B > 0$ and $L : \mathbf{R}^s \to \mathbf{R}^d$ be a linear transformation. Suppose that $\{\sigma_k\}_{k \in Z}$ is a sequence of measures on $\mathbf{R}^d$ satisfying*

(i) $\|\sigma_k\| \lesssim B$ *for* $k \in \mathbf{Z}$

(ii) $|\widehat{\sigma}_k(\xi)| \lesssim B[\min\{(\eta^k|L\xi|)^{\delta_1}, (\eta^k|L\xi|)^{-\delta_2}\}]$ *for* $\xi \in \mathbf{R}^s$ *and* $k \in \mathbf{Z}$

(iii) *For some* $q > 1$*, there exists* $A_q > 0$ *such that*

$$\|\sigma^* f\|_{L^q(R^d)} = \| \sup_{k \in Z} | \, |\sigma_k| * f| \|_{L^q(R^d)} \lesssim B\|f\|_{L^q(R^m)}$$

Then, for every $p \in ((2q/q+1), (2q/q-1))$*, there exists a constant* $C_p = C(p, s, d, \eta, \delta_1, \delta_2)$ *such that*

$$\Big\| \sum_{k \in Z} \sigma_k * f \Big\|_{L^p(R^d)} \leq C_p \, B\|f\|_{L^p(R^d)}$$
$$\Big\| \Big(\sum_{k \in Z} |\sigma_k * f|^2 \Big)^{1/2} \Big\|_{L^p(R^d)} \leq C_p \, B\|f\|_{L^p(R^d)}$$

hold for any $f \in L^p(\mathbf{R}^d)$*. The constant* C_p *is independent of* B *and the linear transformation* L*.*

So we apply Lemma 4.3 to σ_k^m for a fixed $m \in A(\Omega)$. Firstly, we shall estimate $\{|\widehat{\sigma_k^m}(\xi, s)| : k \in \mathbf{Z}\}$ for $(\xi, s) \in \mathbf{R}^n \times \mathbf{R}$. By the equations (4.17), we get that

$$\begin{aligned} |\widehat{\sigma_k^m}(\xi, s)| &\leq \int_{2^{mk}}^{2^{m(k+1)}} \int_H |e^{-i(r\xi \cdot y)} - 1| \frac{|a_m(y)|}{|y|^n} \, d\mu(y) \, \frac{dr}{r} \\ &\lesssim m(2^{mk}|\xi|)\|a_m\|_{L^1(S^{n-1})} \end{aligned}$$

By the same argument as the equation (4.16), we have that $|\widehat{\sigma_k^m}(\xi, s)| \lesssim (2^{mk}|\xi|)^{-\alpha_1}\|a_m\|_{L^2(S^{n-1})}$ for some $\alpha_1 > 0$. Interpolating with $|\widehat{\sigma_k^m}(\xi, s)| \lesssim m\,\|a_m\|_{L^1(S^{n-1})}$, we obtain that

$$|\widehat{\sigma_k^m}(\xi, s)| \lesssim m\,(2^{mk}|\xi|)^{-\alpha/m} \quad \text{for some} \quad \alpha > 0.$$

So we prove this estimate

$$|\widehat{\sigma_k^m}(\xi, s)| \lesssim \min \left[m\,(2^{mk}|\xi|)^{\alpha/m}, m\,(2^{mk}|\xi|)^{-\alpha/m} \right] \tag{4.18}$$

In order to apply Lemma 4.3, we must show L^p-boundedness of $(\sigma^m)^*$ given by

$$(\sigma^m)^*(f)(x,t) = \sup_{k\in Z} |\, |\sigma_k^m| * f(x,t)| \quad \text{for} \quad (x,t) \in \mathbf{R}^n \times \mathbf{R}.$$

Since the proof is similar with those of $\sigma^*_{|K|,1}$ in Theorem 3, we shall give a quick proof. Fix $m \in A(\Omega)$ and choose $\zeta \in C_0^\infty$ such that $\zeta(t) \equiv 1$ for $|t| \le 1/2$ and $\zeta(t) \equiv 0$ for $|t| \ge 1$. For $k \in \mathbf{Z}$, we denote other measures ν_k^m and τ_k^m by

$$\nu_k^m = |(\sigma_k^m)^0| * (\widehat{\zeta}_{2^{mk}} \otimes \delta_R) \quad \text{and} \quad \tau_k^m = |\sigma_k^m| - \nu_k^m$$

where $\widehat{(\sigma_k^m)^0}(\xi,s) = \widehat{\sigma_k^m}(0,s)$ for any $\xi \in \mathbf{R}^n$ and $s \in \mathbf{R}$. Here we note that

$$||\widehat{(\sigma_k^m)^0}|| \lesssim m \quad \text{and} \quad ||\widehat{\sigma_k^m}|(\xi,s)| \lesssim m\,(2^{mk}|\xi|)^{-\alpha/m}.$$

In addition, by (4.18), we have

$$\begin{aligned}
|\widehat{\tau_k^m}(\xi,s)| &\le |\,|\widehat{\sigma_k^m}|(\xi,s)| + |\,|\widehat{(\sigma_k^m)^0}|(\xi,s)|\,|\widehat{\zeta}(2^{mk}\xi)| \\
&\lesssim m(2^{mk}|\xi|)^{-\alpha/m} \\
|\widehat{\tau_k^m}(\xi,s)| &\le |(\widehat{|\sigma_k^m|} - \widehat{|(\sigma_k^m)^0|})(\xi,s)| + |\,|\widehat{(\sigma_k^m)^0}|(\xi,s)|\,|1-\widehat{\zeta}(2^{mk}\xi)| \\
&\lesssim m(2^{mk}|\xi|).
\end{aligned}$$

So, we have that

$$|\widehat{\tau_k^m}(\xi,s)| \lesssim \min\{m(2^{mk}|\xi|)^{-\alpha/m}, m(2^{mk}|\xi|)^{\alpha/m}\}. \tag{4.19}$$

Define that

$$\begin{aligned}
\tau_m^*(f) &= \sup_k |(|\tau_k^m| * f)|, \\
\nu_m^*(f) &= \sup_k |(|\nu_k^m| * f)|, \\
g_\tau(f) &= \left\{ \sum_k [(|\tau_k^m| * f)]^2 \right\}^{1/2}.
\end{aligned}$$

Then

$$(\sigma^m)^*(f) \lesssim g_\tau(f) + \nu_m^*(f),\ \tau_m^*(f) \lesssim g_\tau(f) + 2\nu_m^*(f) \tag{4.20}$$

By the assumption of V_ϕ^m, Lemma 2.6 and its remark for $1 < p < \infty$, we have

$$\|\tau_m^*(f)\|_p \lesssim \Big\| \sup_k |(|(\sigma_k^m)^0| * f)| \Big\|_p \lesssim \|(\mathrm{id}_{R^n} \otimes V_\phi^m) f\|_p \le C_p m \|f\|_p$$

for any $1 < p \le \infty$ where C_p is independent of m. Also from (4.19) and (4.20) $\|g_\tau(f)\|_2 \lesssim \|f\|_2$ and $\|\tau_m^*(f)\|_2 \lesssim \|f\|_2$. And Lemma 4.3 implies

$$\|g_\tau(f)\|_p \lesssim \|f\|_p \quad \text{and} \quad \|(\sigma^m)^*\|_p \lesssim \|f\|_p \quad \text{for any} \quad 1 < p < \infty$$

by using (4.20) and the bootstrap argument. Additionally, by the equations (4.18) and Lemma 4.3, we have that $\|T_m f\|_p = \| \sum_k \sigma_k^m f\|_p \le C_p \, m \, \|f\|_p$ for all $1 < p < \infty$ where C_p is independent of m. Therefore, $\|Tf\|_p \le C_p \|f\|_p$ for all $1 < p < \infty$. □

5. Estimate of oscillatory integral on H. Let us introduce the following lemma.

Lemma 5.1 [**19**]. *If $P(y) = \sum_{|\alpha| \le m} a_\alpha y^\alpha$ is a polynomial of degree m in $\mathbf{R}^n$ and $\varepsilon < 1/m$, then*

$$\int_{|y| \le 1} |P(y)|^{-\varepsilon} \, dy \le A_\varepsilon \bigg(\sum_{|\alpha| \le m} |a_\alpha| \bigg)^{-\varepsilon}.$$

The constant A_ε may depend on ε, m and the dimension n, but it is independent of the coefficients $\{a_\alpha\}$.

Define the operator $S_m = S_{k,m}$ by

$$S_m f(r) = \phi(r/2^k) \int_{R^{n-1}} \exp\{-i \, r F_m(\tilde{z})\} \, \phi(|\tilde{z}/\rho|) f(\tilde{z}\,) \, d\tilde{\mu}(\tilde{z})$$

where $\phi \in C_0^\infty(\mathbf{R})$, $\phi(r) = 1$ for $|r| \le 1/2$ and $\phi(r) = 0$ for $|r| \ge 1$. Then

$$S_m^* S_m f(\tilde{y}) = \int_R L_m(\tilde{y}, \tilde{z}) f(\tilde{z}) \, d\tilde{\mu}(\tilde{z})$$

where

$$\begin{aligned} L_m(\tilde{y}, \tilde{z}) &= 2^k \phi(|\tilde{z}/\rho|) \phi\,(|\tilde{y}/\rho|) \\ &\quad \times \int_R \exp\big\{ - i \, r 2^k (F_m(\tilde{y}) - F_m(\tilde{z})) \big\} \, [\phi(r)]^2 \, dr. \end{aligned}$$

By using Van der Corput's lemma, we get

$$|L_m(\tilde{y},\tilde{z})| \lesssim 2^k \left|\phi(|\tilde{z}/\rho|)\phi\left(|\tilde{y}/\rho|\right)\right| \left[2^k(F_m(\tilde{y}) - F_m(\tilde{z}))\right]^{-1}.$$

Since $|L_m(\tilde{y},\tilde{z})| \lesssim 2^k \left|\phi(|\tilde{z}/\rho|)\phi\left(|\tilde{y}/\rho|\right)\right|$, we get

$$|L_m(\tilde{y},\tilde{z})| \lesssim 2^k \left|\phi(|\tilde{z}/\rho|)\phi\left(|\tilde{y}/\rho|\right)\right| \left[2^k(F_m(\tilde{y}) - F_m(\tilde{z}))\right]^{-1/2m}.$$

It follows from Lemma 5.1 that

$$\begin{aligned}
&\sup_{\tilde{z}\in R^{n-1}} \int_{R^{n-1}} |L_m(\tilde{y},\tilde{z})|\, d\tilde{\mu}(\tilde{y}) \\
&\qquad \lesssim \|h\|_\infty \sup_{\tilde{z}\in R^{n-1}} \int_{R^{n-1}} |\rho^{n-1} L_m(\rho\tilde{y},\rho\tilde{z})|\, d\tilde{y} \\
&\qquad \lesssim \sup_{\tilde{z}\in R^{n-1}} \int_{|\tilde{z}|<1} 2^k \rho^{n-1} \left[2^k(F_m(\rho\tilde{y}) - F_m(\rho\tilde{z}))\right]^{-1/2m} d\tilde{y} \\
&\qquad \lesssim 2^k \rho^{n-1} \left[2^k \rho^m \sum_{|\alpha|=m} |b_\alpha|\right]^{-1/2m}
\end{aligned}$$

and similarly

$$\sup_{\tilde{y}\in R^{n-1}} \int_{R^{n-1}} |L_m(\tilde{y},\tilde{z})|\, d\tilde{\mu}(\tilde{z}) \lesssim 2^k \rho^{n-1} \left[2^k \rho^m \sum_{|\alpha|=m} |b_\alpha|\right]^{-1/2m}.$$

Hence

$$\|S_m\|_{2,2} \le \|S_m^* S_m\|_{2,2}^{1/2} \lesssim 2^{k/2} \rho^{(n-1)/2} \left[2^k \rho^m \sum_{|\alpha|=m} |b_\alpha|\right]^{-1/4m}.$$

By interpolation with $\|S_m\|_{1,\infty} \le C$, we get

$$\|S_m\|_{p,p'} \lesssim 2^{k/p'} \rho^{(n-1)/p'} \left[2^k \rho^m \sum_{|\alpha|=m} |b_\alpha|\right]^{-1/(2mp')} \quad \text{for } 1 \le p \le 2.$$

Now we can estimate the given oscillatory integral by the Hölder inequality and the result above.

$$
\begin{aligned}
&\int_{I_k} \left| \int_{|\tilde{z}|<\rho} e^{-i\,rF_m(\tilde{z})} \tilde{K}_s(\tilde{z})\, d\tilde{\mu}(\tilde{z}) \right| \frac{dr}{r} \\
&\quad \leq \left(\int_{I_k} \left| \int_{|\tilde{z}|<\rho} e^{-i\,rF_m(\tilde{z})} \tilde{K}_s(\tilde{z})\, d\tilde{\mu}(\tilde{z}) \right|^{p'} \frac{dr}{r} \right)^{1/p'} \\
&\quad \lesssim 2^{-k/p'} \|S_m\|_{p,p'} \|K\|_{L^p(H)} \\
&\quad \lesssim 2^{-k/p'} 2^{k/p'} \rho^{(n-1)/p'} \left(2^k \rho^m \sum_{|\alpha|=m} |b_\alpha| \right)^{-1/(2mp')} \rho^{-(n-1)/p'} \\
&\quad \simeq \left(2^k \rho^m \sum_{|\alpha|=m} |b_\alpha| \right)^{-1/(2mp')}
\end{aligned}
$$

So, choosing $p' = 2$ completes the proof of Proposition 3.1.

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