ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 2, 2006

A NOTE ON RIESZ BASES OF EIGENVECTORS FOR A CLASS OF NONANALYTIC OPERATOR FUNCTIONS

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ABSTRACT. Riesz basis properties for a class of self-adjoint and continuous operator functions are studied. A new approach based on the spectral distribution function is presented.

1. Introduction. There is a hypothesis in the spectral theory of operator functions in the following form.

If $L(\alpha)$ is an operator function of the class $C([a,b],S(H))$ such that $L(a) \ll 0$, $L(b) \gg 0$, for all $x \in H \setminus \{0\}$, the function $(L(\alpha)x,x)$ has exactly one zero in (a,b) and $\pi(L) = \{\gamma\} \in (a,b)$, then the eigenvectors of $L(\alpha)$, corresponding to eigenvalues in (a,b) form a Riesz basis for the Hilbert space H or they are complete in H.

Here by $C([a,b],S(H))$ we denote the class of self-adjoint and continuous operator functions defined on the interval $[a,b]$, and $\pi(L)$ is the set of the limit spectrum, i.e.,

$$\pi(L) = \{\lambda \in (a,b) \mid \exists x_n,\ \|x_n\| = 1,\ x_n \to 0\,(\text{weakly}),\ L(\lambda)x_n \to 0\}.$$

The spectrum $\sigma(L)$, the point spectrum or the set of eigenvalues $\sigma_e(L)$ of L are subsets of $[a,b]$ defined as follows: $\lambda \in \sigma(L)$ if $0 \in \sigma(L(\lambda))$ and $\lambda \in \sigma_e(L)$ if $0 \in \sigma_e(L(\lambda))$. A nonzero vector x from the kernel $\ker L(\lambda)$ for $\lambda \in \sigma_e(L)$ is called an eigenvector of L corresponding to λ.

This problem in the finite-dimensional case for the class $C^1([a,b], S(H))$—the class of self-adjoint and continuously differentiable operator functions, was solved in [**1**]. For analytic operator functions the

2000 AMS *Mathematics Subject Classification.* Primary 47A56, 47A75.
Key words and phrases. Riesz basis, operator functions, spectral distribution function, eigenvectors.
Research of the first author supported by NATO B2 program of TUBITAK.
Received by the editors on January 22, 2003, and in revised form on September 5, 2005.

result was given in [**7**, Theorem 30.12], which follows from a representation of the form

$$L(\alpha) = B(\alpha)(\alpha I - Z),$$

where $B(\alpha)$ is invertible on $[a, b]$, $\sigma(Z) \subset (a, b)$ and Z is similar to a self-adjoint operator. A similar result for a class of nonanalytic operator functions was obtained by Markus and Matsaev, see [**8, 9**]. Namely, they proved the following.

Theorem 1 [**9**]. *Let* $L \in C^2([a, b], S(H))$ *and the conditions*:

i) $L(a) \ll 0$, $L(b) \gg 0$,

ii) $\int_0^{t_0} w(t, L'')/t\, dt < +\infty$ *for sufficiently small* t_0, *where* $w(t, L'')$ *is the modulus of continuity for* L'',

iii) *the operator function* L *satisfies the regularity condition, i.e., there exist positive numbers* δ *and* ϵ *such that for every* $\alpha \in [a, b]$ *and* $x \in H$, $\|x\| = 1$, $|(L(\alpha)x, x)| < \varepsilon \Rightarrow (L'(\alpha)x, x) > \delta$

are satisfied. Then L *admits a factorization of the form*

$L(\alpha) = B(\alpha)(\alpha I - Z)$, *where* $B(\alpha)$ *is a continuous and invertible operator function on* $[a, b]$, $\sigma(Z) \subset (a, b)$, Z *is bounded and is similar to a self-adjoint operator.*

Now it follows from the representation $L(\alpha) = B(\alpha)(\alpha I - Z)$ that the hypothesis is true for the class given in this theorem. A similar factorization theorem under weaker conditions was given in the paper of Azizov, Dijksma and Sukhocheva [**2**]. This paper also contains some sufficient conditions on L under which the closed linear span of all eigenvectors, corresponding to eigenvalues from $[a, b]$, has a Riesz basis consisting of eigenvectors of L, see also Matsaev and Spigel [**10**].

In this paper we study this problem and prove that the assertion holds in a dense subspace of the space $C([a, b], S(H))$ if the number of eigenvalues either at the right or at the left of $\{\gamma\}$ is finite. We denote this class by $C_\gamma^F([a, b], S(H))$.

2. Main results. In what follows we suppose

Assumption 1. *$L(a) \ll 0$, $L(b) \gg 0$, for all $x \in H\backslash\{0\}$ the function $(L(\alpha)x, x)$ has exactly one zero in (a, b) and $\pi(L) = \{\gamma\} \in (a, b)$.*

Note that all operators in this paper are bounded and we say that an operator is invertible if it is boundedly invertible. The main result given in this paper is connected with the notion of approximate Riesz basis in the following sense.

Definition 1. Let $L \in C([a, b], S(H))$. We say that the eigenvectors of the operator function L form an approximate Riesz basis if there is a sequence $\{L_n\}_{n=1}^{\infty} \in C([a, b], S(H))$ of operator functions such that $L_n \Rightarrow L$(uniformly) as $n \to \infty$, and the eigenvectors of L_n for all n form a Riesz basis for H.

Recall that a collection $\{f_\alpha\}$, $\alpha \in I$, of elements of H is called a Riesz basis of H if there is an invertible, bounded operator G such that $\{Gf_\alpha\}$, $\alpha \in I$, is an orthonormal basis of H, see [**4, 7**].

Denote by $PW([a, b], S(H))$ the class of piece-wise linear and continuous operator functions [**5, 6**] and define the subspace $PW_\gamma^F([a, b], S(H))$ by the same way as the subspace $C_\gamma^F([a, b], S(H))$. We use mainly an approximation method given in [**5, 6**]. According to this method a continuous operator function, satisfying the Rayleigh system axioms, see [**1, 6**], can be approximated by piece-wise linear ones from the same class. Note that operator functions studied in this note form a subclass of Rayleigh systems. For this reason we consider first the basis problem for the piece-wise linear operator functions.

Theorem 2. *Let L be an operator function from $PW([a, b], S(H))$ satisfying Assumption* 1*. If L is differentiable at the point γ or $L \in PW_\gamma^F([a, b], S(H))$, then the eigenvectors corresponding to eigenvalues in $[a, b]$ form a Riesz basis for H.*

Proof. By the conditions of the theorem $L \in PW([a, b], S(H))$. Consequently, there are a finite number (denoted by k) of points of discontinuities of derivative $L'(\alpha)$. We use the principle of induction and therefore at the beginning we prove this theorem for $k = 1$, i.e., for operator-valued functions of the form

$$L(\alpha) = \begin{cases} \alpha B_+ - A & \alpha \geq 0, \\ \alpha B_- - A & \alpha \leq 0 \end{cases} \qquad \alpha \in [a, b].$$

Here by shifting the argument we assume that the point of discontinuity of $L'(\alpha)$ is 0. The proof is completely based on a variational approach. Namely, we use variational principles for the spectral distribution function $N(\lambda, L)$ to prove this theorem. $N(\lambda, L)$ is the number of eigenvalues of L strictly larger than λ. For pencils of the form $L(\alpha) = \alpha B - A$, where $B > 0$ or $B < 0$ (the definite case) we use the classical variational principles for $N(\lambda, L)$. If neither $B > 0$ nor $B < 0$ is satisfied, then it is an indefinite case and we use a variational principle for pencils in the indefinite case.

Let us first prove the theorem under the definiteness conditions $B_\pm \gg 0$. Since $\pi(L) = \{\gamma\}$ the spectrum $\sigma(L) \setminus \{\gamma\}$ is discrete, see [**6**]. We consider two cases: $\gamma = 0$ and $\gamma \neq 0$. Now suppose $\gamma = 0$ and $L \in PW_\gamma^F([a, b], S(H))$. Then either $N(0, L) < +\infty$ or there is a finite number of negative eigenvalues. We suppose $N(0, L) = n < +\infty$. Let us construct a self-adjoint operator function of the form

$$F(\alpha) = \begin{cases} \alpha I - B_+^{-1/2} A B_+^{-1/2} & \alpha \geq 0, \\ \alpha I - B_-^{-1/2} A B_-^{-1/2} & \alpha \leq 0. \end{cases}$$

We denote by $M_+(L)$ ($M_-(L)$) the closed linear span of eigenvectors, corresponding to positive (nonpositive) eigenvalues of the operator function $L(\alpha)$. Let $H_+(U)$ ($H_-(U)$) be the closed linear span of the eigenvectors, corresponding to positive (nonpositive) eigenvalues of an operator U. We have

$$\sigma(F) = \sigma(L), \qquad \sigma_e(F) = \sigma_e(L),$$

and

$$B_+^{1/2} : M_+(L) \longrightarrow H_+(T); \qquad B_-^{1/2} : M_-(L) \longrightarrow H_-(S), \tag{1}$$

where $T = B_+^{-1/2} A B_+^{-1/2}$ and $S = B_-^{-1/2} A B_-^{-1/2}$. Denoting by $L_\pm(\alpha) = \alpha B_\pm - A$ we obtain, see [**1, 5**], the following equality for the spectral distribution function of the operator functions $L_{\pm}(\alpha)$ which plays the key role in the proof of this case.

$$\begin{aligned}\dim H_+(B_-^{-1}A) &= N(0, L_-) \\ &= \max\dim\left\{E \;\middle|\; \frac{(Au,u)}{(B_-u,u)} > 0,\, u \in E\setminus\{0\}\right\} \\ &= \max\dim\{E \mid (Au,u) > 0\} \\ &= \max\dim\left\{E \;\middle|\; \frac{(Au,u)}{(B_+u,u)} > 0,\, u \in E\setminus\{0\}\right\} \\ &= N(0, L_+) = n.\end{aligned} \tag{2}$$

We can write

$$H_+(S) = B_-^{1/2} H_+(B_-^{-1}A). \tag{3}$$

It follows from (2) and (3) that $\dim H_+(S) = n$. Consequently, S is a compact and self-adjoint operator. Thus $H = H_-(S) \oplus H_+(S)$ and $\dim H_-^{\perp}(S) = \dim H_+(S) = n$. Because of the invertibility of the operator $B_-^{-1/2}$ we have

$$\dim\left[B_-^{-1/2}[H_-(S)]\right]^{\perp} = \dim[H_-(S)]^{\perp} = n. \tag{4}$$

Now, writing the formula $H = M_-(L) \oplus M_-(L)^{\perp}$ in the form

$$H = \left[B_-^{-1/2}[H_-(S)]\right] \oplus \left[B_-^{-1/2}[H_-(S)]\right]^{\perp}$$

we obtain by (4) $\dim M_-(L)^{\perp} = n$. On the other hand, using the condition $N(0, L) = n < +\infty$, we have $\dim M_+(L) = n$ and $M_-(L) \cap M_+(L) = \{0\}$, see [**7**, Theorem 32.8]. Therefore,

$$H = M_-(L) \dotplus M_+(L). \tag{5}$$

Now it follows from (1) and (5), see [**4**], that the eigenvectors of the operator-valued function $L(\alpha)$ form a Riesz basis of H.

If L is differentiable at the point γ, then $\gamma \neq 0$ and, under the conditions of the theorem, the spectrum of operator-valued function

$L(\alpha)$ in (a, b) consists of the point γ and at most countable number of eigenvalues of finite multiplicity, see [**6**]. If the set is infinite, then it converges to γ. Thus, here we have the same situation as in the case $\gamma = 0$ and the Riesz basis property in the case follows from the above proved case.

Now let us consider the indefinite case. It means that we do not suppose that the conditions $B_\pm \gg 0$ are satisfied. We have

$$L(\alpha) = \begin{cases} \alpha B_+ - A & \alpha \geq 0, \\ \alpha B_- - A & \alpha \leq 0 \end{cases} \qquad \alpha \in [a, b].$$

Here $a < 0$, $b > 0$ and, by Assumption 1, $L(a) \ll 0$, $L(b) \gg 0$. Recall that we have assumed that $N(0, L) < \infty$.

i) if $A \leq 0$, then $\sigma_e^+(L) := \sigma_e(L) \cap (0, +\infty) = \varnothing$. Therefore, $\sigma_e(L) = \sigma_e(L_-)$ and the pencil $L_-(\alpha) = \alpha B_- - A$ satisfies Assumption 1 on $[a, b]$. Thus the eigenvectors of $L_-(\alpha)$ and consequently the eigenvectors of $L(\alpha)$ form a Riesz basis of H.

ii) if $A \geq 0$, then $\sigma_e^-(L) := (-\infty, 0) \cap \sigma_e(L) = \varnothing$, $\sigma_e(L) = \sigma_e(L_+)$, and we have the same situation as in the case i) if we consider L_+ instead of L_-. In this case, since eigenvectors of L (or L_+) form a basis of H, it follows from the condition $N(0, L) < +\infty$ that $\gamma = 0$ is an eigenvalue of infinite multiplicity.

iii) Let A be a self-adjoint operator. Define the cones

$$C_-^A = \{x \mid (Ax, x) < 0\} \quad \text{and} \quad C_+^A = \{x \mid (Ax, x) > 0\}.$$

It remains to consider the case $C_\pm^A \neq \varnothing$. Now using variational principles in the indefinite case, we have

$$N(0, L) = \max\dim \left\{E \mid E \subset C_+^A\right\}$$

and

$$N(0, L_-) = \max\dim \left\{E \mid E \subset C_+^A \cap C_+^{B_-}\right\}.$$

It follows from these formulae that

$$N(0, L_-) \leq N(0, L) = n. \tag{6}$$

Hence the spectrum of the linear pencil $L_-(\alpha) = \alpha B_- - A$ is discrete and eigenvectors form a Riesz basis of H. We can write

$$H = M_-(L_-) \dot{+} M_+(L_-). \tag{7}$$

On the other hand,

$$M_+(L) \cap M_-(L_-) = \{0\}. \tag{8}$$

Now we obtain from (6), (7) and (8) that

$$N(0, L_-) = N(0, L) = n$$

and

$$H = M_-(L_-) \dot{+} M_+(L) = M_-(L) \dot{+} M_+(L).$$

The general case. We turn now to the inductive step from $n-1$ to n, so for $k = n-1$ we assume that all eigenvectors of L, corresponding to eigenvalues from $[a, b]$, form a Riesz basis of H. Let $\{t_k\}_1^n$ be points of discontinuity of the derivative of L. Setting $t_0 = a$ and $t_{n+1} = b$ the piece-wise linear pencil L (it is denoted here by L_n to indicate that L' has n points of discontinuity) can be written in the form

$$L_n(\alpha) = \sum_{k=0}^{n} \left[\frac{\alpha - t_k}{t_{k+1} - t_k} \, (A_{k+1} - A_k) + A_k \right] \times \chi_{[t_k, t_{k+1})},$$

where $\chi_{[t_k, t_{k+1})}$ is the characteristic function of the interval $[t_k, t_{k+1})$ and $\{A_k\}_0^{n+1}$ are bounded operators. By shifting the argument we may assume that $t_n = 0$. Despite the fact that the operator function $L_n(\alpha)$ satisfies Assumption 1, in general the function $L_{n-1}(\alpha)$ (it is linearly extended on $[t_n, b]$) does not satisfy it. But it can be extended to $[a, b]$, satisfying the main assumption and having the same number of points of discontinuities of the derivative and such that

$$\sigma_e^-(L_n) = \sigma_e^-(\tilde{L}_{n-1}), \qquad N(0, \tilde{L}_{n-1}) = N(0, L_n). \tag{9}$$

Now we have

$$M_-(\widetilde{L}_{n-1}) = M_-(L_n)$$

and by the assumption for $k = n - 1$

$$H = M_-(\widetilde{L}_{n-1}) \dot{+} M_+(\widetilde{L}_{n-1}). \tag{10}$$

We obtain from (9) that

$$\dim M_+(\widetilde{L}_{n-1}) = N(0, \widetilde{L}_{n-1}) = N(0, L_n) = \dim M_+(L_n) = n.$$

The needed result follows by taking into account (10) and the fact that $M_-(L_n) \cap M_+(L_n) = \{0\}$. $\square$

Now we are ready to prove an approximate basis property in the class $C_\gamma^F([a,b],\, S(H))$.

Theorem 3. *If an operator-valued function L satisfies Assumption* 1 *and $L \in C_\gamma^F([a,b],\, S(H))$, then the eigenvectors of L form an approximate Riesz basis.*

Proof. Since $L \in C([a,b], S(H))$ and $\pi(L) = \{\gamma\}$, there exists a sequence $\{L_n\}_{n=1}^\infty$ such that:

a) $L_n \rightrightarrows L$ and $P_n \rightrightarrows P$ (uniformly). Here by $P_n(x)$ and $P(x)$ are denoted roots of the equations $(L_n(\alpha)x, x) = 0$ and $(L(\alpha)x, x) = 0$, respectively.

b) $L_n \in PW([a,b], S(H))$, having the properties given in Assumption 1, see [**5**, Theorem 2.2 and Lemma 3.2].

Moreover, it follows from the condition $L \in C_\gamma^F([a,b], S(H))$ that $L_n \in PW_\gamma^F([a,b],\, S(H))$, i.e., $N(\gamma, L_n) < \infty$. Indeed, $N(\gamma, L) < \infty$ and $\|P_n - P\| < \varepsilon$ for all $\varepsilon > 0$ and sufficiently large n. Then, denoting

$$\lambda_1 := \min_{\sigma_e(L) \cap [\gamma, b]} \lambda,$$

we have $\lambda_1 > \gamma$ and

$$N(\theta, L) = N(\theta - 2\varepsilon, L), \tag{11}$$

for $\theta \in (\gamma, \lambda_1)$ and sufficiently small ε. On the other hand, using the following formula for the spectral distribution function, see [**1**],

$$N(\theta, L) = \max \dim \Big\{ E \mid P(x) > \theta, x \in E \setminus \{0\} \Big\}$$

and choosing ε from the inequality $\|P_n - P\| < \varepsilon$, we obtain from (11) that

$$N(\theta - \varepsilon, L_n) = N(\theta, L) = N(\gamma, L) < +\infty.$$

It means that $N(\gamma, L_n) < \infty$. Consequently, the sequence $\{L_n\}_{n=1}^{\infty}$ (we can choose n sufficiently large) satisfies the conditions of Theorem 2 and the needed results follow immediately from Theorem 2. $\square$

Remark. The completeness of the eigenvectors in Theorem 2 in the definite case can be proved by using an indefinite scalar product [**3**]. Here we illustrate it for a model problem of the form

$$L(\alpha) = \begin{cases} \alpha B_+ - A & \alpha \geq 0, \\ \alpha B_- - A & \alpha \leq 0, \end{cases} \qquad \alpha \in [a, b].$$

$0 \neq f \perp M := \operatorname{span}\{M_{\pm}(L)\} \Leftrightarrow f \perp M_+(L)$ and $f \perp M_-(L)$. It is clear that $0 \neq f \perp M_-(L)$ implies $B_-^{-1/2} f \in H_+(S)$ and $f \in R(A)$. Define $[x, y] := (A^{-1}x, y)$, $\ker A \neq 0$. We obtain

$$[f, f] > 0. \tag{12}$$

On the other hand, using the condition $0 \neq f \perp M_+(L)$, we have $B_+^{-1/2} f \in H_+^{\perp}(T) = H_-(T)$. Finally, since $f \in R(A)$ it is easy to check that

$$[f, f] = (A^{-1}f, f) < 0. \tag{13}$$

The inequality (13) contradicts the inequality (12). Consequently, $f = 0$.

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