ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 2, 2006

INTEGRABILITY OF PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS THROUGH LINEAR DIFFERENTIAL EQUATIONS

H. GIACOMINI, J. GINÉ AND M. GRAU

ABSTRACT. In this work we consider rational ordinary differential equations $dy/dx = Q(x,y)/P(x,y)$, with $Q(x,y)$ and $P(x,y)$ coprime polynomials with real coefficients. We give a method to construct equations of this type for which a first integral can be expressed from two independent solutions of a second-order homogeneous linear differential equation. This first integral is, in general, given by a non Liouvillian function.

We show that all the known families of quadratic systems with an irreducible invariant algebraic curve of arbitrarily high degree and without a rational first integral, can be constructed by using this method. We also present a new example of this kind of family.

We give an analogous method for constructing rational equations but by means of a linear differential equation of first order.

1. Introduction. This paper deals with rational ordinary differential equations such as

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}, \tag{1}$$

where $Q(x,y)$ and $P(x,y)$ are coprime polynomials with real coefficients. We associate to this rational equation a planar polynomial differential system by introducing an independent variable t usually called *time*. Denoting by $\dot{} = d/dt$, we have

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{2}$$

2000 AMS *Mathematics Subject Classification.* Primary 34A05, 34C05, 34C20.
Key words and phrases. Planar polynomial system, first integral, invariant curves, Darboux integrability.
The second and third authors are partially supported by a MCYT grant number BFM 2002-04236-C02-01. The second author is partially supported by DURSI of Government of Catalonia's Acció Integrada ACI2002-24 and by the Distinció de la Generalitat de Catalunya per a la promoció de la recerca universitària.
Received by the editors on May 26, 2003, and in revised form on October 8, 2003.

where $(x,y) \in \mathbf{R}^2$. This system defines the vector field $\mathcal{X} = P(x,y)(\partial/\partial x) + Q(x,y)(\partial/\partial y)$ over $\mathbf{R}^2$ and, equivalently, the one-form $\omega = Q(x,y)\,dx - P(x,y)\,dy$. We indistinctively talk about equation (1) and system (2). Let d be the maximum degree of P and Q. We say that system (2) is of degree d. When d $= 2$, we say that (2) is a *quadratic system*.

In order to simplify notation, we define $\mathbf{R}[x,y]$ as the ring of polynomials in two variables with real coefficients and $\mathbf{R}(x,y)$ as the field of rational functions in two variables with real coefficients, that is, the quotient field of the previous ring. Analogous definitions stand for $\mathbf{R}[x]$ and $\mathbf{R}(x)$.

We have an equation (1) defined in a certain class of functions, in this case, the rational functions with real coefficients $\mathbf{R}(x,y)$ and we consider the problem whether there is a first integral in another, possibly larger, class. For instance, as we will discuss later on, Poincaré stated the problem of determining when a system (2) has a rational first integral. A $\mathcal{C}^k$ function $H : \mathcal{U} \to \mathbf{R}$ such that it is constant on each trajectory of (2) and it is not locally constant is called a *first integral* of system (2) of class k, and the equation $H(x,y) = c$ for a fixed $c \in \mathbf{R}$ gives a set of trajectories of the system, but in an implicit way. When $k \geq 1$, these conditions are equivalent to $\omega \wedge dH = 0$ and H not locally constant. The problem of finding such a first integral and the functional class it must belong to is what we call the *integrability problem*.

To find an integrating factor or an inverse integrating factor for system (2) is closely related to finding a first integral for it. When considering the integrability problem we also address whether an (inverse) integrating factor belongs to a certain given class of functions.

Definition 1. Let $\mathcal{W}$ be an open set of $\mathbf{R}^2$. A function $\mu : \mathcal{W} \to \mathbf{R}$ of class $\mathcal{C}^k(\mathcal{W})$, $k > 1$, that satisfies the linear partial differential equation

$$\omega \wedge d\mu = \mu\, d\omega, \tag{3}$$

is called an *integrating factor* of system (2) on $\mathcal{W}$.

It has been shown than an easier function to find which also gives additional properties for a differential system (2) is the inverse of an integrating factor, that is, $V = 1/\mu$, which is called the *inverse integrating factor*.

We note that $\{V = 0\}$ is formed by orbits of system (2). The function $\mu = 1/V$ defines on $\mathcal{W} \setminus \{V = 0\}$ an integrating factor of system (2), which allows the computation of a first integral of the system on $\mathcal{W} \setminus \{V = 0\}$. The *first integral H associated to the inverse integrating factor V* can be computed through the integral $H(x, y) = \int \omega/V$, and the condition (3) for $\mu = 1/V$ ensures that this line integral is well defined.

The inverse integrating factors play an important role in two of the most difficult open problems of qualitative theory of planar polynomial vector fields, which are the center problem and 16th Hilbert problem. In [**6**], it has been noticed that for many polynomial differential systems with a center at the origin there is always an inverse integrating factor V globally defined in all $\mathbf{R}^2$, which is usually a polynomial. However, the first integral for a polynomial differential system with a center at the origin can be very complicated.

We say that a function $f(x, y)$ is an *invariant* for a system (2) if $\omega \wedge df = kf$ with $k(x, y)$ a polynomial of degree lower or equal than $\mathrm{d} - 1$, where d is the degree of the system. This polynomial $k(x, y)$ is called the *cofactor* of $f(x, y)$. In the previous equality and all along this paper we use the convention of identifying the space of functions over $\mathbf{R}^2$ and the space of two-forms over $\mathbf{R}^2$. In case $f(x, y) = 0$ defines a curve in the real plane, this definition implies that $\omega \wedge df$ equals zero on the points such that $f(x, y) = 0$. In case $f(x, y)$ is a polynomial we say that $f(x, y) = 0$ is an *invariant algebraic curve* for system (2).

Let us consider $f(x, y) = 0$ an invariant algebraic curve for system (2). We will always assume that $f(x, y)$ is an irreducible polynomial in $\mathbf{R}[x, y]$. Otherwise, it can be shown that each of its factors is an invariant algebraic curve for system (2). We will denote by n the degree of the polynomial $f(x, y)$.

In [**12**], Darboux gives a method for finding an explicit first integral for a system (2) in the case that $\mathrm{d}(\mathrm{d} + 1)/2 + 1$ different irreducible invariant algebraic curves are known, where d is the degree of the system. In this case, a first integral of the form $H = f_1^{\lambda_1} f_2^{\lambda_2} \cdots f_r^{\lambda_r}$, where $f_i(x, y) = 0$ is an invariant algebraic curve for system (2) and $\lambda_i \in \mathbf{C}$, not all of them null, for $i = 1, 2, \ldots, r$, $r \in \mathbf{N}$, can be defined in the open set $\mathbf{R}^2 \setminus \Sigma$, where $\Sigma = \{(x,y) \in \mathbf{R}^2 \mid (f_1 \cdot f_2 \cdot \cdots \cdot f_r)(x,y) = 0\}$. The functions of this type are called *Darboux* functions. We remark that, particularly, if $\lambda_i \in \mathbf{Z}$, for all $i = 1, 2, \ldots, r$, H is a *rational first integral* for system (2). In this sense Jouanoulou [**15**], showed that if at least $\mathrm{d}(\mathrm{d}+1)+2$ different irreducible invariant algebraic curves are known, then there exists a rational first integral.

The main fact used to prove Darboux's theorem, and Jouanoulou's improvement, is that the cofactor corresponding to each invariant algebraic curve is a polynomial of degree $\leq \mathrm{d} - 1$. Invariant functions can also be used in order to find a first integral for the system. This observation permits a generalization of Darboux's theory which is given in [**14**] where, for instance, nonalgebraic invariant curves with an algebraic cofactor for a polynomial system of degree 4 are presented. In our work, we give other families of systems with such invariant curves.

Christopher, in [**9**], studies the multiplicity of an invariant algebraic curve and gives the definition for exponential factor, which is a particular case of invariant for system (2).

Definition 2. Let h, g be two coprime polynomials. The function $e^{h/g}$ is called an *exponential factor* for system (2) if for some polynomial k of degree at most $\mathrm{d} - 1$, where d is the degree of the system, the following relation is fulfilled: $\omega \wedge d\left(e^{h/g}\right) = k e^{h/g}$. As before, we say that $k(x,y)$ is the *cofactor* of the exponential factor $e^{h/g}$.

We note that an exponential factor $e^{h/g}$ does not define an invariant curve, but the next proposition, proved in [**9**], gives the relationship between both notions.

Proposition 3 [**9**]. *If $F = e^{h/g}$ is an exponential factor and g is not a constant, then $g = 0$ is an invariant algebraic curve, and h satisfies the equation $\omega \wedge dh = (hk_g + gk_F)$ where k_g and k_F are the cofactors of g and F, respectively.*

All these previous results are closely related to a result of Singer [**25**] which represents an important progress in the resolution of the integrability problem when considering first integrals for a system (2) in the class of Liouvillian functions. Roughly speaking, we can define a *Liouvillian function* or a function which can be expressed by means of quadratures, as a function constructed from rational functions using composition, exponentiation, integration, and algebraic functions. A precise definition of this class of functions is given in [**25**].

Theorem 4 [**25**]. *Let us consider the polynomial one-form* $\omega = Q\,dx - P\,dy$ *related to system* (2)*. System* (2) *has a Liouvillian first integral if, and only if,* ω *has an inverse integrating factor of the form* $V = \exp \int \xi$*, where* ξ *is a closed rational one-form.*

We notice that the conditions on the function V given in this theorem can be restated as $d\omega = \xi \wedge \omega$ and $d\xi = 0$.

Taking into account Theorem 4, Christopher [**10**] gives the following result, which makes precise the form of the inverse integrating factor.

Theorem 5 [**10**]. *If system* (2) *has an inverse integrating factor of the form* $\exp \int \xi$ *with* $d\xi = 0$ *and* $\xi = \xi_1\,dx + \xi_2\,dy$ *where* ξ_i*,* $i = 1, 2$*, are rational functions in* x *and* y*, then there exists an inverse integrating factor of system* (2) *of the form*

$$V = \exp\{D/E\} \prod C_i^{l_i},$$

where D*,* E *and the* C_i *are polynomials in* x *and* y*, and* $l_i \in \mathbf{C}$*.*

Theorem 5 states that the search for Liouvillian first integrals can be reduced to the search of invariant algebraic curves and exponential factors.

In [**21**], Poincaré stated the following problem concerning the integration of an equation (1): *Give conditions on the polynomials* P *and* Q *to recognize when there exists a rational first integral.* As the same Poincaré noticed, a sufficient condition to solve this problem consists on finding an upper bound for the degree of the invariant algebraic curves for a given system (2). From Darboux's result, it is known that for every polynomial vector field, there exists an upper bound for the possible degrees of irreducible invariant algebraic curves. However, it is a hard problem to explicitly determine such an upper bound. Some bounds have been given under certain conditions on the invariant curves, see the work of Cerveau and Lins Neto [**5**], or on the local behavior of critical points, see Carnicer's work [**4**].

In this sense, Lins Neto conjectured [**16**] that a polynomial system (2) of degree d with an invariant algebraic curve of degree high enough (where this bound only depends on d) would have a rational first integral. This conjecture has been shown to be false by several counterexamples. Moulin-Ollagnier [**19**] gives a family of quadratic Lotka-Volterra systems, each with an invariant algebraic curve of degree 2ℓ, where ℓ is the parameter of the family, without rational first integral. A simpler example is given by Christopher and Llibre in [**11**]. In [**8**] a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree without a Darboux first integral nor a Darboux inverse integrating factor is given. All these counterexamples exhibit a Liouvillian first integral.

The natural conjecture at this step, also given by Lins Neto, see [**17**], after the counterexample of Moulin-Ollagnier appeared, is that a polynomial system (2) of degree d with an invariant algebraic curve of degree high enough (where this bound only depends on d) has a Liouvillian first integral.

In this work we show a relationship between solutions of a class of systems (2) and linear homogeneous ordinary differential equations of order 2 of the form

$$A_2(x)\,w''(x) + A_1(x)\,w'(x) + A_0(x)\,w(x) = 0, \tag{4}$$

where $x \in \mathbf{R}$, $w'(x) = dw(x)/dx$ and $w''(x) = dw'(x)/dx$. We only consider equations (4) where $A_i(x) \in \mathbf{R}[x]$ for $0 \le i \le 2$ and $A_2(x) \not\equiv 0$.

By means of a change of variable we rewrite an equation (4) as a polynomial differential system such that it has an invariant related to $w(x)$. In case $w(x)$ is a polynomial we get an invariant algebraic curve.

Moreover, we give an explicit first integral for all the systems built up by this method by means of two independent solutions of equation (4).

We give analogous results for a linear homogeneous ordinary differential equation of order 1 such as

$$w'(x) + A(x)w(x) = 0, \tag{5}$$

where $x \in \mathbf{R}$, $w'(x) = dw(x)/dx$ and $A(x) \in \mathbf{R}(x)$. All these results are given in Section 2.

In Section 3 we consider all the families of quadratic systems with an algebraic curve of arbitrarily high degree known until the moment of composition of this paper and we show that they all belong to the construction explained in Section 2. The families of quadratic systems with an algebraic curve of arbitrarily high degree studied in this paper are the ones appearing in [**7, 8, 11, 19**] and one example more first appearing in this work. This new example consists of a biparametrical family of quadratic systems, which we give an explicit expression of a first integral for, such that when one of the parameters is a natural number, say n, the system exhibits an irreducible invariant algebraic curve of degree n.

We give the explicit expression for the first integral of a certain system (2) by means of invariant functions for it, and applying the Generalized Darboux's theory as explained in [**14**] where a new kind of first integrals, not only the Liouvillian ones as in classical theories, is described. We exemplify this result with the families of systems depending on parameters described in Section 3. We remark that the first integrals that we give in Section 2 are not, in general, of Liouvillian type. However, these first integrals are Liouvillian at the values of parameters which correspond to the systems with algebraic solutions. In the subsection 3.4, we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed by the method appearing in Section 2 from an equation (5).

A question suggested by these examples is whether there are polynomial systems which are not reversible nor Liouvillian integrable which have a center and can be integrated by means of Theorem 6, see Section 2. The work [**3**] is related to this question as it gives an example of an analytic system, not polynomial, with a center which is not reversible nor Liouvillian integrable. All the known families of polynomial vector fields with a center at the origin are either Liouvillian integrable or reversible, see [**27, 28**] for the definition of reversibility. In [**27, 28**], Żołądek classifies all the reversible cubic systems with a center. The reversible systems may have a first integral not given by Liouvillian functions or no explicit form of a first integral may be known. For instance the reversible system $\dot{x} = -y + x^4$, $\dot{y} = x$ has a first integral composed by Airy functions, see [**14**], and no Liouvillian first integral exists. The system $\dot{x} = -y^3 + x^2y^2/2$, $\dot{y} = x^3$ is an example given by Moussu, see [**20**], which has a center at the origin since it is a monodromic and reversible singular point and no explicit first integral is known for this system.

Since some examples of polynomial systems, which can be integrated by the method described in Section 2, appear after a birational transformation, another suggested open question is if all the polynomial systems with a center are birationally equivalent to one derived from Theorem 6 or from Theorem 11.

2. Homogeneous linear differential equations of order ≤ 2 **and planar polynomial systems.** Let us consider a homogeneous linear differential equation of order 2:

$$A_2(x)\, w''(x) + A_1(x)\, w'(x) + A_0(x)\, w(x) = 0, \tag{6}$$

where $w'(x) = dw(x)/dx$, $w''(x) = dw'(x)/dx$, $A_i(x) \in \mathbf{R}[x]$, $i = 0, 1, 2$, and $A_2(x) \not\equiv 0$.

Theorem 6. *Given $g(x,y) = g_0(x,y)/g_1(x,y)$ with $g_i(x,y) \in \mathbf{R}[x,y]$, $g_1(x,y) \not\equiv 0$ and $\partial g/\partial y \not\equiv 0$, each nonzero solution $w(x)$ of equation* (6) *is related to a finite number of solutions $y = y(x)$ of the rational equation*

$$\frac{dy}{dx} = \frac{A_0(x)g_1^2 + A_1(x)g_1g_0 + A_2(x)g_0^2 + A_2(x)\,(g_1(\partial g_0/\partial x) - g_0(\partial g_1/\partial x))}{A_2(x)\,(g_0(\partial g_1/\partial y) - g_1(\partial g_0/\partial y))}, \tag{7}$$

by the functional change $dw/dx = g(x,y)w(x)$, which implicitly defines y as a function of x.

Proof. Let us consider equation (6) and the functional change $dw/dx = g(x,y)w(x)$ where $y = y(x)$, that is, y is implicitly defined as a function of x. This change may also be written as $w(x) = \exp(\int_{x_0}^{x} g(s, y(s))\, ds)$, where x_0 is any constant, and it is injective. We see that it is not necessarily bijective unless the maximum degree of $g_1(x,y)$ and $g_0(x,y)$ in the variable y equals 1. But it defines a finite number of functions $y(x)$.

By this functional change, equation (6) becomes

$$w(x)\left(A_0(x)+g\,A_1(x)+g^2A_2(x)+A_2(x)\,\frac{dy}{dx}\,\frac{\partial g}{\partial y}+A_2(x)\,\frac{\partial g}{\partial x}\right)=0.$$

We have that $w(x)$ is a nonzero solution of (6) so this equation is equivalent to the ordinary differential equation of first order (7). Therefore, each non-zero solution $w(x)$ of equation (6) corresponds to a finite number of solutions $y=y(x)$ of the planar polynomial system (7). □

Theorem 7. *We consider the* 1*-form related to equation* (7)

$$\omega=\left(A_0(x)g_1^2+A_1(x)g_1\,g_0+A_2(x)g_0^2+A_2(x)\left(g_1\,\frac{\partial g_0}{\partial x}-g_0\,\frac{\partial g_1}{\partial x}\right)\right)dx$$
$$-\,A_2(x)\left(g_0\,\frac{\partial g_1}{\partial y}-g_1\,\frac{\partial g_0}{\partial y}\right)dy.$$

Let $w(x)$ be any nonzero solution of equation (6). *Then the curve defined by $f(x,y)=0$, with $f(x,y):=g_1(x,y)w'(x)-g_0(x,y)w(x)$ is invariant for system* (7) *and has the polynomial cofactor*

$$k(x,y)=\left(A_0(x)\,\frac{\partial g_1}{\partial y}+A_1(x)\,\frac{\partial g_0}{\partial y}\right)g_1+A_2(x)\,g_0\frac{\partial g_0}{\partial y}$$
$$+\,A_2(x)\left(\frac{\partial g_1}{\partial y}\,\frac{\partial g_0}{\partial x}-\frac{\partial g_0}{\partial y}\,\frac{\partial g_1}{\partial x}\right).$$

Proof. Let us consider $f(x,y)$ as defined above, and let us compute $\omega\wedge df$:

$$\omega\wedge df=\left(\frac{\partial g_1}{\partial x}\,w'(x)+g_1w''(x)-\frac{\partial g_0}{\partial x}\,w(x)-g_0w'(x)\right)A_2(x)$$
$$\cdot\left(g_0\frac{\partial g_1}{\partial y}-g_1\frac{\partial g_0}{\partial y}\right)+\left(\frac{\partial g_1}{\partial y}\,w'(x)-\frac{\partial g_0}{\partial y}\,w(x)\right)$$
$$\cdot\left[A_0(x)g_1^2+A_1(x)g_1\,g_0+A_2(x)g_0^2+A_2(x)\left(g_1\,\frac{\partial g_0}{\partial x}-g_0\,\frac{\partial g_1}{\partial x}\right)\right]$$

$$
\begin{aligned}
&= g_1 \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y} \right) A_2(x)\, w''(x) \\
&\quad + g_1 \left(A_1(x)\, g_0 + A_0(x)\, g_1 \right) \left(\frac{\partial g_1}{\partial y}\, w'(x) - \frac{\partial g_0}{\partial y}\, w(x) \right) \\
&\quad + \left(A_2(x) \left(\frac{\partial g_0}{\partial x} \frac{\partial g_1}{\partial y} - \frac{\partial g_1}{\partial x} \frac{\partial g_0}{\partial y} \right) + A_2(x)\, g_0 \frac{\partial g_0}{\partial y} \right) \\
&\quad \times \left(g_1 w'(x) - g_0 w(x) \right)
\end{aligned}
$$

Since $w(x)$ is a solution of (6), we can substitute $A_2(x)w''(x)$ by $-A_1(x)w'(x) - A_0(x)w(x)$. Therefore,

$$
\begin{aligned}
\omega \wedge df = \Bigg[& \left(A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y} \right) g_1 \\
& + A_2(x)\, g_0 \frac{\partial g_0}{\partial y} + A_2(x) \left(\frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right) \Bigg] f(x,y).
\end{aligned}
$$

Then, we have that the function $f(x,y)$ is an invariant for system (7) and has the written polynomial cofactor. □

Theorem 8. *Let* $\{w_1(x), w_2(x)\}$ *be a set of fundamental solutions of equation* (6). *We define* $f_i(x,y) := g_1(x,y)w_i'(x) - g_0(x,y)w_i(x)$, $i = 1,2$. *Then, system* (7) *has a first integral* $H(x,y)$ *defined by*

$$
H(x,y) := \frac{f_1(x,y)}{f_2(x,y)} = \frac{g_1(x,y)\, w_1'(x) \,-\, g_0(x,y)\, w_1(x)}{g_1(x,y)\, w_2'(x) \,-\, g_0(x,y)\, w_2(x)}.
$$

Proof. By Theorem 7, we have that $f_i(x,y) := g_1(x,y)w_i'(x) - g_0(x,y)w_i(x)$, $i = 1,2$, are invariants for system (7), both with the polynomial cofactor

$$
\begin{aligned}
k(x,y) = & \left(A_0(x) \frac{\partial g_1}{\partial y} + A_1(x) \frac{\partial g_0}{\partial y} \right) g_1 + A_2(x)\, g_0 \frac{\partial g_0}{\partial y} \\
& + A_2(x) \left(\frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x} \right).
\end{aligned}
$$

We remark that f_1/f_2 cannot be constant since the two solutions $w_i(x)$, $i = 1,2$, are independent. Therefore,

$$
\omega \wedge dH = \frac{f_2(\omega \wedge df_1) - f_1(\omega \wedge df_2)}{f_2^2} = \frac{f_2 k f_1 - f_1 k f_2}{f_2^2} \equiv 0.
$$

So, $H(x,y)$ is a first integral of system (7). $\square$

Lemma 9. *The function defined by*

$$q(x) := A_2(x) \exp\left(\int_{x_0}^{x} \frac{A_1(s)}{A_2(s)}\, ds\right)$$

is an invariant for system (7), *with cofactor*

$$(A_1(x) + A_2'(x)) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y}\right).$$

We notice that $q(x)$ is a product of invariant algebraic curves and exponential factors for system (8), with complex exponents.

Proof. We compute $\omega \wedge dq$ and we have

$$\begin{aligned} \omega \wedge dq = {} & \omega \wedge \frac{A_1(x) + A_2'(x)}{A_2(x)}\, q\, dx \\ & = (A_1(x) + A_2'(x)) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y}\right) q. \end{aligned}$$

We notice that this algebraic cofactor has degree $\le \mathrm{d}-1$ provided that system (7) has degree d. $\square$

Proposition 10. *We use the same notation as in Theorem* 6. *Let $w(x)$ be a nonzero solution of* (6), *and we define $f(x,y) := w'(x) - g(x,y)w(x)$ and $q(x)$ as in Lemma* 9. *The function $V(x,y) = q(x)f(x,y)^2$ is an inverse integrating factor of system* (7).

Proof. We only need to verify that $\omega \wedge dV + V d\omega = 0$. We have that

$$\begin{aligned} d\omega = -\Bigg[& 2\, A_0(x)\, g_1 \frac{\partial g_1}{\partial y} + A_1(x) \left(g_0 \frac{\partial g_1}{\partial y} + g_1 \frac{\partial g_0}{\partial y}\right) + 2\, A_2(x)\, g_0 \frac{\partial g_0}{\partial y} \\ & + 2\, A_2(x) \left(\frac{\partial g_1}{\partial y} \frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y} \frac{\partial g_1}{\partial x}\right) + A_2'(x) \left(g_0 \frac{\partial g_1}{\partial y} - g_1 \frac{\partial g_0}{\partial y}\right)\Bigg]. \end{aligned}$$

Then,

$$
\begin{aligned}
\omega \wedge dV &= \omega \wedge (2\,q\,f\,df + f^2\,dq) \\
&= 2\,q\,f\,(\omega \wedge df) + f^2\,(\omega \wedge dq) \\
&= \Bigg[2\,A_0(x)\,g_1\,\frac{\partial g_1}{\partial y} + 2\,A_1(x)\,g_1\,\frac{\partial g_0}{\partial y} \\
&\qquad + 2\,A_2(x)\,g_0\,\frac{\partial g_0}{\partial y} + 2\,A_2(x)\left(\frac{\partial g_1}{\partial y}\,\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\,\frac{\partial g_1}{\partial x}\right) \\
&\qquad + A_1(x)\,g_0\,\frac{\partial g_1}{\partial y} - A_1(x)\,g_1\,\frac{\partial g_0}{\partial y} \\
&\qquad + A_2'(x)\left(g_0\,\frac{\partial g_1}{\partial y} - g_1\,\frac{\partial g_0}{\partial y}\right)\Bigg]\,V \\
&= \Bigg[2\,A_0(x)\,g_1\,\frac{\partial g_1}{\partial y} + A_1(x)\,\left(g_0\,\frac{\partial g_1}{\partial y} + g_1\,\frac{\partial g_0}{\partial y}\right) \\
&\qquad + 2\,A_2(x)\,g_0\,\frac{\partial g_0}{\partial y} + 2\,A_2(x)\left(\frac{\partial g_1}{\partial y}\,\frac{\partial g_0}{\partial x} - \frac{\partial g_0}{\partial y}\,\frac{\partial g_1}{\partial x}\right) \\
&\qquad + A_2'(x)\left(g_0\,\frac{\partial g_1}{\partial y} - g_1\,\frac{\partial g_0}{\partial y}\right)\Bigg]\,V \\
&= -V\,d\omega. \qquad \square
\end{aligned}
$$

We remark that Theorem 8 gives, in general, a non-Liouvillian first integral for the planar polynomial systems (7). In Section 3 we analyze some polynomial systems constructed from Theorem 8 that have no Liouvillian first integral.

We consider now a linear homogeneous ordinary differential equation of order 1 such as

$$w'(x) + A(x)\,w(x) = 0, \tag{8}$$

where $x \in \mathbf{R}$, $w'(x) = dw(x)/dx$ and $A(x) = A_0(x)/A_1(x)$ with $A_i(x) \in \mathbf{R}[x]$ and $A_1(x) \not\equiv 0$. We give analogous results for this case whose proofs are not given to avoid non-useful repetitions.

Theorem 11. *Given $g(x,y) = g_0(x,y)/g_1(x,y)$ with $g_i(x,y) \in \mathbf{R}[x,y]$, $g_1(x,y) \not\equiv 0$ and $\partial g/\partial y \not\equiv 0$ and $h(x) = h_0(x)/h_1(x)$ with*

$h_i(x) \in \mathbf{R}[x]$ and $h_1(x) \not\equiv 0$, each nonzero solution $w(x)$ of equation (8) *is related to a finite number of solutions $y = y(x)$ of the rational equation*

$$(9)\quad \frac{dy}{dx} = \frac{A_1(x)h_0(x)g_1^2 - A_0(x)h_1(x)g_0g_1 - A_1(x)h_1(x)\left(g_1(\partial g_0/\partial x) - g_0(\partial g_1/\partial x)\right)}{A_1(x)\,h_1(x)\left(g_1(\partial g_0/\partial y) - g_0(\partial g_1/\partial y)\right)},$$

by the functional change

$$w(x) = g(x,y) - \exp\left(-\int_0^x A(s)\,ds\right)\left[\int_0^x \exp\left(\int_0^s A(r)\,dr\right)h(s)\,ds\right].$$

Theorem 12. *We consider the* 1*-form related to equation* (9)

$$\omega = \left[A_1(x)h_0(x)g_1^2 - A_0(x)h_1(x)g_0g_1 - A_1(x)h_1(x)\left(g_1\frac{\partial g_0}{\partial x} - g_0\frac{\partial g_1}{\partial x}\right)\right]dx - A_1(x)\,h_1(x)\left(g_1\frac{\partial g_0}{\partial y} - g_0\frac{\partial g_1}{\partial y}\right)dy.$$

Let $w(x)$ be any nonzero solution of equation (8)*, that is, for $C \in \mathbf{R} - \{0\}$ we have $w(x) = C\exp\left(-\int_0^x A(s)\,ds\right)$.*

Then, the function

$$f(x,y) := g_1\,w(x) - g_0 + g_1\exp\left(-\int_0^x A(s)\,ds\right)\left[\int_0^x \exp\left(\int_0^s A(r)\,dr\right)h(s)\,ds\right]$$

is invariant for the polynomial system (9)*, with the polynomial cofactor*

$$k(x,y) = -A_0(x)\,h_1(x)\,g_1\frac{\partial g_0}{\partial y} + A_1(x)\,h_0(x)\,g_1\frac{\partial g_1}{\partial y} + A_1(x)\,h_1(x)\left(\frac{\partial g_0}{\partial y}\frac{\partial g_1}{\partial x} - \frac{\partial g_1}{\partial y}\frac{\partial g_0}{\partial x}\right).$$

Lemma 13. *The function $q(x,y) = g_1(x,y)\exp\left(\int_0^x -A(s)\,ds\right)$ is an invariant for system* (9) *with the same polynomial cofactor as $f(x,y)$.*

Theorem 14. *We use the same notation as in Theorem* 12 *and Lemma* 13. *The function* $H(x,y)$ *defined by* $H(x,y) := f(x,y)/q(x,y)$ *is a first integral for system* (9) *and the function* $V(x,y) := A_1(x) \times h_1(x) g_1(x,y)\, q(x,y)$ *is an inverse integrating factor.*

We remark that $H(x,y)$ is a Liouvillian function and, therefore, a system (9) has always a Liouvillian first integral.

In Section 3 we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed following Theorem 11.

3. Examples of families of quadratic systems.

3.1 Quadratic systems with invariant algebraic curves of arbitrarily high degree linear in one variable. We first consider the examples of families of quadratic systems with algebraic solutions of arbitrarily high degree appearing in [**7**]. In that work all the invariant algebraic curves linear in the variable y, that is, defined by $f(x,y) = p_1(x)y + p_2(x)$, where p_1 and p_2 are polynomials, are determined.

The example appearing in [**8**] is a further study of an example appearing in [**7**], and the example given in [**11**] is also described in [**7**]. We show that all these quadratic systems, with an invariant algebraic curve of arbitrary degree can be constructed by the method explained in the previous section. Moreover, we give the explicit expression of a first integral for any value of the parameter n, even in the case when n is not a natural number. If n is a natural number, we obtain the invariant algebraic curves of arbitrary degree and a Liouvillian first integral. However, when $n \notin \mathbf{N}$ we obtain polynomial systems with a non Liouvillian first integral.

As it is shown in [**7**], all these families of systems can be written, after an affine change of variables if necessary, in the form

$$(10) \qquad \begin{aligned} \dot{x} &= \Omega_1(x), \\ \dot{y} &= (2n+1)\, L'(x)\, \Omega_1(x) - \frac{n(n+1)}{2}\, \Omega_1(x)\, \Omega_1''(x) - L(x)^2 + y^2, \end{aligned}$$

where $\Omega_1(x)$ is any quadratic polynomial, $L(x)$ is any linear polynomial and $' = d/dx$. We have that system (10) has an invariant curve $f(x,y) = 0$, where $f(x,y) := p_1(x)y + \Omega_1(x)p_1'(x) - L(x)p_1(x)$, with a cofactor $y + L(x)$, where $p_1(x)$ is a solution of the second order linear differential equation

$$\begin{aligned}\Omega_1(x)\, w''(x) + (\Omega_1'(x) - 2L(x))\, w'(x) \\ + \frac{n}{2}\,(4\,L'(x) - (n+1)\,\Omega_1''(x))\, w(x) = 0.\end{aligned} \tag{11}$$

In [**7**] it is shown that, in case $n \in \mathbf{N}$, an irreducible polynomial of degree n belonging to a family of orthogonal polynomials is a solution of equation (11). For instance, when $\Omega_1(x) = 1$, we get the Hermite polynomials, when $\Omega_1(x) = x$, we get the Generalized Laguerre polynomials and when $\Omega_1(x) = 1 - x^2$, we get the Jacobi polynomials.

We consider again the general case in which $n \in \mathbf{R}$, and we define $A_2(x) := \Omega_1(x)$, $A_1(x) := \Omega_1'(x) - 2L(x)$ and $A_0(x) := n(4L'(x) - (n+1)\Omega_1''(x))/2$. We have the linear differential equation (11) in the same notation as in Theorem 6 and we consider

$$g(x,y) := \frac{L(x) - y}{A_2(x)}.$$

The system obtained by the method explained in Section 2 exactly coincides with system (10). We consider a set of fundamental solutions of equation (11) $\{w_1(x), w_2(x)\}$ and applying Theorem 8, we have a first integral for system (10) for any value of the parameter $n \in \mathbf{R}$.

In case $n \in \mathbf{N}$ we have that $w_1(x)$ degenerates to a polynomial and $w_1'(x) - g(x,y)w_1(x) = 0$ coincides with the algebraic curve given in the work [**7**].

We explicitly give the first integral for each of the families described in [**7**] and for $n \in \mathbf{R}$. We have that $A_2(x)$ is a non null quadratic polynomial in this case and, depending on its number of roots, we can transform it by a real affine change of variable to one of the following forms: $A_2(x) = 1$, x, x^2, $1 - x^2$, $1 + x^2$.

If $A_2(x) = 1$, we can choose $L(x) = x$ by an affine change of coordinates. A set of fundamental solutions $\{w_1(x), w_2(x)\}$ for (11) with $n \in \mathbf{R}$ is

$$w_1(x) = 2^n\sqrt{\pi}\left(\frac{1}{\Gamma\left((1-n)/2\right)}\,{}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right) - \frac{2x}{\Gamma\left(-n/2\right)} \times {}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right)\right),$$

$$w_2(x) = 2^n\sqrt{\pi}\left(\frac{1}{\Gamma\left((1-n)/2\right)}\,{}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right) + \frac{2x}{\Gamma\left(-n/2\right)} \times {}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right)\right),$$

where $\Gamma(x)$ is the Euler's-Gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt$ and ${}_1F_1(a;b;x)$ is the confluent hypergeometric function defined by the series

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\,\frac{x^k}{k!},$$

with $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, the Pochhammer symbol. See [**1**] for further information about these functions. So, a first integral for this system is the expression given in Theorem 8: $H(x,y) := f_1(x,y)/f_2(x,y)$, where

$$\begin{aligned} f_{1,2}(x,y) = &\pm\Gamma\left(\frac{1-n}{2}\right)\left[6\left(xy - x^2 + 1\right)\,{}_1F_1\left(\frac{1-n}{2};\frac{3}{2};x^2\right) - 4(n-1)x^2\,{}_1F_1\left(\frac{3-n}{2};\frac{5}{2};x^2\right)\right] \\ &+ 3\Gamma\left(-\frac{n}{2}\right) \\ &\times\left[2nx\,{}_1F_1\left(1-\frac{n}{2};\frac{3}{2};x^2\right) + (x-y)\,{}_1F_1\left(-\frac{n}{2};\frac{1}{2};x^2\right)\right]. \end{aligned}$$

When $n \in \mathbf{N}$, we have that (11) corresponds to the equation for Hermite polynomials and $w_1(x)$ coincides with the Hermite polynomial of degree n. The invariant algebraic curve given in [**7**] corresponds to $f_1(x,y) = 0$.

If $A_2(x) = x$, we choose $L(x) = (x - \alpha)/2$, where α is an arbitrary real constant, and a set of fundamental solutions for (11) is:

$$w_1(x) = \frac{(\alpha+1)_n}{\Gamma(n+1)}\ {}_1F_1\left(-n; \alpha+1; x\right),$$
$$w_2(x) = x^{-\alpha}\ {}_1F_1\left(-\alpha-n; 1-\alpha; x\right).$$

The first integral for this system is $H(x,y) = x^{\alpha} h_1(x,y)/h_2(x,y)$ with

$$\begin{aligned} h_1(x,y) &= (2y - x + \alpha)\,(\alpha+1)\ {}_1F_1\left(-n; \alpha+1; x\right) \\ &\quad - 2\,n\,x\ {}_1F_1\left(1-n; \alpha+2; x\right), \\ h_2(x,y) &= (2y - x + \alpha)\,(\alpha-1)\ {}_1F_1\left(-\alpha-n; 1-\alpha; x\right) \\ &\quad - 2\,(\alpha+n)\,x\ {}_1F_1\left(1-\alpha-n; 2-\alpha; x\right). \end{aligned}$$

The first integral as given in Theorem 8 is $f_1(x,y)/f_2(x,y)$ and we notice that $H(x,y) = c f_1(x,y)/f_2(x,y)$ where $c \in \mathbf{R} - \{0\}$. We do not write c in terms of the parameters of the system to simplify notation.

When $n \in \mathbf{N}$, we have that (11) corresponds to the equation of generalized Laguerre polynomials and $w_1(x)$ coincides with the generalized Laguerre polynomial $L_n^{(\alpha)}$. The invariant algebraic curve given in [**7**] corresponds to $f_1(x,y) = 0$, where $f_1(x,y) := w_1'(x) - g(x,y)w_1(x)$.

If $A_2(x) = x^2$, the birrational transformation yet described in [**7**], $x = 1/X$ and $y = (1/X)(1/2 - Y)$ makes this case equivalent to the previous one.

If $A_2(x) = 1-x^2$, we choose $L(x) = ((\alpha+\beta)x + (\alpha-\beta))/2$, where α, β are two arbitrary real constants, and a set of fundamental solutions for (11) is:

$$w_1(x) = \frac{(\alpha+1)_n}{\Gamma(n+1)}\ {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right),$$
$$w_2(x) = (1-x)^{-\alpha}\ {}_2F_1\left(-\alpha-n, 1+\beta+n; 1-\alpha; \frac{1-x}{2}\right),$$

where ${}_2F_1\left(a_1, a_2; b; x\right)$ is the hypergeometric function defined by

$${}_2F_1\left(a_1, a_2; b; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k}\,\frac{x^k}{k!}.$$

The first integral given in Theorem 8 is $H(x,y) = (1-x)^\alpha h_1(x,y)/h_2(x,y)$, where

$$\begin{aligned} h_1 &= n\,(1+\alpha+\beta+n)\,(x^2-1) \\ &\quad \times {}_2F_1\left(1-n, 2+\alpha+\beta+n; 2+\alpha; \frac{1-x}{2}\right) \\ &\quad + (\alpha+1)\,((\alpha+\beta)x + (\alpha-\beta) - 2y) \\ &\quad \times {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right), \\ h_2 &= (\alpha-1)\,((\alpha-\beta)x + (\alpha+b) + 2y) \\ &\quad \times {}_2F_1\left(-\alpha-n, 1+\beta+n; 1-\alpha; \frac{1-x}{2}\right) \\ &\quad + (\alpha+n)\,(1+\beta+n)\,(x^2-1) \\ &\quad \times {}_2F_1\left(1-\alpha-n, 2+\beta+n; 2-\alpha; \frac{1-x}{2}\right). \end{aligned}$$

The first integral as given in Theorem 8 is $f_1(x,y)/f_2(x,y)$, and we notice that $H(x,y) = cf_1(x,y)/f_2(x,y)$ where $c \in \mathbf{R} - \{0\}$. As before, we do not write c in terms of the parameters of the system to simplify notation.

When $n \in \mathbf{N}$, we have that (11) corresponds to the equation of Jacobi polynomials and $w_1(x)$ coincides with the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, and the invariant algebraic curve given in [**7**] corresponds to $f_1(x,y) = 0$, where $f_1(x,y) := w_1'(x) - g(x,y)w_1(x)$.

If $A_2(x) = 1+x^2$ the complex affine change of variable $x = \mathrm{i}X$ makes this case equivalent to the previous one, as it is shown in [**7**].

We have re-encountered by this method all the examples appearing in [**7**] from a unified point of view. In addition, in this work we have given an explicit expression of a first integral for each case and for any value of the parameter $n \in \mathbf{R}$. To this end, we have found invariants for the system and we have applied the generalization of Darboux's method as explained in [**14**] to be able to construct a first integral which is, in general, of non-Liouvillian type.

3.2 A Lotka-Volterra system. As it has been explained in the introduction, the first counterexample to Lins Neto conjecture was given by Moulin-Ollagnier in [**19**]. His example is a quadratic system with two invariant straight lines and an irreducible invariant algebraic curve $f(x,y) = 0$ of degree 2ℓ when $\ell \in \mathbf{N}$. This gives a family of systems depending on the parameter ℓ which have a Darboux inverse integrating factor when $\ell \in \mathbf{N}$ but no rational first integral. The method used in [**19**] only shows the existence of such invariant algebraic curve but no closed formula to compute it is given. We give an explicit expression for an invariant by means of Bessel functions for any value of $\ell \in \mathbf{R} - \{1/2\}$ which, in the particular case $\ell \in \mathbf{N}$, degenerates to the algebraic curve encountered in [**19**].

We show that, after a birrational transformation, this example coincides with a system constructed with the method explained in Section 2. A *birrational transformation* is a rational change of variables whose inverse is also rational. This kind of transformation brings polynomial systems such as (2) to polynomial systems and do not change the Liouvillian or non-Liouvillian character of the first integral.

Let us consider the system appearing in [**19**] but assuming that $\ell \in \mathbf{R} - \{1/2\}$

$$\dot{x} = x\left(1 - \frac{x}{2} + y\right), \quad \dot{y} = y\left(-\frac{2\ell+1}{2\ell-1} + \frac{x}{2} - y\right). \tag{12}$$

We make the birrational transformation

$$x = \frac{4uv}{1-2\ell}, \quad y = \frac{1-2\ell}{4v},$$

whose inverse is

$$u = xy, \quad v = \frac{1-2\ell}{4y}.$$

By this transformation, system (12) becomes

$$\dot{u} = \frac{2u}{1-2\ell}, \quad \dot{v} = \frac{1-2\ell}{4} + \frac{2\ell+1}{2\ell-1}\,v + \frac{2u}{2\ell-1}\,v^2. \tag{13}$$

We notice that the equation for the orbits satisfied by the variable v as a function of u is a Ricatti equation.

Let us consider $g(u,v) := v$ and the linear differential equation of order 2 given by

$$u\, w''(u) + \frac{1}{2}\,(1+2\ell)\, w'(u) - \frac{1}{8}\,(1-2\ell)^2\, w(u) = 0. \tag{14}$$

Applying the method given in the previous section, this linear differential equation gives system (13) modulus a change of time.

A set of two fundamental solutions for equation (14) is given by

$$\begin{aligned} w_1(u) &= u^{(1-2\ell)/4}\, I_{(1/2)-\ell}\left((1-2\ell)\sqrt{\frac{u}{2}}\right), \\ w_2(u) &= u^{(1-2\ell)/4}\, I_{\ell-(1/2)}\left((1-2\ell)\sqrt{\frac{u}{2}}\right), \end{aligned} \tag{15}$$

provided that ℓ is not of the form $(1-2r)/2$, with r an integer number, because in this case w_1 and w_2 are linearly dependent. The function $I_\nu(u)$ is the Modified Bessel function defined by the solution of the second order differential equation

$$u^2\, w''(u) + u\, w'(u) - (u^2+\nu^2)\, w(u) = 0, \tag{16}$$

and being bounded when $u \to 0$ in any bounded range of $\arg(u)$ with $\mathbf{R}e(u) \geq 0$. See [**1**] for further information about this function.

Hence, by Theorem 8 we have that $H(u,v) = f_1(u,v)/f_2(u,v)$, where $f_i(u,v) := w_i'(u) - v w_i(u)$ for $i = 1,2$, is a first integral for system (13). For the sake of simplicity, we consider the following renaming of the independent variable $u = 2z^2/(1-2\ell)^2$. This is not a birrational transformation and that's why we only use it to simplify notation. The function H becomes:

$$H = \frac{(1-2\ell)^2 I_{((1+2\ell)/2)}(z) - 4vz\, I_{((2\ell-1)/2)}(z)}{(1-2\ell)^2 I_{-((1+2\ell)/2)}(z) - 4vz\, I_{-((2\ell-1)/2)}(z)}. \tag{17}$$

By Theorem 7 we have that $f_i(u,v)$, $i = 1,2$ are invariants with the same polynomial cofactor k for system (13), so the curve $f(u,v) = 0$ given by $f(u,v) = \pi z^{2\ell+1}(f_1^2(u,v) - f_2^2(u,v))$ is also an invariant. We multiply by π only for esthetic reasons.

Now we assume that $\ell \in \mathbf{N}$ and we show that $f = 0$ defines an invariant *algebraic* curve. To this end we use the following formulas appearing in [**1, 26**]. When $\nu - 1/2 = n \in \mathbf{Z}$, we define $c(n) = -n\pi\sqrt{-1}/2$ and the following relation is satisfied:

$$
\begin{aligned}
I_\nu(z) = -\frac{1}{\sqrt{z}}\, \mathrm{e}^{c(n)} \sqrt{\frac{2}{\pi}} \Bigg\{ & \sinh\,(c(n) - z) \sum_{k=0}^{\lfloor (2|\nu|-1)/4 \rfloor} \\
& \times \frac{(|\nu| + 2k - (1/2))!}{(2k)!(|\nu| - 2k - (1/2))!(2z)^{2k}} \; + \cosh\,(c(n) - z) \\
& \times \sum_{k=0}^{\lfloor (2|\nu|-3)/4 \rfloor} \frac{(|\nu| + 2k + (1/2))!}{(2k+1)!(|\nu| - 2k - (3/2))!(2z)^{2k+1}} \Bigg\},
\end{aligned} \tag{18}
$$

where $\lfloor x \rfloor$ stands for the greatest integer k such that $k \leq x$ and $|\nu|$ stands for the absolute value.

From the former equation we obtain the following two equalities, with $\nu - (1/2) = n \in \mathbf{Z}$ and $\ell \in \mathbf{N}$,

$$
I_\nu^2(z) - I_{-\nu}^2(z) = \frac{2}{\pi z} \sum_{k=0}^{n} (-1)^{k+1} \frac{(2n-k)!(2n-2k)!}{k!((n-k)!)^2} \left(\frac{1}{2z}\right)^{2(n-k)}, \tag{19}
$$

$$
\begin{aligned}
& I_{\ell+(1/2)}(z)\, I_{\ell-(1/2)}(z) - I_{-(\ell+(1/2))}(z)\, I_{-(\ell-(1/2))}(z) \\
& \quad = (-1)^\ell \frac{2}{\pi z} \left[\left(\sum_{i=0}^{\lfloor \ell/2 \rfloor} \frac{(\ell+2i)!}{(2i)!(\ell-2i)!} \left(\frac{1}{2z}\right)^{2i} \right) \right. \\
& \qquad \times \left(\sum_{j=0}^{\lfloor (\ell-2)/2 \rfloor} \frac{(\ell+2j)!}{(2j+1)!(\ell-2j-2)!} \left(\frac{1}{2z}\right)^{2j-1} \right) \\
& \qquad - \left(\sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \frac{(\ell+2i+1)!}{(2i+1)!(\ell-2i-1)!} \left(\frac{1}{2z}\right)^{2i-1} \right) \\
& \qquad\quad \left. \times \left(\sum_{j=0}^{\lfloor (\ell-1)/2 \rfloor} \frac{(\ell+2j-1)!}{(2j)!(\ell-2j-1)!} \left(\frac{1}{2z}\right)^{2j} \right) \right].
\end{aligned} \tag{20}
$$

Then, we have that $f_1(z,v) = (1-2\ell)^2\, I_{\ell+(1/2)}(z) - 4vz\, I_{\ell-(1/2)}(z)$ and $f_2(z,v) = (1-2\ell)^2\, I_{-(\ell+(1/2))}(z) - 4vz\, I_{-(\ell-(1/2))}(z)$, and we write f arranged in powers of v:

$$\begin{aligned} f(z,v) = \pi z^{2\ell+1}\Big(& (1-2\ell)^4 (I^2_{\ell+(1/2)}(z) - I^2_{-(\ell+(1/2))}(z)) \\ & - 8vz(1-2\ell)^2 (I_{\ell+(1/2)}(z)\, I_{\ell-(1/2)}(z) \\ & - I_{-(\ell+(1/2))}(z)\, I_{-(\ell-(1/2))}(z)) \\ & + 16v^2z^2 (I^2_{\ell-(1/2)}(z) - I^2_{-(\ell-(1/2))}(z))\Big). \end{aligned}$$

Let us consider each coefficient of v in $f(z,v)$ separately, and we will show that it is an even polynomial in the variable z. The coefficient in $f(z,v)$ of v^0 is:

$$\pi z^{2\ell+1}(1-2\ell)^4 (I^2_{\ell+(1/2)}(z) - I^2_{-(\ell+(1/2))}(z)),$$

which by equation (19) is an even polynomial in the variable z of degree 2ℓ. The coefficient in $f(z,v)$ of v^2 is:

$$16\pi z^{2\ell+3} (I^2_{\ell-(1/2)}(z) - I^2_{-(\ell-(1/2))}(z)),$$

which also by equation (19) is an even polynomial in the variable z of degree $2\ell+2$. Finally, the coefficient in $f(z,v)$ of v^1 is:

(21)

$$8\pi(1-2\ell)^2 z^{2\ell+2} (I_{\ell+(1/2)}(z) I_{\ell-(1/2)}(z) - I_{-(\ell+(1/2))}(z) I_{-(\ell-(1/2))}(z)),$$

which by equation (20) is an even polynomial in the variable z of degree 2ℓ.

Hence, we have that $f(z,v)$ is an even polynomial in the variable z of total degree $2\ell+4$. When rewriting $z = (1-2\ell)\sqrt{u}/\sqrt{2}$ we have that $f(u,v)$ is a polynomial of total degree $\ell+2$ which is irreducible. The fact of being irreducible is easily seen because it is a polynomial of degree two in v and it cannot be decomposed in linear factors (the discriminant is not a polynomial raised to an even power) and the coefficients of v^0 and v^2 do not have any root in common.

Undoing the birrational change of variables we get that $f(x,y)$ is an irreducible polynomial of degree 2ℓ given by:

$$\begin{aligned} f(x,y) \\ = x^{\ell+(1/2)}\, y^{\ell-(1/2)} \Big[2y \left(I^2_{\ell+(1/2)}(z) - I^2_{-(\ell+(1/2))}(z) \right) \\ - 2\sqrt{2}\,\sqrt{xy}\left(I_{\ell+(1/2)}(z)\, I_{\ell-(1/2)}(z) - I_{-(\ell+(1/2))}(z)\, I_{-(\ell-(1/2))}(z) \right) \\ + x\left(I^2_{\ell-(1/2)}(z) - I^2_{-(\ell-(1/2))}(z) \right) \Big], \end{aligned}$$

where z is the same variable as before, that is, $z = ((1-2\ell)\sqrt{xy})/\sqrt{2}$.

By equation (17) we can write the first integral for system (12) for any value of $\ell \in \mathbf{R} - \{(1/2)(1-2r) \mid r \in \mathbf{N}\}$:

$$H(x,y) = \frac{\sqrt{2y}\, I_{((1+2\ell)/2)}(z) - \sqrt{x}\, I_{((2\ell-1)/2)}(z)}{\sqrt{2y}\, I_{-((1+2\ell)/2)}(z) - \sqrt{x}\, I_{-((2\ell-1)/2)}(z)}.$$

We have studied system (12) for any value of the parameter $\ell \in \mathbf{R} - \{(1/2)(1-2r) \mid r \in \mathbf{N}\}$ giving an explicit expression for a first integral using Theorem 6 and the generalized Darboux's theory as explained in [**14**]. This first integral is not of Liouvillian type. Moreover, we give one of its invariants with a polynomial cofactor. In the particular case $\ell \in \mathbf{N}$, this invariant is the invariant algebraic curve whose existence was proved in [**19**].

3.3 A new example of a family of quadratic systems with an invariant algebraic curve of arbitrarily high degree. We give another example of a family of quadratic systems with an irreducible invariant algebraic curve of degree 2ℓ when $\ell \in \mathbf{N}$, where ℓ is a parameter of the family. This family also depends on the parameter $a \in \mathbf{R}$.

Let us consider the quadratic system

(22)

$$\begin{aligned} \dot{x} &= (2a-1)\ell x - a(2\ell-1)y + 2a(a-\ell)(2\ell-1)x^2 - 2a^2(2\ell-1)^2 xy, \\ \dot{y} &= y(2(2a-1)\ell + 2a(2a-2\ell-1)(2\ell-1)x - 4a^2(2\ell-1)^2 y), \end{aligned}$$

where $a, \ell \in \mathbf{R}$ which satisfy $a \neq 0$, $\ell \neq 1/2$ and $(2\ell-1)a^2 - 2\ell \neq 0$. A straightforward computation shows that system (22) has $y = 0$ and $y - x^2 = 0$ as invariant algebraic curves.

Let us consider the following birrational transformation $x = Y$, $y = XY^2$ whose inverse is $X = y/x^2$ and $Y = x$. In these new variables system (22) becomes

(23)
$$\begin{aligned}\dot{X} &= 2a(2\ell-1)(X-1)XY,\\ \dot{Y} &= \left((2a-1)\ell + a(2\ell-1)(2a-2\ell-X)Y - 2a^2(2\ell-1)^2XY^2\right)Y.\end{aligned}$$

By a change of the time variable we can divide this system by Y, and the resulting system coincides with the one described in Theorem 6 taking $A_2(X) := 2X(X-1)^2$, $A_1(X) := (2\ell - 2a + 3X)(X-1)$, $A_0(X) := \ell(1-2a)$ and $g(X,Y) := a(2\ell-1)Y/(X-1)$. The equation $A_2(X)w''(X) + A_1(X)w'(X) + A_0(X)w(X) = 0$ has the following set of fundamental solutions in this case:

$$\begin{aligned}w_1(X) &= (X-1)^{-\ell}\ {}_2F_1\left(\frac{1}{2}-\ell, -\ell; a-\ell; X\right),\\ w_2(X) &= (X-1)^{-\ell}X^{1-a+\ell}\ {}_2F_1\left(1-a, \frac{3}{2}-\ell; 2-a+\ell; X\right).\end{aligned}$$

By Theorem 7, $f_i(X,Y) = w_i'(X) - g(X,Y)w_i(X)$, $i = 1, 2$, define invariants with a polynomial cofactor for system (23). Moreover, by Theorem 8 we have a non-Liouvillian first integral given by $H(X,Y) = f_1(X,Y)/f_2(X,Y)$.

In the particular case $\ell \in \mathbf{N}$, we notice that $f_1(X,Y) = 0$ is a rational function. It is an easy computation to show that this rational function is a polynomial when rewritten in coordinates x and y. This polynomial gives place to an invariant algebraic curve of degree 2ℓ for system (22). That is, by undoing the birrational transformation, we deduce that $f_1(x,y)$ is an irreducible invariant algebraic curve for system (22), given by:

$$\begin{aligned}f_1(x,y) = 2(a-\ell)(\ell + (2\ell-1)ax)x^{2\ell-1}\ {}_2F_1\left(\frac{1}{2}-\ell, -\ell; a-\ell; \frac{y}{x^2}\right)\\ + \ell(2\ell-1)x^{2\ell-3}(x^2-y)\ {}_2F_1\left(\frac{3}{2}-\ell, 1-\ell; 1+a-\ell; \frac{y}{x^2}\right).\end{aligned}$$

It is easy to see that the polynomial $f_1(x,y)$ has degree 2ℓ, and the cofactor associated to the invariant algebraic curve $f_1(x,y) = 0$ is $\ell(2\ell-1)((2a-1) + 4a(a-\ell)x - 4(2\ell-1)a^2y)$.

The first integral for (22) is given by $H(x,y) = y^{a-\ell} f_1(x,y)/h(x,y)$, where

$$\begin{aligned} h(x,y) = 2\,(a-\ell-2)\, &\big[(a-\ell-1)\,x^2 \\ &+ (1-a-a(2\ell-1)x)\,y\big]\, x^{7-2a} \\ &\times {}_2F_1\left(1-a, \frac{3}{2}-a; 2-a-\ell; \frac{y}{x^2}\right) \\ &+ (a-1)\,(2a-3)\,x^{5-2a}\,(x^2-y)\,y \\ &\times {}_2F_1\left(2-a, \frac{5}{2}-a; 3-a+\ell; \frac{y}{x^2}\right). \end{aligned}$$

We notice that when both a and ℓ belong to the set of natural numbers, we have that $h(x,y) = 0$ is an invariant algebraic curve different from $f_1(x,y) = 0$. Then we have a quadratic system with a rational first integral $H(x,y)$ with arbitrary degree.

3.4 A complete family of quadratic systems with a center at the origin. In this subsection we give an example of a 3-parameter family of quadratic systems with a center at the origin which can be constructed using Theorem 11. The family encountered corresponds to the reversible case, see [**23**].

The family of quadratic systems depends on 12 parameters but, up to affine transformations and positive time rescaling, we get a family of 5 essential parameters. We have taken a system (9) and we have chosen $g(x,y) := y^2$, $h(x) := 2x(d\,x-1)/(1+ax)$ and $A(x) := 2b/(1+ax)$, where a, b, d are real parameters. Using Theorem 11, we have encountered the three-parameter family of quadratic systems next described. We remark that, in spite of the simplicity of the chosen polynomials $g(x,y)$, $A(x)$ and $h(x)$, we amazingly obtain the complete family of quadratic systems with a reversible center at the origin. We notice that other choices of the functions $g(x,y)$, $A(x)$ and $h(x)$ would give place to other families of polynomial systems.

Let us recall that a *center* is an isolated singular point of an equation (2) with a neighborhood foliated of periodic orbits. When the linear approximation of an equation (2) near a singular point has non null purely imaginary eigenvalues, the point can be a center or a focus. To distinguish between these two possibilities is the so-called *center* *problem*. Poincaré gave a method to solve it by defining a numerable set of values, called Liapunov-Poincaré constants, which are all zero when the singular point is a center and at least one of them is not null when it is a focus. When these constants are computed from a family of systems, they are polynomials on the coefficients of the family. Hilbert's Nullstellensatz ensures that there always exists a finite number of independent polynomials which generates the whole ideal made up with all these Liapunov-Poincaré polynomials. The zero-set of these independent polynomials gives place to the center subfamilies. The reader is referred to [**23, 24**] for a survey on this subject.

The computation of these center cases for the family of quadratic systems was done by Dulac [**13**] for the case of complex systems and a proof for real systems is given in [**18**]. We also refer to Bautin [**2**] who showed the existence of only three independent constants. The computation of the zero set of these three independent values gives place to four complete families of quadratic systems with a center at the origin which are described in [**24**].

Let us now consider an equation (9) such as $(1+ax)\,w'(x)+2b\,w(x) = 0$ and $g(x,y)$ and $h(x)$ as formerly defined. The rational equation as constructed in Theorem 11 is

$$\frac{dy}{dx} = \frac{-x + dx^2 - by^2}{y + axy},$$

which gives the corresponding quadratic planar system

$$\dot{x} = y + axy, \quad \dot{y} = -x + dx^2 - by^2. \tag{24}$$

We suppose that $ab(a+b)(a+2b)(a+b+d) \neq 0$. In case this value is zero, the origin of system (24) is still a center but with a Darboux integrating factor instead of a Darboux first integral. This particular case can also be studied by our method, but we do not write it to avoid giving examples without essential differences.

By Theorem 12 we have that $f(x,y)$ is an invariant of system (24) with cofactor $-2by$, where $f(x,y)$ is given by

$$\begin{aligned} f(x,y) = {} & b\,(a+b)\,(a+2b)\,w(x) - (a+b+d)\,(1+ax)^{-2b/a} \\ & - b\,(a+b)\,(a+2b)\,y^2 \;+\; b\,(a+2b)\,d\,x^2 \\ & - 2\,b\,(a+b+d)\,x \;+\; a \;+\; b \;+\; d, \end{aligned} \tag{25}$$

with $w(x)$ any nonzero solution of $(1+ax)w'(x)+2bw(x)=0$, that is, $w(x)=C(1+ax)^{-2b/a}$.

Choosing $C=(a+b+d)\,(b(a+b)(a+2b))^{-1}$ we get an invariant conic. System (24) has two invariant algebraic curves, the former conic with cofactor $-2by$ and an invariant straight line given by $1+ax=0$ with cofactor y. The Darboux first integral

$$H(x,y)=(1+ax)^{2b/a}f(x,y)$$

coincides with the first integral described in Theorem 14.

The origin of this system is a center since it is a monodromic singular point with a continuous first integral defined in a neighborhood of it. This example addresses the issue that other families of polynomial systems of higher degree with a center at the origin can be easily obtained by this method, avoiding the cumbersome computation of Poincaré-Liapunov constants.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (Milton Abramowitz and Irene A. Stegun, eds.), Dover Publ., Inc., New York, 1992 (reprint of the 1972 edition).

2. N.N. Bautin, *On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or centre type*, Trans. Amer. Math. Soc. **100** (1954), 397–413.

3. M. Berthier, D. Cerveau and A. Lins Neto, *Sur les feuilletages analytiques réels et le problème du centre*, J. Differential Equations **131** (1996), 244–266.

4. M. Carnicer, *The Poincaré problem in the nondicritical case*, Ann. Math. **140** (1994), 289–294.

5. D. Cerveau and A. Lins Neto, *Holomorphic foliations in* **CP**(2) *having an invariant algebraic curve*, Ann. Inst. Fourier **41** (1991), 883–903.

6. J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *On the integrability of two-dimensional flows*, J. Differential Equations **157** (1999), 163–182.

7. J. Chavarriga and M. Grau, *Invariant algebraic curves linear in one variable for planar real quadratic systems*, Appl. Math. Comput. **138** (2003), 291–308.

8. ———, *A family of quadratic polynomial differential systems non-integrables Darboux and with algebraic solutions of arbitrarily high degree*, Appl. Math. Lett. **16** (2003), 833–837.

9. C. Christopher, *Invariant algebraic curves and conditions for a centre*, Proc. Roy. Soc. Edinburgh **124** (1994), 1209–1229.

10. ———, *Liouvillian first integrals of second order polynomial differential systems*, Electronic J. Differential Equations **1999** 1–7.

11. C. Christopher and J. Llibre, *A family of quadratic polynomial differential systems with invariant algebraic curves of arbitrarily high degree without rational first integrals*, Proc. Amer. Math. Soc. **130** (2002), 2025–2030 (electronic).

12. G. Darboux, *Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré* (*Mélanges*), Bull. Sci. Math. **32** (1878), 60–96; 123–144; 151–200.

13. H. Dulac, *Détermination et intégration d'une certaine classe d'équations différentielles ayant pour point singulier un centre*, Bull. Sciences Math. **32** (1908), 230–252.

14. I.A. García and J. Giné, *Generalized cofactors and nonlinear superposition principles*, Appl. Math. Lett. **16** (2003), 1137–1141.

15. J.P. Jouanolou, *Équations de Pfaff algébriques*, Lecture Notes in Math., vol. 708, Springer, Berlin, 1979.

16. A. Lins Neto, *Folheaçõesem* **CP**(2) *com curva algébrica invariante e singularidades radiais*, Instituto de Estudios con Iberoamerica y Portugal, Seminarios Temáticos, Vol. **2**, Fasc. III, 1998.

17. ———, *Some examples for the Poincaré and Painlevé problems*, Ann. Scient. Éc. Norm. Sup. **35** (2002), 231–266.

18. V.A. Lunkevich and K.S. Sibirskiĭ, *Integrals of a general quadratic differential system in cases of a center*, Diff. Equations **18** (1982), 563–568.

19. J. Moulin Ollagnier, *About a conjecture on quadratic vector fields*, J. Pure Appl. Algebra **165** (2001), 227–234.

20. R. Moussu, *Une démonstration d'un théorème de Lyapunov-Poincaré*, Astérisque **98–99** (1982), 216–223.

21. H. Poincaré, *Sur l'intégration algébrique des équations différentielles du premier ordre et du premier degré* Rend. Circ. Mat. Palermo **5** (1891), 161–191.

22. M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. **279** (1983), 215–229.

23. D. Schlomiuk, *Algebraic paticular integrals, integrability and the problem of center*, Trans. Amer. Math. Soc. **338** (1993), 799–841.

24. ———, *Algebraic and geometric aspects of the theory of polynomial vector fields*, in *Bifurcations and periodic orbits of vector fields*, (Montreal, PQ, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **408**, Kluwer Acad. Publ., Dordrecht, 1993, pp. 429–467.

25. M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

26. Wolfram Research, Inc., `http://functions.wolfram.com`, 2000–2003.

27. H. Żołądek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal. **4** (1994), 79–136.

28. ———, *Remarks on: "The classification of reversible cubic systems with center,"* Topol. Methods Nonlinear Anal. **8** (1996), 335–342.

Laboratoire de Mathématiques et Physique Théorique, C.N.R.S. UMR 6083, Faculté des Sciences et Techniques, Université de Tours, Parc de Grandmont 37200 Tours, France
E-mail address: giacomin@celfi.phys.univ-tours.fr

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain
E-mail address: gine@eps.udl.es

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain
E-mail address: mtgrau@matematica.udl.es