

## REGULAR COMPONENTS OF MODULI SPACES OF STABLE MAPS AND $K$ -GONAL CURVES

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ABSTRACT. Here we prove for certain integers  $g, r, d$  and  $k$  the existence of a generically smooth irreducible component of the moduli space of stable maps  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, (k, d))$  with the expected dimension. As a byproduct, we obtain the existence of a generically smooth component of dimension  $\rho(g, r, d) := g - (r + 1)(g + r - d)$  for the Brill-Noether locus  $W_d^r(C)$  of a general  $k$ -gonal curve  $C$  of genus  $g$ .

**1. The statements.** For any complex projective variety  $Y$  and any class  $\beta \in H_2(Y, \mathbf{Z})$ , one considers the moduli space  $M_g^-(Y, \beta)$  of all stable maps  $f : C \rightarrow Y$ , with  $C$  a reduced connected nodal curve of arithmetic genus  $g$  and  $f_*([C]) = \beta$  (see [7] for the construction of these moduli spaces). The *expected dimension* of the algebraic stack  $M_g^-(Y, \beta)$  is  $\dim(Y)(1 - g) + 3g - 3 - b \cdot \omega_Y$ . For all integers  $g, r, d$ , set  $\rho(g, r, d) := g - (r + 1)(g + r - d) = (r + 1)d - rg - r(r + 1)$  (the so-called Brill-Noether number). As in [6] we are interested in the case in which  $Y = \mathbf{P}^1 \times \mathbf{P}^r$ , and we look for irreducible components,  $V$ , of  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, \beta)$  which are *good*, i.e., such that  $V$  is generically smooth and with the expected dimension. When  $Y = \mathbf{P}^1 \times \mathbf{P}^r$  the class  $\beta$  is given by a pair  $(k, d)$  of non-negative integers and in this case the dimension of a good component of  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, \beta)$  is  $\rho(g, r, d) + 3g - 3 + 2k - g - 2$ . The main aim of this paper is the proof of the following result.

**Theorem 1.1.** *Fix positive integers  $g, r, d$  and  $k$  such that  $(g+2)/2 \geq k \geq r + 3 \geq 6$ ,  $\rho(g, r, d) \geq 0$ , and  $g \leq (r + 1)\lfloor d/r \rfloor - r - 3$ . Then there exists a good component of  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, (k, d))$ .*

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As in [1, 2, 6], we will use in an essential way the smoothing results for reducible nodal curves in  $\mathbf{P}^r$  proved in [8, 11]. The case  $\rho(g, r, d) \geq 0$  of Theorem 1.1 is related to [4, Corollary 2.3.2]. To make explicit this connection, in Section 3 we will prove the following result.

**Theorem 1.2.** *Fix positive integers  $g, r, d$  and  $k$  with  $k \geq 3$ ,  $(g + 2)/2 > k \geq r + 2 \geq 5$ ,  $d \geq r(r + 3)$  and  $\rho(g, r, d) \geq 0$ . Let  $C$  be the general  $k$ -gonal curve of genus  $g$ . Let  $R \in \text{Pic}^k(C)$  be the  $k$ -gonal pencil. Then there exists an irreducible and generically smooth component,  $V$ , of  $W_d^r(C)$  with  $\dim(V) = \rho(g, r, d)$  and such that for a general  $L \in V$ ,  $L$  is very ample,  $h^0(X, L) = r + 1$ ,  $h^0(X, L \otimes R^*) = 0$ .*

Except for the very ampleness of a general  $L \in V$  and the generic smoothness of  $V$  Theorem 1.2 is a particular case of [4, Corollary 2.3.2]. Nevertheless, we believe that Theorem 1.2 has some interest because the methods of [3, 4] seem to be adapted to dimensional computations but not to tangent space computations, and hence it seems that those methods cannot be used to check the generic smoothness of some components of a Brill-Noether scheme  $W_d^r(C)$ .

**2. Proof of Theorem 1.1.** For any locally complete intersection subscheme  $T \subset \mathbf{P}^r$ , let  $N_T$  be its normal bundle. For any locally complete intersection subscheme  $T \subset \mathbf{P}^1 \times \mathbf{P}^r$ , let  $N_{T/(1,r)}$  be its normal bundle. A finite subset  $S$  of  $\mathbf{P}^r$  is said to be in linearly general position if, for every subset  $A$  of  $S$ , the linear span,  $\langle A \rangle$ , of  $A$  has dimension  $\min\{r, \text{card}(A) - 1\}$ .  $M(g; k)$  will denote the set of all smooth  $k$ -gonal curves of genus  $g$  and  $M(g; k)^-$  its closure in  $M_g^-$ . Hence if  $2 \leq k \leq (g + 2)/2$ , then  $M(g; k)$  and  $M(g; k)^-$  are irreducible and of dimension  $2g + 2k - 5$ . We need the following well-known lemma [10].

**Lemma 2.1.** *Let  $D \subset \mathbf{P}^r$  be a rational normal curve. Then  $N_D$  is the direct sum of  $r - 1$  line bundles of degree  $r + 2$ .*

We recall that for any  $S \subset \mathbf{P}^r$  with  $\text{card}(S) = r + 3$  and  $S$  in linearly general position there is a unique rational normal curve  $D \subset \mathbf{P}^r$  with  $S \subset D$ .

**Lemma 2.2.** *Fix integers  $r, t$  with  $r \geq 3$  and  $1 \leq t \leq r + 3$ . Let  $C \subset \mathbf{P}^r$  be a locally complete intersection curve and  $S \subset C_{\text{reg}}$  with  $\text{card}(S) = t$ . Assume  $h^1(C, N_C) = 0, S$  in linearly general position and that the general rational normal curve  $D \subset \mathbf{P}^r$  with  $S \subset D$  is transversal to  $C$ . Set  $X := C \cup D$ . Then  $p_a(X) = p_a(C) + t - 1, \text{deg}(X) = \text{deg}(C) + r$  and  $h^1(X, N_X) = 0$ . If  $C$  is smooth and connected and  $h^1(C, N_C(-S)) = 0$ , then the nodal curve  $X$  is smoothable inside  $\mathbf{P}^r$ . Furthermore, for a general  $S' \subset C_{\text{reg}}$  with  $\text{card}(S') = r + 3$ , we have  $h^1(X, N_X(-S')) = 0$ .*

*Proof.* The case  $t \leq r + 2$  was proved in [1, Lemma I.2], for  $r = 3$  and [2, Lemma 2.3] for  $t = r + 2$  and iteration of Lemma 2.2 for  $t \leq r + 1$ , for  $r \geq 4$ . By [8, Corollary 3.2, Theorem 4.1] or [11, pages 30–31], the restricted normal bundle  $N_X|C$  (respectively  $N_X|D$ ) is obtained from  $N_C$  (respectively  $N_D$ ) making  $t$  positive elementary transformations, one for each point of  $S$ ; for each  $P \in S$  the associated positive elementary transformation is uniquely determined by the tangent line to  $D$  (respectively to  $C$ ) at  $P$ , or, seen  $N_X|C$  (respectively  $N_X|D$ ) as a quotient of  $TP^r|C$  (respectively  $TP^r|D$ ), as the quotient by the plane spanned by the two tangent lines to  $C$  and to  $D$  at  $P$ . Hence  $N_C$  (respectively  $N_D$ ) is a subsheaf of  $N_X|C$  (respectively  $N_X|D$ ) and the associated quotient sheaf is a skyscraper sheaf supported by  $S$  and with length  $\text{card}(S)$ . Thus we have  $h^1(C, N_X|C) = 0$  and  $N_X|D$  is the direct sum of line bundles of degree at least  $r + 2$  (Lemma 2.1). Hence,  $h^1(D, N_X|D) = 0$  and the restriction map  $H^0(D, N_X|D) \rightarrow N_X|S$  is surjective; here we use  $t \leq r + 3$ . From the Mayer-Vietoris exact sequence

$$(1) \quad 0 \longrightarrow N_X \longrightarrow N_X|C \oplus N_X|D \longrightarrow N_X|S \longrightarrow 0$$

we obtain  $h^1(X, N_X) = 0$ . Now we check the last part of Lemma 2.2. By semi-continuity if  $h^1(C, N_C(-S)) = 0$ , then  $h^1(C, N_C(-S')) = 0$  for a general  $S'$ . By twisting the Mayer-Vietoris exact sequence (1) with  $O_X(-S')$  we obtain  $h^1(X, N_X(-S')) = 0$ . The smoothability of  $X$  follows from [8, Theorem 4.1].

The case  $t = r + 3$  of Lemma 2.2 could be done for  $r = 3$  using the case  $k = 3$  of [1, Lemma I.2]. To apply Lemma 2.2 we need to find curves,  $C$ , with  $h^1(C, N_C(-S)) = 0$ . We will use the following observation.

*Remark 2.3.* Fix integers  $r, d$  with  $d \geq r \geq 3$ . Let  $C \subset \mathbf{P}^r$  be the general smooth rational curve of degree  $d$ . Since  $C \cong \mathbf{P}^1$ ,  $N_C$  is the direct sum of  $r - 1$  line bundles, say of degree  $a_1, \dots, a_{r-1}$  with  $a_1 \geq \dots \geq a_{r-1}$ . Since  $\deg(N_C) = (r + 1)d - 2$ , we have  $\sum_{i=1}^{r-1} a_i = (r + 1)d - 2$ . By [10] the vector bundle  $N_C$  is rigid, i.e.,  $a_1 - a_{r-1}^2 = 1$ . Hence  $a_{r-1} = [(r + 1)d - 2]/(r - 1)^3 r + 2$ . Hence for any integer  $t$  with  $1^2 t^2 r + 3$  and any subset  $S \subset C$  with  $\text{card}(S) = t$  we have  $h^1(C, N_C(-S)) = 0$ .

The assertion concerning  $N_{D/(1,r)}$  in the next lemma is the key tool for our proof of Theorem 1.1.

**Lemma 2.4.** *Fix integers  $t$  and  $r$  with  $r \geq 3$  and  $1 \leq t \leq r + 3$ . Fix  $P \in \mathbf{P}^1$  and set  $M := \{P\} \times \mathbf{P}^r$ . Let  $Y \subset \mathbf{P}^1 \times \mathbf{P}^r$  be a nodal curve intersecting transversally  $M$  and  $S \subseteq Y \cap M$  with  $\text{card}(S) = t$  and such that  $S$  is in linearly general position as a subset of  $M \cong \mathbf{P}^r$ . Assume that the general rational normal curve  $D \subset M$  with  $S \subseteq D$  has  $Y \cap D = S$ . Fix any such  $D$ , and set  $X = Y \cup D$ . Hence  $X$  is a nodal curve,  $p_a(X) = p_a(Y) + t - 1$  and if  $Y$  has type  $(k, d)$ , then  $X$  has type  $(k, d + r)$ . Assume the existence of  $S' \subseteq S$  with  $\text{card}(S') = \min\{t, r + 2\}$  and such that for the datum  $(S', D)$  the tangent directions to  $Y$  at the points of  $S'$  are general (as lines through the origin in the corresponding tangent spaces to  $\mathbf{P}^1 \times \mathbf{P}^r$ ). Then  $N_{X/(1,r)}|_D$  is the direct sum of  $r - 1$  line bundles of degree  $r + 2$  and one line bundle of degree  $t$ .*

*Proof.*  $X$  has type  $(k, d + r)$  because  $D$  has type  $(0, r)$ . Since  $Y$  is transversal to  $M$ ,  $S \subset Y_{\text{reg}}$  and  $X$  is nodal. Hence  $p_a(X) = p_a(Y) + t - 1$ . By Lemma 2.1  $N_{D/(1,r)}$  is the direct sum of  $r - 1$  line bundles of degree  $r + 2$  and a trivial line bundle. By [8, Corollary 3.2] or [11, pages 30–31],  $N_{X/(1,r)}|_D$  is obtained from  $N_{D/(1,r)}$  making  $t$  positive elementary transformations. At each  $P \in S$  the corresponding positive elementary transformation is uniquely determined by the tangent direction to  $Y$  at  $P$  and if this tangent direction (for fixed  $S$  and  $D$ ) is general, then the positive elementary transformation is general and hence it increases by one the lower degree rank one factor of  $N_{D/(1,r)}$ . If  $t \leq r + 2$ , then by assumption all the positive elementary transformations are independently general and hence we conclude. If  $t = r + 3$ , at least this is true for a subset  $S'$  of  $S$  with  $\text{card}(S') = r + 2$ . Call  $N$  the

subsheaf of  $N_{X/(1,r)}|D$  obtained making only the positive elementary transformations associated to  $S'$ . By the first part  $N$  is the direct sum of  $r$  line bundles of degree  $r + 2$ . Since  $N_{X/(1,r)}|D$  is obtained from  $N$  making one positive elementary transformation, we conclude.

*Remark 2.5.* For fixed  $D, S$  and  $S'$  the assumption on the tangent directions to  $Y$  at the points of  $S'$  given in Lemma 2.4 is satisfied by a sufficiently near curve  $Y' \subset \mathbf{P}^1 \times \mathbf{P}^r$  with  $S \subseteq Y'$  if  $h^1(Y, N_{Y/(1,r)}(-2S')) = 0$  (see [9, 1.5], for a similar situation).

*Remark 2.6.* Fix  $P \in \mathbf{P}^1$  and set  $M := \{P\}'\mathbf{P}^r$ . Fix  $S \subset M$  with  $\text{card}(S) = r + 3$  and  $S$  in linearly general position. Fix  $S' \subset S$  with  $\text{card}(S') = r + 2$ . Let  $D \subset M$  be the unique rational normal curve containing  $S$ . Let  $Y = Y_1 \cup \dots \cup Y_{r+2} \subset \mathbf{P}^1 \times \mathbf{P}^r$  be the nodal curve of type  $(r+1, r(r+2))$  obtained in the following way. Order the points  $P_i$ ,  $1 \leq i \leq r+3$ , of  $S$  with  $P_{r+3} \notin S'$ . Let  $Y_1$  be the general curve of type  $(1, r)$  containing  $P_1$  and  $P_{r+3}$ . For  $2 \leq i \leq r+1$ , let  $Y_i$  be the general curve of type  $(1, r)$  containing  $P_i$  and intersecting  $Y_{i-1}$ . Thus  $Y_i \cong \mathbf{P}^1$  for every  $i$ ,  $Y$  is nodal and connected,  $p_a(Y) = 0$ ,  $\text{deg}(Y) = r(r+2)$ ,  $Y$  is transversal to  $M$  and  $S \subset Y$ . For the fixed set  $S'$  and the fixed curve  $D$  it is easy to check that the tangent directions to  $Y$  along the points of  $S'$  are general; indeed, given  $r + 2$  general tangent directions  $L_i$ ,  $1 \leq i \leq r + 2$ , in the tangent space of  $\mathbf{P}^1 \times \mathbf{P}^r$  at  $P_i$  we may find  $Y$  with that tangent directions. It is also easy to check the condition  $h^1(Y, N_{Y/(1,r)}(-2S')) = 0$  considered in Remark 2.5.

*Proof of Theorem 1.1.* Set  $c := \lfloor g/(r+1) \rfloor$  and  $b := g - c(r+1)$ . Hence  $0 \leq b \leq r$ . Set  $m := d - cr - b$ . First assume  $b > 0$ . Let  $C_0 \subset \mathbf{P}^r$  be a general smooth rational curve of degree  $\mu$ . Take a smooth rational curve  $D \subset \mathbf{P}^r$  with  $\text{deg}(D) = b$  and intersecting quasi-transversally  $C_0$  at exactly  $b + 1$  points spanning a  $b$ -dimensional linear space. By Remark 2.3 we may apply Lemma 2.2 to the pair  $(C_0, D)$ . Let  $Y$  be a sufficiently general smoothing of  $C_0 \cup D$ . Take a smooth rational normal curve  $D' \subset \mathbf{P}^r$  intersecting  $Y$  at exactly  $r + 2$  general points and quasi-transversally. We may apply Lemma 2.2 to the pair  $(Y, D')$ . If  $c = 1$ , we stop. If  $c > 1$  we continue  $c - 1$  times, each time adding a rational normal curve  $D''$  intersecting the previous curve  $Y''$  at exactly  $r + 2$  general points and then smoothing  $Y'' \cup D''$ . By Lemma 2.2 we

conclude. If  $b = 0$  we omit the first step; we start with  $C_0$  and just add  $c$  rational normal curves, each of them intersecting the previous curve exactly at  $r + 2$  points and quasi-transversally. In this way we cover exactly the set of triples  $(g, r, d)$  considered in the statement of Theorem 1.1. Now we work with a smooth curve  $W \subset \mathbf{P}^1 \times \mathbf{P}^r$  and we fix  $r + 2$  points of  $W$  which are in the same fiber of the degree  $k$  pencil. We assume that the projection of  $\mathbf{P}^1 \times \mathbf{P}^r$  onto the second factor maps  $W$  isomorphically onto a curve  $Y$ . We fix a general subset,  $S$ , of  $W$  with  $\text{card}(S) = r + 2$  and call  $S'$  its image in  $\mathbf{P}^r$ . We may assume that  $S'$  is in linearly general position. Let  $D \subset \mathbf{P}^r$  be the unique rational normal curve containing  $S'$ . Hence  $S'$  is a degree  $r + 2$  effective divisor on  $D \cong \mathbf{P}^1$ . We see  $D$  as a curve,  $A$ , of type  $(0, r)$  in  $\mathbf{P}^1 \times \mathbf{P}^r$ . We apply Lemma 2.4 to  $A$ . For general  $W$  and  $S'$  we may assume that  $D$  intersects quasi-transversally  $Y$  and that  $Y \cap D = S'$ . Set  $X := W \cup A$ . By construction  $X$  is a nodal curve. Now we repeat the proof of Lemma 2.2 using  $\mathbf{P}^1 \times \mathbf{P}^r$  as the ambient variety instead of  $\mathbf{P}^r$  and using Lemma 2.4 instead of Lemma 2.2. We use Remark 2.6 and (if necessary) the union of a curve given by Remark 2.6 of at most  $r - 1$  lines to start the inductive construction. As in the first part of the proof, in each inductive step we preserve the condition  $h^1(Y, N_{Y/(1,r)}(-2S')) = 0$ .

It is the use of Remark 2.6 to start the inductive proof of Theorem 1.1 which gave the restriction  $g \leq (r + 1)\lfloor d/r \rfloor - r - 3$  instead of, say,  $g \leq (r + 1)\lfloor d/r \rfloor - 1$ .

*Remark 2.7.* We claim that at least for  $g \gg r$  we may assume that the line bundle  $M \in \text{Pic}^d(X)$  corresponding to the general element of the good component,  $V$ , of  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, (k, d))$  constructed in the proof of Theorem 1.1 has  $h^0(X, M) = r + 1$  if  $d^2g + r$  and  $h^0(X, M) = d + 1 - g$  (i.e.,  $h^1(X, M) = 0$ ) if  $d^3g + r$ . We shall use the following observation to check the claim when at the last step we added a rational normal curve intersecting the previous curve at  $t^2r$  points or when  $d$  is not divisible by  $r$ . Our component  $V$  has in the boundary a reducible element in which after the first  $r + 2$  rational normal curves we insert another rational normal curve,  $E$ , intersecting the previous curve at  $b + 1$  points. We add also  $d - \lfloor d/r \rfloor r$  lines, say  $D_i$ ,  $1 \leq i \leq d - \lfloor d/r \rfloor r$ . Then we add  $\lfloor d/r \rfloor - r - 3$  rational normal curves intersecting the previous configuration at  $r + 3$  points in such a way that at least  $r - b$  of them intersects  $E$  and every

$D_i$  intersects one of these curves. The reducible nodal curve  $T$  obtained in this way has  $h^0(T, \mathbf{O}_T(1)) = r + 1$ . Hence, by semi-continuity, we have the claim for the general  $X \in V$  with  $V$  with the numerical invariants of  $T$ . For  $g \gg r$  this condition will not introduce any further restriction on the numerical invariants covered by Theorem 1.1. The line bundle  $R \in \text{Pic}^k(X)$  corresponding to the general element of the good component of  $M_g^-(\mathbf{P}^1 \times \mathbf{P}^r, (k, d))$  constructed in the proof of Theorem 1.1 has  $h^0(X, R) = 2$ . The case  $b = 0$  is similar and hence omitted.

**3. The general  $k$ -gonal curve: Proof of Theorem 1.2.** In this section we consider the scheme-theoretic structure of the irreducible components of the Brill-Noether locus  $W_d^r(C)$ , where  $C$  is a general  $k$ -gonal curve of genus  $g$ . The aim is the proof of Theorem 1.2.

*Proof of Theorem 1.2.* Notice that  $\rho(g+r+1, r, d+r) = \rho(g, r, d)$ . We make the inductive construction made in the proof of Theorem 1.1 with respect to the integer  $t = r + 2$ , say starting with a smooth curve  $Y''$  and adding a smooth rational normal curve  $D''$  intersecting  $Y''$  quasi-transversally and at exactly  $r + 2$  points. We assume that the unique irreducible component,  $A$ , of  $\text{Hilb}(\mathbf{P}^r)$  containing  $Y''$  contains curves with general moduli. By [2, first part of the proof of 3.1], the unique irreducible component,  $B$ , of  $\text{Hilb}(\mathbf{P}^r)$  containing  $Y'' \cup D''$  contains curves with general moduli. Set  $q := p_a(Y)$  and  $y := \deg(Y'')$ . We assume that  $Y''$  is a general  $k$ -gonal curve of genus  $q$ . Hence, as in the proof of Theorem 1.1, i.e., as in [6, proof of 3.1],  $Y'' \cup D''$  is a generalized covering of degree  $k$  and in particular it is a flat limit of a family of smooth  $k$ -gonal curves. The proof of the first part of [2, 3.1], gives that the two rational maps  $\alpha : A \rightarrow M_q^-$  and  $\beta : Y \rightarrow M_{q+r}^-$ , defined respectively in a neighborhood of  $Y''$  and in a neighborhood of  $Y'' \cup D''$ , have the same dimension near  $Y''$  and near  $Y'' \cup D''$ ; by the inductive assumption on  $Y''$  this dimension is  $\rho(q, y, r) + r^2 + 2r = \rho(q + r + 1, y + r, r) + r^2 + 2r$ . Since  $\text{Hilb}(\mathbf{P}^r)$  is smooth (and hence locally Cohen-Macaulay) at  $Y'' \cup D''$ , then  $\beta$  is flat (and in particular open) near  $Y'' \cup D''$ . If  $\text{Aut}(Y'')$  is trivial, it is easy to obtain  $D''$  and  $Y'' \cap D'' \subset D''$  such that  $\text{Aut}(Y'' \cup D'')$  is trivial. Thus  $Y'' \cup D''$  is a smooth point of  $M_{q+r+1}^-$ . Call  $B'$  the unique irreducible component containing  $Y'' \cup D''$ , when we see  $Y'' \cup D''$  as

a point of  $M_{q+r+1}^-(\mathbf{P}^1 \times \mathbf{P}^r, (k, y+r))$  using the construction of the proof of 1.1, i.e., the construction in [6], i.e., taking  $D'' \cap Y''$  in the same fiber of the degree  $k$  pencil  $Y'' \rightarrow \mathbf{P}^1$  and  $D''$  of type  $(0, r)$ , i.e.,  $D''$  contained in a slice  $M = \{P\} \times \mathbf{P}^r$  of  $\mathbf{P}^1 \times \mathbf{P}^r$ . We could easily obtain  $\text{Aut}(Y'')$  trivial for  $q \geq 3$ ; if  $q \leq 2$  we use  $Y'' \cap D''$  to obtain the triviality of  $\text{Aut}(Y'' \cup D'')$ ; alternatively, if  $\text{Aut}(Y'' \cup D'')$  is not trivial, we use a finite covering of  $M_{q+r+1}^-$  which is smooth near the counter images of the point representing  $Y'' \cup D''$ . Hence, by the openness of  $\beta$  near  $Y'' \cup D''$ ,  $B$  contains the general  $k$ -gonal curve, i.e., the induced rational map  $\gamma : B' \rightarrow M(q+r+1; k)^-$  is dominant. Hence, for a general  $C \in M(q+r+1; k)$  the fiber  $\gamma^{-1}(C)$  is not empty and of dimension  $\rho(q+r+1, r, y+r) = \rho(q+r+1, r, y+1)$  and it is an irreducible component of  $G_{q+r+1}^r(C)$ . In characteristic zero  $\gamma^{-1}(C)$  is generically smooth because  $B'$  is generically smooth [5, Corollary 16.23]. Hence we obtain Theorem 1.2 for the invariants  $g' := q+r+1$  and  $d' := y+r$  having it for the invariants  $q$  and  $y$ . We may obtain the same for  $q+t-1$  with  $1 \leq t \leq r+1$ , taking  $\text{card}(Y'' \cap D'') = t$ . In the cases  $t = r+1$  and  $t = r+2$  if  $h^0(Y'', \mathbf{O}_{Y''}(1)) = r+1$  we obtain  $h^0(Y'' \cup D'', \mathbf{O}_{Y'' \cup D''}(1)) = r+1$  and then by semi-continuity  $h^0(Y, \mathbf{O}_Y(1)) = r+1$  for a general smoothing,  $Y$ , of  $Y'' \cup D''$ . If  $t \leq r$  even if  $h^0(Y, \mathbf{O}_Y(1)) = r+1$  we could have  $h^0(Y'' \cup D'', \mathbf{O}_{Y'' \cup D''}(1)) > r+1$ . This gives no problem for  $G_{q+r+1}^r(C)$  but does not give (a priori) a result for  $W_{q+r+1}^r(C)$ . However, we are forced at the final step to be in this situation only if  $d > g+r$  and in this case the result is obvious taking non-special divisors. We have to start this inductive procedure. Using Remark 2.6 for doing that we are forced to assume  $d \geq r(r+3)$  (or a similar very unpleasant bound).

We will outline another proof of Theorem 1.2 which uses heavily [6]. The following result was proved in the case  $\rho(g, r, d) < 0$  in [6] (see [6, Proposition 3.1]); the same proof works in the case  $\rho(g, r, d) \geq 0$ ; just note that we use the word “injective” instead of “surjective” for the Petri maps because in [11, 3.3], it is assumed for  $C$  and proved for  $Y$  that the Petri map has maximal rank and for  $\rho \geq 0$  (respectively  $r < 0$ ) the Petri map has maximal rank if and only if it is injective (respectively surjective).



**Lemma 3.1.** *Fix non-negative integers  $g, r, d$  and  $k$  with  $d \geq r \geq 3$ ,  $k \geq r + 2$  and  $\rho(g, r, d) \geq 0$ . Assume the existence of a smooth non-degenerate curve  $C \subset \mathbf{P}^r$  with  $\deg(C) = d$ ,  $p_a(C) = g$ ,  $h^1(C, N_C) = 0$ ,  $h^0(C, \mathbf{O}_C(1)) = r + 1$  and such that the Petri map  $H^0(C, \mathbf{O}_C(1)) \otimes H^0(C, K_C(-1)) \rightarrow H^0(C, K_C)$  is injective. Assume the existence of a base-point free and simple  $R \in \text{Pic}^k(C)$  such that  $h^0(C, \mathbf{O}_C(1) \otimes R^*) = 0$  and  $R$  satisfies condition (5) of [6]. Then there exists a smooth non-degenerate curve  $Y \subset \mathbf{P}^r$  with  $\deg(Y) = d + r$ ,  $p_a(Y) = g + r + 1$ ,  $h^1(Y, N_Y) = 0$ ,  $h^0(Y, \mathbf{O}_Y(1)) = r + 1$  and such that the Petri map  $H^0(Y, \mathbf{O}_Y(1)) \otimes H^0(Y, K_Y(-1)) \rightarrow H^0(C, K_C)$  is injective and a base-point free and simple  $A \in \text{Pic}^k(Y)$  such that  $h^0(Y, \mathbf{O}_Y(1) \otimes A^*) = 0$  and  $A$  satisfies condition (5) of [6].*

In a very similar way (it corresponds to the case  $1 \leq t \leq r + 1$  of Lemma 2.2, while Lemma 3.1 corresponds to the case  $t = r + 2$ ) we have the following lemma.

**Lemma 3.2.** *Fix non-negative integers  $g, r, d, t$  and  $k$  with  $1 \leq t \leq r + 1$ ,  $d \geq r \geq 3$ ,  $k \geq r + 2$  and  $\rho(g, r, d) \geq 0$ . Assume the existence of a smooth non-degenerate curve  $C \subset \mathbf{P}^r$  with  $\deg(C) = d$ ,  $p_a(C) = g$ ,  $h^1(C, N_C) = 0$ ,  $h^0(C, \mathbf{O}_C(1)) = r + 1$  and such that the Petri map  $H^0(C, \mathbf{O}_C(1)) \otimes H^0(C, K_C(-1)) \rightarrow H^0(C, K_C)$  is injective. Assume the existence of a base-point free and simple  $R \in \text{Pic}^k(C)$  such that  $h^0(C, \mathbf{O}_C(1) \otimes R^*) = 0$  and  $R$  satisfies condition (5) of [6]. Then there exists a smooth non-degenerate curve  $Y \subset \mathbf{P}^r$  with  $\deg(Y) = d + r$ ,  $p_a(Y) = g + t - 1$ ,  $h^1(Y, N_Y) = 0$  and such that the Petri map  $H^0(Y, \mathbf{O}_Y(1)) \otimes H^0(Y, K_Y(-1)) \rightarrow H^0(C, K_C)$  is injective and a base-point free and simple  $A \in \text{Pic}^k(Y)$  such that  $h^0(Y, \mathbf{O}_Y(1) \otimes A^*) = 0$  and  $A$  satisfies condition (5) of [6]. If  $t = r + 1$ , we may find  $Y$  with  $h^0(Y, \mathbf{O}_Y(1)) = r + 1$ .*

We stress that to apply Proposition 3.3 of [11] on the Petri map we need complete linear systems. Hence to apply Lemmas 3.1 and 3.2 and copy the proof of [6, Theorem 2], we need to start the induction with several linearly normal curves in  $\mathbf{P}^r$ . To apply Remark 2.5 we need to take an integer  $q_0 \gg r$  and for each integer  $q$  with  $0 \leq q \leq q_0$  take the general linearly normal non-special curve of genus  $q$  and degree  $q + r$

in  $\mathbf{P}^r$ . It seems that this stupid problem gives an awful bound for Theorem 1.2, say  $g \gg r$ .

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#### REFERENCES

1. E. Ballico and Ph. Ellia, *Beyond the maximal rank conjecture for curves in  $\mathbf{P}^3$* , in *Space curves*, Lecture Notes in Math., vol. 1266, Springer-Verlag, Berlin, 1987, pp. 1–23.
2. E. Ballico and Ph. Ellia, *On the existence of curves with maximal rank in  $\mathbf{P}^n$* , *J. Reine Angew. Math.* **397** (1989), 1–22.
3. M. Coppens and G. Martens, *Linear series on 4-gonal curves*, *Math. Nachr.* **213** (2000), 35–55.
4. ———, *Linear series on a general  $k$ -gonal curve*, *Abh. Math. Sem. Univ. Hamburg* **69** (1999), 347–361.
5. D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, Berlin, 1999.
6. G. Farkas, *Regular components of moduli spaces of stable maps*, *Proc. Amer. Math. Soc.* **131** (2003), 2027–2036.
7. W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, in *Algebraic geometry* (J. Kollar, R. Lazarsfeld and D. Morrison, eds.), *Proc. Sympos. Pure Math.* (Santa Cruz 62), 1995; Part 2, 1997, pp. 45–96.
8. R. Hartshorne and A. Hirschowitz, *Smoothing algebraic space curves*, in *Algebraic geometry* (Sitges, 1983), *Lecture Notes in Math.*, vol. 1124, Springer-Verlag, Berlin, 1985, pp. 98–131.
9. D. Perrin, *Courbes passant par  $m$  points généraux de  $\mathbf{P}^3$* , *Bull. Soc. Math. France* **28/29** (1987).
10. G. Sacchiero, *Fibrati normali di curve razionali dello spazio proiettivo*, *Ann. Univ. Ferrara Sc. Math.* **26** (1980), 33–40.
11. E. Sernesi, *On the existence of certain families of curves*, *Invent. Math.* **75** (1984), 25–57.

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