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A FOURTH-ORDER FOUR-POINT RIGHT FOCAL BOUNDARY VALUE PROBLEM

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ABSTRACT. We are concerned with the unit interval right focal boundary value problem $-x^{(4)}(t) = f(x(t))$, $x(0) = x'(q) = x''(r) = x'''(1) = 0$. Under various assumptions on f and the real numbers $0 < q < r < 1$ we prove the existence of positive solutions for this boundary value problem by applying a generalization of the Leggett-Williams fixed point theorem, the Five Functionals Fixed-Point Theorem.

1. Introduction. The literature on positive solutions to boundary value problems is extensive. The recent book by Agarwal, Wong and O'Regan [**2**] gives a good overview for much of the work which has been done and the methods used. More specifically the monograph by Agarwal [**1**] gives a thorough discussion of previously known results related to right focal boundary value problems. Anderson [**3**] extended these known results by finding and giving conditions for the positivity of the Green's function for an n-point right focal boundary value problem. We will use these results in conjunction with the Five Functionals Fixed Point theorem [**8**] to give sufficient conditions for the existence of three positive solutions to a fourth-order four-point right focal boundary value problem. This theorem has been used successfully on similar third-order three-point right focal problems [**4**–**7**], and on certain second-order boundary value problems [**9, 10, 13**].

2. Preliminaries. We are concerned with the existence of three positive solutions to the fourth-order boundary value problem:

$$-x^{(4)}(t) = f(x(t)) \quad \text{for all} \quad t \in [0,1] \tag{1}$$

$$x(0) = x'(q) = x''(r) = x'''(1) = 0, \tag{2}$$

where $f : \mathbf{R} \to \mathbf{R}$ is continuous and $f(x) \geq 0$ for $x \geq 0$. Here

$$0 < q < r < 1;$$

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further conditions on the distances between boundary points will be imposed later in the paper. A solution of (1), (2) is nonnegative on $[0,1]$, nondecreasing on $[0,q]$, and nonincreasing on $[q,1]$.

In this paper we will assume the reader has an understanding of Green's functions and their applications. For the remainder of this section we will state the generalization of the Leggett-Williams fixed point theorem [**8**], which will be used to prove our main result, and provide some background results and definitions. For more on the Leggett-Williams and other fixed point theorems, see [**11, 12, 14**].

Definition 1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P,\ \lambda \geq 0$ implies $\lambda x \in P$;

(ii) $x \in P,\ -x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Definition 3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if

$$\alpha : P \longrightarrow [0,\infty)$$

is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if

$$\beta : P \longrightarrow [0,\infty)$$

is continuous and

$$\beta(tx + (1-t)y) \le t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P; then for nonnegative real numbers h, a, b, d and c we define the following convex sets:

$$\begin{aligned} P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in P : a \le \alpha(x),\ \ \gamma(x) \le c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \le d,\ \ \gamma(x) \le c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \le \alpha(x),\ \ \theta(x) \le b,\ \ \gamma(x) \le c\}, \end{aligned}$$

and

$$Q(\gamma, \beta, \psi, h, d, c) = \{x \in P : h \le \psi(x),\ \ \beta(x) \le d,\ \ \gamma(x) \le c\}.$$

The following fixed point theorem is a generalization of the Leggett-Williams fixed point theorem due to Avery [**8**].

Theorem 1. *Let P be a cone in a real Banach space E, c and M positive numbers, α and ψ nonnegative continuous concave functionals on P, and γ, β and θ nonnegative continuous convex functionals on P with*

$$\alpha(x) \le \beta(x) \quad \text{and} \quad \|x\| \le M\gamma(x)$$

for all $x \in \overline{P(\gamma, c)}$. Suppose

$$A : \overline{P(\gamma, c)} \longrightarrow \overline{P(\gamma, c)}$$

is completely continuous and there exists nonnegative numbers h, d, a, b with $0 < d < a$ such that:

(i) *$\{x \in P(\gamma, \theta, \alpha, a, b, c) : \alpha(x) > a\} \neq \varnothing$ and $\alpha(Ax) > a$ for $x \in P(\gamma, \theta, \alpha, a, b, c)$;*

(ii) *$\{x \in Q(\gamma, \beta, \psi, h, d, c) : \beta(x) < d\} \neq \varnothing$ and $\beta(Ax) < d$ for $x \in Q(\gamma, \beta, \psi, h, d, c)$;*

(iii) $\alpha(Ax) > a$ *for* $x \in P(\gamma, \alpha, a, c)$ *with* $\theta(Ax) > b$;

(iv) $\beta(Ax) < d$ *for* $x \in Q(\gamma, \beta, d, c)$ *with* $\psi(Ax) < h$.

Then A *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *such that*

$$\beta(x_1) < d, \quad a < \alpha(x_2), \quad \textit{and} \quad d < \beta(x_3) \quad \textit{with} \quad \alpha(x_3) < a.$$

3. Introduction to the $(4,4)$ **BVP.** We are concerned with proving the existence of three positive solutions of the fourth-order nonlinear right focal boundary value problem (1), (2), where $f : \mathbf{R} \to \mathbf{R}$ is continuous and $f(x) \geq 0$ for $x \geq 0$. The solutions of (1), (2) are the fixed points of the operator A defined by

$$Ax(t) = \int_0^1 G(t,s) f(x(s))\, ds.$$

Here $G(t,s)$, the Green's function for the related homogeneous equation

$$-x^{(4)}(t) = 0$$

satisfying boundary conditions (2), is given [**3**] by

$$G(t,s) = \begin{cases} s \in [0,q] : \begin{cases} u_1(t,s) & : t \leq s \\ u_1(t,s) - \dfrac{1}{6}(t-s)^3 & : t \geq s \end{cases} \\ s \in [q,r] : \begin{cases} u_2(t,s) & : t \leq s \\ u_2(t,s) - \dfrac{1}{6}(t-s)^3 & : t \geq s, \end{cases} \\ s \in [r,1] : \begin{cases} u_3(t,s) & : t \leq s \\ u_3(t,s) - \dfrac{1}{6}(t-s)^3 & : t \geq s, \end{cases} \end{cases} \tag{3}$$

where

$$u_i(t,s) = -\begin{vmatrix} 0 & t & (1/2)t^2 & (1/6)t^3 \\ (1/2)(q-s)^2 U(1-i) & 1 & q & (1/2)q^2 \\ (r-s)U(2-i) & 0 & 1 & r \\ 1 & 0 & 0 & 1 \end{vmatrix}$$

for $i = 1, 2, 3$, and the function U is the unit step function

$$U(k) = \begin{cases} 0 & : k < 0 \\ 1 & : k \geq 0. \end{cases}$$

The physical motivation for the fourth-order, four-point problem is a uniform cantilever beam with free vibration such that the left end is clamped and the right end is free with vanishing bending moment and shearing force, see Meirovitch [**15**]. That problem is formulated as

$$W^{(4)}(x) = f(W(x)) \quad \text{for all} \quad x \in [0,1] \tag{4}$$

$$W(0) = W'(0) = W''(1) = W'''(1) = 0. \tag{5}$$

In taking the limit in boundary conditions (2) as $q \to 0$ and $r \to 1$, we arrive at (5). Likewise the middle pair of functions in the Green's function (3) for (1), (2) approaches the Green's function for (4), (5), if we adjust for the negative sign in (1) that is absent in (4).

Lemma 1. *For all $t \in (0,1]$ and all $s \in (0,1]$,*

$$G(q,s) \geq G(t,s) > 0 \tag{6}$$

if $1 - q < \min\{q, 2(r-q)\}$, *i.e.,* $1/2 < q < 1/2 + q/2 < r < 1$.

Proof. From [**3**] we have that if $q \geq 1 - q$, $G(t,s) > 0$ for $t \in (0,1]$, $s \in (0,1]$. In the rest of the discussion, we will refer to the derivative of G with respect to t for fixed s; it is given by

$$\frac{d}{dt}\,G(t,s) = \begin{cases} s \in [0,q] : \begin{cases} \frac{1}{2}\,(t-s)^2 & : t < s \\ 0 & : t \geq s \end{cases} \\ s \in [q,r] : \begin{cases} \frac{1}{2}\,(q-t)(2s-t-q) & : t < s \\ -\frac{1}{2}\,(s-q)^2 & : t \geq s, \end{cases} \\ s \in [r,1] : \begin{cases} \frac{1}{2}\,(t-q)(t-2r+q) \\ \frac{1}{2}\,(t-q)(t-2r+q) - \frac{1}{2}\,(t-s)^2. \end{cases} \end{cases} \tag{7}$$

Fix $s \in [0,q]$. Then $G(0,s) = 0$ by the first boundary condition and $(d/dt)G(t,s)$ as noted above is such that G is monotone increasing on $[0,s)$ and constant on $[s,1]$. In particular, $G(s,s) = G(q,s) \geq G(t,s)$ for all $t \in [0,1]$.

Now choose $s \in [q,r]$. Again we have $G(0,s) = 0$ and $(d/dt)G(t,s) > 0$ on $[0,q)$; the second boundary condition yields $(d/dt)G(q,s) = 0$. For $t \in [q,1]$, (7) shows that $(d/dt)G(t,s) \leq 0$. Consequently, $G(q,s) \geq G(t,s)$.

Finally, suppose $s \in [r,1]$. As in the previous two cases, $G(0,s) = 0$ and $(d/dt)G(t,s) \geq 0$ for $t \in [0,q]$. Using (7), $(d/dt)G(t,s) \leq 0$ for $t \in [q,1]$ if $1 \leq 2r - q$. $\square$

Remark. In light of the previous lemma, throughout this paper we assume that the boundary points satisfy

$$\frac{1}{2} < q < \frac{1+q}{2} < r < 1. \tag{8}$$

Lemma 2. *Pick a real number $h \in [0, 1-q]$. Then*

$$G(q-h,s) \leq G(q+h,s)$$

for all $s \in [0,1]$.

Proof. We proceed by cases based on the branches of the Green's function (3), and a computer algebra system.

(i) $s \in [0, q-h]$:

$$\begin{aligned} G(q+h,s) - G(q-h,s) &= \left[u_1(q+h,s) - \frac{1}{6}(q+h-s)^3\right] \\ &\quad - \left[u_1(q-h,s) - \frac{1}{6}(q-h-s)^3\right] \\ &= 0. \end{aligned}$$

(ii) $s \in [q-h, q]$:

$$\begin{aligned} G(q+h,s) - G(q-h,s) &= u_1(q+h,s) - \frac{1}{6}(q+h-s)^3 - u_1(q-h,s) \\ &= \frac{1}{6}(s+h-q)^3 \geq 0. \end{aligned}$$

(iii) $s \in [q, r]$, $s \leq q + h$:

$$G(q+h,s) - G(q-h,s) = u_2(q+h,s) - \frac{1}{6}(q+h-s)^3 - u_2(q-h,s)$$
$$= \frac{1}{6}[2h^3 - (q+h-s)^3] \geq 0$$

since $s \geq q$ implies $h^3 \geq (q+h-s)^3$.

(iv) $s \in [q+h, r]$:

$$G(q+h,s) - G(q-h,s) = u_2(q+h,s) - u_2(q-h,s)$$
$$= \frac{1}{3}h^3 \geq 0.$$

(v) $s \in [r, 1]$, $s \geq q + h$:

$$G(q+h,s) - G(q-h,s) = u_3(q+h,s) - u_3(q-h,s)$$
$$= \frac{1}{3}h^3 \geq 0.$$

(vi) $s \in [r, q+h]$:

$$G(q+h,s) - G(q-h,s) = u_3(q+h,s) - \frac{1}{6}(q+h-s)^3 - u_3(q-h,s)$$
$$= \frac{1}{6}[2h^3 - (q+h-s)^3] \geq 0. \qquad \square$$

4. Inequalities and equalities needed in the existence theorems. Define the Banach space E by

$$E = \{x \mid x \in C[0,1],\ x(0) = 0\}$$

with the supnorm, and define the cone P of E by

$$P = \left\{ x \in E \;\middle|\; \begin{array}{l} x \text{ is nondecreasing on } [0,q], \text{ nonincreasing on } [q,1]; \\ x \text{ is nonnegative valued on } [0,1]; \\ x(q+h) \geq x(q-h) \quad \text{and} \quad x(q+h) \geq m_2\|x\| \\ \text{for all } h \in [0, 1-q]. \end{array} \right\},$$

where m_2 is given in (9) below.

For integers h, k_1, k_2 with

$$0 \leq h \leq 1 - q$$

and

$$0 < k_1 \leq k_2 < 1 - q,$$

define the concave functionals on the cone P

$$\alpha(x) := \min_{t \in [q-k_2, q-k_1] \cup [q+k_1, q+k_2]} x(t) = x(q - k_2)$$

and

$$\psi(x) := \min_{t \in [q-h, q+h]} x(t) = x(q - h),$$

and the convex functionals on the cone P

$$\begin{aligned} \beta(x) &:= \max_{t \in [q-h, q+h]} x(t) = x(q), \\ \gamma(x) &:= \max_{t \in [0, q-h] \cup [q+h, 1]} x(t) = x(q + h), \end{aligned}$$

and

$$\theta(x) := \max_{t \in [q-k_2, q-k_1] \cup [q+k_1, q+k_2]} x(t) = x(q + k_1).$$

We will make use of various properties and constants associated with the Green's function (3), which include the values

$$\begin{aligned} C_1 &:= \int_0^1 G(q + h, s)\, ds \\ &= \frac{1}{24}[3q^4 - h^4 + 4h^3(1-q) + 2q^2(6r - 3r^2 - 4q) + 6h^2(r-q)(r+q-2)], \end{aligned}$$

$$C_2 := \int_0^{q-h} G(q, s)\, ds = \frac{1}{24}\,(q - h)^4,$$

$$C_3 := \int_{q+h}^1 G(q, s)\, ds$$

$$= \begin{cases} \dfrac{1}{12}\, q^2[(r-q-h)(3r+3h-q)+2(1-r)(3r-2q)] & : q+h \le r \\ \dfrac{1}{6}\, q^2(3r-2q)(1-q-h) & : q+h \ge r, \end{cases}$$

$$C_4 := \int_{q-h}^{q+h} G(q,s)\,ds$$
$$= \begin{cases} \dfrac{1}{24}\,(8hq^3+4h^3q-h^4) & : q+h \le r \\ \dfrac{1}{24}\,[2q^2(6hr-4hq-3(r-q)^2)+q^4-(q-h)^4] & : q+h \ge r, \end{cases}$$

$$C_5 := \int_{q-k_2}^{q-k_1} G(q-k_2,s)\,ds + \int_{q+k_1}^{q+k_2} G(q-k_2,s)\,ds$$

and the constants

$$M := \max_{0\le s\le 1} \frac{G(q,s)}{G(q-h,s)} = \frac{q^2(3r-2q)}{(2q-1)(3r+q^2-4q+1)}$$
$$m_1 := \min_{0\le s\le 1} \frac{G(q-k_2,s)}{G(q+k_1,s)}$$
$$m_2 := \min_{0\le s\le 1} \frac{G(q+h,s)}{G(q,s)} = \frac{r^3-3(r-q)^2}{q^2(3r-2q)}$$

for all $h \in [0,1-q]$ and $0<k_1\le k_2<1-q$.

5. Theorem on the existence of three positive solutions. In this section we state and prove a theorem on the existence of three positive solutions to the BVP (1), (2). By a positive solution of the BVP (1), (2) we mean a solution which is in the cone defined in the proof of the following theorem.

Theorem 2. *Suppose a, b, and c are nonnegative real numbers with $0<a<b<b/m_1\le c$ such that nonnegative continuous f satisfies the following conditions*:

(i) $f(x) < (a-(c(C_2+C_3))/(C_1))/C_4$ *for all* $x \in [a/M, a]$,

(ii) $f(x) > b/C_5$ *for* $x \in [b,(b/m_1)]$,

(iii) $f(x) \le (c/C_1)$ *for* $x \in [0,(c/m_2)]$.

Then, the $(4,4)$ *boundary value problem* (1), (2) *has three positive solutions,* x_1*,* x_2*,* $x_3 \in P(\gamma, c)$*.*

Proof. Define the completely continuous operator A by

$$Ax(t) = \int_0^1 G(t,s) f(x(s))\, ds.$$

We seek fixed points of A which satisfy the conclusion of the theorem. We note first, if $x \in P$, then from properties of $G(t,s)$:

$$\begin{aligned} &Ax(t) \geq 0, \\ &\frac{d}{dt} Ax(t) \geq 0 \quad \text{for} \quad t \in [0,q], \\ &\frac{d}{dt} Ax(t) \leq 0 \quad \text{for} \quad t \in [q,1], \end{aligned}$$

$$Ax(q-h) \leq Ax(q+h) \quad \text{for} \quad h \in [0, 1-q],$$

and

$$Ax(q+h) \geq m_2\, Ax(q) = m_2 \|Ax\|.$$

Consequently, $Ax \in P$, that is, $A : P \to P$.

Note that for all $x \in P$,

$$\alpha(x) = x(q-k_2) \leq x(q) = \beta(x)$$

and

$$\|x\| \leq \frac{1}{m_2}\, x(q+h) = \frac{1}{m_2}\, \gamma(x).$$

If $x \in \overline{P(\gamma, c)}$, then $\|x\| \leq 1/m_2\, \gamma(x) \leq c/m_2$, and by assumption (iii) we have

$$\begin{aligned} \gamma(Ax) &= \max_{t \in [0,q-h] \cup [q+h,1]} \int_0^1 G(t,s) f(x(s))\, ds \\ &= \int_0^1 G(q+h,s) f(x(s))\, ds \\ &\leq \left(\frac{c}{C_1}\right) \int_0^1 G(q+h,s)\, ds \\ &= c. \end{aligned}$$

Therefore,

$$A : \overline{P(\gamma, c)} \longrightarrow \overline{P(\gamma, c)}.$$

Let

$$x_P(t) \equiv \frac{b}{2}\left(1 + \frac{1}{m_1}\right) \quad \text{and} \quad x_Q(t) \equiv \frac{a}{2}\left(1 + \frac{1}{M}\right)$$

for all $t \in [0, 1]$. It is clear that

$$x_P \in \left\{ x \in P\left(\gamma, \theta, \alpha, b, \frac{b}{m_1}, c\right) : \alpha(x) > b \right\} \neq \varnothing$$

and

$$x_Q \in \left\{ x \in Q\left(\gamma, \beta, \psi, \frac{a}{M}, a, c\right) : \beta(x) < a \right\} \neq \varnothing.$$

In the following claims we verify the remaining conditions of the generalized Leggett-Williams fixed point theorem.

Claim 1. *If $x \in Q(\gamma, \beta, a, c)$ with $\psi(Ax) < a/M$, then $\beta(Ax) < a$.*

$$\begin{aligned}
\beta(Ax) &= \max_{t \in [q-h, q+h]} \int_0^1 G(t, s) f(x(s)) ds \\
&= \int_0^1 G(q, s) f(x(s)) ds \\
&= \int_0^1 \frac{G(q, s)}{G(q-h, s)} \, G(q-h, s) f(x(s)) ds \\
&\leq M \int_0^1 G(q-h, s) f(x(s)) ds \\
&= M \psi(Ax) \\
&< a.
\end{aligned}$$

Claim 2. *If* $x \in Q(\gamma, \beta, \psi, (a/M), a, c)$ *then* $\beta(Ax) < a$.

$$\begin{aligned}
\beta(Ax) &= \max_{t\in[q-h,q+h]} \int_0^1 G(t,s) f(x(s))\, ds \\
&= \int_0^{q-h} G(q,s) f(x(s)) ds + \int_{q-h}^{q+h} G(q,s) f(x(s))\, ds \\
&\quad + \int_{q+h}^1 G(q,s) f(x(s))\, ds \\
&< \left(\frac{c}{C_1}\right) \int_0^{q-h} G(q,s)\, ds + \left(\frac{a - (c(C_2+C_3))/C_1}{C_4}\right) \\
&\quad \times \int_{q-h}^{q+h} G(q,s)\, ds + \left(\frac{c}{C_1}\right) \int_{q+h}^1 G(q,s)\, ds \\
&= \left(\frac{c}{C_1}\right)(C_2 + C_3) + \left(\frac{a - (c(C_2+C_3))/C_1}{C_4}\right) C_4 \\
&= a.
\end{aligned}$$

Claim 3. *If* $x \in P(\gamma, \alpha, b, c)$ *with* $\theta(Ax) > b/m_1$*, then* $\alpha(Ax) > b$.

$$\begin{aligned}
\alpha(Ax) &= \min_{t\in[q-k_2,q-k_1]\cup[q+k_1,q+k_2]} \int_0^1 G(t,s) f(x(s))\, ds \\
&= \int_0^1 G\left(q-k_2, s\right) f(x(s))\, ds \\
&= \int_0^1 \left(\frac{G(q-k_2,s)}{G(q+k_1,s)}\right) G(q+k_1,s) f(x(s))\, ds \\
&\geq m_1 \int_0^1 G(q+k_1,s) f(x(s))\, ds \\
&= m_1 \theta(Ax) \\
&> b.
\end{aligned}$$

Claim 4. *If* $x \in P(\gamma, \theta, \alpha, b, (b/m_1), c)$ *then* $\alpha(Ax) > b$.

$$\begin{aligned}\alpha(Ax) &= \min_{t\in[q-k_2,q-k_1]\cup[q+k_1,q+k_2]} \int_0^1 G(t,s)f(x(s))\,ds \\ &\geq \int_{q-k_2}^{q-k_1} G(q-k_2,s)f(x(s))\,ds + \int_{q+k_1}^{q+k_2} G(q-k_2,s)f(x(s))\,ds \\ &> \left(\frac{b}{C_5}\right)\left(\int_{q-k_2}^{q-k_1} G(q-k_2,s)\,ds + \int_{q+k_1}^{q+k_2} G(q-k_2,s)\,ds\right) \\ &= \left(\frac{b}{C_5}\right) C_5 \\ &= b.\end{aligned}$$

Therefore the hypotheses of the generalized Leggett-Williams fixed point theorem (Five Functionals Fixed-Point theorem) are satisfied, and there exist three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ for the $(4,4)$ right focal boundary value problem such that:

$$\alpha(x_1) > b,$$
$$\beta(x_2) < a,$$

and

$$\alpha(x_3) < b \quad \text{with} \quad \beta(x_3) > a. \qquad \square$$

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