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FOURTH-ORDER SCHEMES OF EXPONENTIAL TYPE FOR SINGULARLY PERTURBED PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We present a class of difference schemes of exponential type for solving singularly perturbed parabolic partial differential equations. This class includes a scheme of fourth order of accuracy when the perturbation parameter, ε, is fixed. For small ε, the orders of accuracy are verified experimentally. Stability analysis for these schemes are also presented. Numerical results and comparisons with other schemes are considered.

1. Introduction. We consider the following parabolic partial differential equation

$$Lu(x,t) \equiv \frac{\partial u}{\partial t} - \varepsilon \frac{\partial^2 u}{\partial x^2} - b(x,t)\frac{\partial u}{\partial x} + d(x,t)u = f(x,t), \quad (x,t) \in \Omega \equiv (0,1) \times (0,T], \tag{1.1a}$$

$$u(x,0) = g(x), \quad x \in [0,1], \tag{1.1b}$$

$$u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad \forall\, t \geq 0, \tag{1.1c}$$

where ε is a parameter in $(0,1]$. The functions g_0 and g_1 are continuous and bounded as $t \to \infty$ and b, d and f are sufficiently smooth functions of x and t. Also, we assume that $b(x,t) \geq \beta > 0$ and $d(x,t) \geq 0$ on $\overline{\Omega}$. For $\varepsilon \to 0^+$ the exact solution of equation (1.1) exhibits a boundary layer at $x = 0$. In the case of $b(x,t) \leq -\beta < 0$ the problem can be transformed to the problem (1.1) by making the change of variable $x \to 1 - x$. Problems of this type arise, for example, in the modeling

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of steady and unsteady viscous flow problems with large Reynolds number, convective heat transport problems with large Peclet numbers and electromagnetic field problems in moving media.

The accuracy of the numerical solution and the computational efficiency are highly dependent on the numerical methods used to solve these kind of partial differential equations. Standard three-point finite difference methods of approximating spatial derivatives may work well for smooth solutions, but they fail when $\varepsilon \to 0^+$. Most computational techniques for solving equation (1.1) involve first- or second-order methods, which have been proposed by [**2, 5, 16, 17, 21, 24, 29**].

There are three principal approaches to solve numerically the singularly perturbed boundary-value problems, namely, the finite difference methods, the finite element methods and the spline approximation methods. For more details of using these methods, we may refer to [**11, 14, 22, 28**]. Also, for solving equation (1.1), we may refer to [**1, 4, 8, 10, 12, 20, 25–27**].

In this paper we derive the exponential tridiagonal schemes for solving equation (1.1) by computing the local truncation error. The resulting coefficients are polynomials in $z = h/\varepsilon$, where h is the mesh width. Finally, we write the coefficients in the exponential form.

The outline of this paper is as follows. In Section 2 we explain tridiagonal finite-difference schemes for solving problem (1.1). Stability analysis and local truncation error are discussed in Section 3. In Section 4 we introduce the exponential scheme for solving problem (1.1). The final section presents some numerical results and comparisons with the other schemes.

2. Derivation of fourth-order schemes. We will consider the following tridiagonal finite-difference schemes for (1.1) in the form

$$\begin{gathered}\frac{1}{\Delta t} S(U_i^{j+1} - U_i^j) - \frac{\varepsilon}{h^2} R(U_i^{j+1/2}) = Q(f_i^{j+1/2}),\\ i = 1, 2, \dots, (N-1), \quad j = 1, 2, \dots,\end{gathered} \tag{2.1}$$

where R and Q are operators in the form

$$\begin{aligned} R(U_i^{j+1/2}) &= R^{j+1}(U_i^{j+1}) + R^j(U_i^j),\\ Q(U_i^{j+1/2}) &= Q^{j+1}(U_i^{j+1}) + Q^j(U_i^j).\end{aligned} \tag{2.2}$$

In this case the operator R^j is a tridiagonal displacement operator, namely,

$$R^j(U_i^j) = r_j^- U_{i-1}^j + r_j^0 U_i^j + r_j^+ U_{i+1}^j. \tag{2.3}$$

In the same manner, we define the operators R^{j+1}, Q^j, Q^{j+1} and S. Here and throughout the paper, N is a positive integer and $x_i = ih$, $i = 0, 1, \dots, N$, with the mesh width $h = 1/N$. The j index indicates the t dependence $(t_{j+1} = t_j + \Delta t, t_{j+1/2} = t_j + (\Delta t/2))$, U_i^j denotes the approximate value for $u(x_i, t_j)$ and $f_i^j = f(x_i, t_j)$. The choice of the coefficients $r_j^{-,0,+}$ (meaning r_j^-, r_j^0 and r_j^+), $r_{j+1}^{-,0,+}$, $q_j^{-,0,+}$, $q_{j+1}^{-,0,+}$ and $s_j^{-,0,+}$ determines the particular scheme. Special cases from (2.1) are in [**6**] and [**7**].

We derive the present schemes by computing the local truncation error as follows:

$$\begin{aligned} \tau_i^{j+1/2} \equiv {} & \frac{1}{\Delta t} S(u(x_i, t_{j+1}) - u(x_i, t_j)) - \frac{\varepsilon}{h^2} R(u(x_i, t_{j+1/2})) \\ & - Q(Lu(x_i, t_{j+1/2})). \end{aligned} \tag{2.4}$$

For a sufficiently smooth function $u(x, t)$, the standard Taylor development of $\tau_i^{j+1/2}$ is given by

$$\begin{aligned} \tau_i^{j+1/2} = {} & T^{0,0} u(x_i, t_{j+1/2}) + T^{1,0} u^{(1)}(x_i, t_{j+1/2}) \\ & + T^{2,0} u^{(2)}(x_i, t_{j+1/2}) + \dots + T^{6,0} u^{(6)}(x_i, t_{j+1/2}) \\ & + T^{0,1} u_t(x_i, t_{j+1/2}) + T^{1,1} u_t^{(1)}(x_i, t_{j+1/2}) \\ & + T^{2,1} u_t^{(2)}(x_i, t_{j+1/2}) + \dots + T^{6,1} u_t^{(6)}(x_i, t_{j+1/2}) \\ & + T^{0,2} u_{tt}(x_i, t_{j+1/2}) + T^{1,2} u_{tt}^{(1)}(x_i, t_{j+1/2}) \\ & + T^{2,2} u_{tt}^{(2)}(x_i, t_{j+1/2}) + \dots + T^{6,2} u_{tt}^{(6)}(x_i, t_{j+1/2}) \\ & + O((\Delta t)^3 + h^5). \end{aligned} \tag{2.5}$$

We compute the coefficients $T^{0,0}, T^{1,0}, \ldots, T^{6,0}$ in the form

$$
\begin{aligned}
T^{0,0} &= -\frac{\varepsilon}{h^2}\,[r^+_{j+1} + r^0_{j+1} + r^-_{j+1} + r^+_j + r^0_j + r^-_j \\
&\quad + hz(q^+_{j+1}d^{j+1}_{i+1} + q^0_{j+1}d^{j+1}_i + q^-_{j+1}d^{j+1}_{i-1} \\
&\quad + q^+_j d^j_{i+1} + q^0_j d^j_i + q^-_j d^j_{i-1})], \\
T^{1,0} &= -\frac{\varepsilon}{h}\,[r^+_{j+1} - r^-_{j+1} + r^+_j - r^-_j - z(q^+_{j+1}b^{j+1}_{i+1} + q^0_{j+1}b^{j+1}_i \\
&\quad + q^-_{j+1}b^{j+1}_{i-1} + q^+_j b^j_{i+1} + q^0_j b^j_i + q^-_j b^j_{i-1}) \\
&\quad + hz(q^+_{j+1}d^{j+1}_{i+1} - q^-_{j+1}d^{j+1}_{i-1} + q^+_j d^j_{i+1} - q^-_j d^j_{i-1})], \\
T^{2,0} &= -\frac{\varepsilon}{2}\,[r^+_{j+1} + r^-_{j+1} + r^+_j + r^-_j \\
&\quad - 2(q^+_{j+1} + q^0_{j+1} + q^-_{j+1} + q^+_j + q^0_j + q^-_j) \\
&\quad - 2z(q^+_{j+1}b^{j+1}_{i+1} - q^-_{j+1}b^{j+1}_{i-1} + q^+_{j+1}b^j_{i+1} - q^-_{j+1}b^j_{i-1}) \\
&\quad + hz(q^+_{j+1}d^{j+1}_{i+1} + q^-_{j+1}d^{j+1}_{i-1} + q^+_j d^j_{i+1} + q^-_j d^j_{i-1})];
\end{aligned}
\tag{2.6a}
$$

(2.6b)

$$
\begin{aligned}
T^{\nu,0} = -\frac{\varepsilon h^{\nu-2}}{\nu!}\,&\{r^+_{j+1} + (-1)^\nu r^-_{j+1} + r^+_j + (-1)^\nu r^-_j - \nu(\nu-1) \\
&\times [q^+_{j+1} + (-1)^\nu q^-_{j+1} + q^+_j + (-1)^\nu q^-_j] \\
&- \nu z[q^+_{j+1}b^{j+1}_{i+1} + (-1)^{\nu-1}q^-_{j+1}b^{j+1}_{i-1} + q^+_j b^j_{i+1} + (-1)^{\nu-1}q^-_j b^j_{i-1}] \\
&+ hz[q^+_{j+1}d^{j+1}_{i+1} + (-1)^\nu q^-_{j+1}d^{j+1}_{i+1} + q^+_j d^j_{i+1} + (-1)^\nu q^-_j d^j_{i-1}]\}, \\
&\qquad \nu = 3,4,5,6.
\end{aligned}
$$

Also, we compute the remaining coefficients in the form

(2.7a)

$$
\begin{aligned}
T^{0,\mu} = -\left(\frac{\Delta t}{2}\right)^{\mu}\frac{\varepsilon}{\mu h^2}\,&\{r^+_{j+1} + r^0_{j+1} + r^-_{j+1} + (-1)^\mu(r^+_j + r^0_j + r^-_j) \\
&+ hz[q^+_{j+1}d^{j+1}_{i+1} + q^0_{j+1}d^{j+1}_i + q^-_{j+1}d^{j+1}_{i-1} \\
&+ (-1)^\mu(q^+_j d^j_{i+1} + q^0_j d^j_i + q^-_j d^j_{i-1})] \\
&- \kappa[(2-\mu)(s^+_j + s^0_j + s^-_j) \\
&- \mu(q^+_{j+1} + q^0_{j+1} + q^-_{j+1} + (-1)^{\mu-1}(q^+_j + q^0_j + q^-_j))]\},
\end{aligned}
$$

$$
\begin{aligned}
T^{1,\mu} = -\left(\frac{\Delta t}{2}\right)^{\mu} \frac{\varepsilon}{\mu h} \{ & r^+_{j+1} - r^-_{j+1} + (-1)^{\mu}(r^+_j - r^-_j) \\
& - z[q^+_{j+1} b^{j+1}_{i+1} + q^0_{j+1} b^{j+1}_i + q^-_{j+1} b^{j+1}_{i-1} \\
& + (-1)^{\mu}(q^+_j b^j_{i+1} + q^0_j b^j_i + q^-_j b^j_{i-1})] \\
& + hz[q^+_{j+1} d^{j+1}_{i+1} - q^-_{j+1} d^{j+1}_{i-1} + (-1)^{\mu}(q^+_j d^j_{i+1} - q^-_j d^j_{i-1})] \\
& - \kappa[(2-\mu)(s^+ - s^-) - \mu(q^+_{j+1} - q^-_{j+1} + (-1)^{\mu-1}(q^+_j - q^-_j))]\} \\
T^{2,\mu} = -\left(\frac{\Delta t}{2}\right)^{\mu} \frac{\varepsilon}{2\mu} \{ & r^+_{j+1} + r^-_{j+1} + (-1)^{\mu}(r^+_j + r^-_j) \\
& - 2[q^+_{j+1} + q^0_{j+1} + q^-_{j+1} + (-1)^{\mu}(q^+_j + q^0_j + q^-_j)] \\
& - z[q^+_{j+1} b^{j+1}_{i+1} - q^-_{j+1} b^{j+1}_{i-1} + (-1)^{\mu}(q^+_{j+1} b^j_{i+1} - q^-_{j+1} b^j_{i-1})] \\
& + hz[q^+_{j+1} d^{j+1}_{i+1} + q^-_{j+1} d^{j+1}_{i-1} + (-1)^{\mu}(q^+_j d^j_{i+1} + q^-_j d^j_{i-1})] \\
& - \kappa[(2-\mu)(s^+_j + s^-_j) - \mu(q^+_{j+1} + q^-_{j+1} + (-1)^{\mu-1}(q^+_j + q^-_j))]\};
\end{aligned}
$$

(2.7b)

$$
\begin{aligned}
T^{\nu,\mu} = -\left(\frac{\Delta t}{2}\right)^{\mu} \frac{\varepsilon h^{\nu-2}}{\mu \nu!} \{ & r^+_{j+1} + (-1)^{\nu} r^-_{j+1} + (-1)^{\mu}(r^+_j + (-1)^{\nu} r^-_j) \\
& - \nu(\nu-1)[q^+_{j+1} + (-1)^{\nu} q^-_{j+1} + (-1)^{\mu}(q^+_j + (-1)^{\nu} q^-_j)] \\
& - \nu z[q^+_{j+1} b^{j+1}_{i+1} + (-1)^{\nu-1} q^-_{j+1} b^{j+1}_{i-1} + (-1)^{\mu}(q^+_j b^j_{i+1} \\
& + (-1)^{\nu-1} q^-_j b^j_{i-1})] \\
& + hz[q^+_{j+1} d^{j+1}_{i+1} + (-1)^{\nu} q^-_{j+1} d^{j+1}_{i+1} + (-1)^{\mu}(q^+_j d^j_{i+1} + (-1)^{\nu} q^-_j d^j_{i-1})] \\
& - \kappa[(2-\mu)(s^+_j + (-1)^{\mu} s^-_j) - \mu(q^+_{j+1} + (-1)^{\nu} q^-_{j+1} \\
& + (-1)^{\mu-1}(q^+_j + (-1)^{\nu} q^-_j))]\}, \\
& \qquad \nu = 3,4,5,6, \quad \mu = 1,2,
\end{aligned}
$$

where $\kappa = 2h^2/(\varepsilon \Delta t)$.

Now we describe the fourth-order schemes for solving equation (1.1). The important feature of these schemes is the fact that we solve only tridiagonal equations and that fictitious points are not needed at each time step along the boundary. One can derive these schemes by using the following possibility

$$
T^{\nu,\mu} = 0, \quad \nu = 0,1,\dots,4, \quad \mu = 0,1. \tag{2.8}
$$

The other possibility is to weaken the previous conditions as follows:

$$T^{\nu,\mu} = O((\Delta t)^2 + h^4), \quad \nu = 0, 1, \dots, 4, \quad \mu = 0, 1. \tag{2.9}$$

The conditions (2.8) or (2.9) represent ten equations in the coefficients $r_j^{-,0,+}, \dots, s_j^{-,0,+}$. The above conditions lead to relatively complicated equations of the above coefficients. For the present schemes to be considered here, we separate (2.6) and (2.7) into two directions x and t in the form

(2.10a)

$$\begin{aligned}
T_x^{0,k} &= -\frac{\varepsilon}{h^2}[r_{j+k}^{+} + r_{j+k}^{0} + r_{j+k}^{-} + hz(q_{j+k}^{+} d_{i+1}^{j+k} + q_{j+k}^{0} d_{i}^{j+k} + q_{j+k}^{-} d_{i-1}^{j+k})],\\
T_x^{1,k} &= -\frac{\varepsilon}{h}[r_{j+k}^{+} - r_{j+k}^{-} - z(q_{j+k}^{+} b_{i+1}^{j+k} + q_{j+k}^{0} b_{i}^{j+k} + q_{j+k}^{-} b_{i-1}^{j+k})\\
&\quad + hz(q_{j+k}^{+} d_{i+1}^{j+k} - q_{j+k}^{-} d_{i-1}^{j+k})],\\
T_x^{2,k} &= -\frac{\varepsilon}{2}[r_{j+k}^{+} + r_{j+k}^{-} - 2(q_{j+k}^{+} + q_{j+k}^{0} + q_{j+k}^{-}) - 2z(q_{j+k}^{+} b_{i+1}^{j+k} - q_{j+k}^{-} b_{i-1}^{j+k})\\
&\quad + hz(q_{j+k}^{+} d_{i+1}^{j+k} + q_{j+k}^{-} d_{i-1}^{j+k})];
\end{aligned}$$

(2.10b)

$$\begin{aligned}
T_x^{\nu,k} &= -\frac{\varepsilon h^{\nu-2}}{\nu!}\{r_{j+k}^{+} + (-1)^{\nu} r_{j+k}^{-} - \nu(\nu-1)[q_{j+k}^{+} + (-1)^{\nu} q_{j+k}^{-}]\\
&\quad - \nu z[q_{j+k}^{+} b_{i+1}^{j+k} + (-1)^{\nu-1} q_{j+k}^{-} b_{i-1}^{j+k}]\\
&\quad + hz[q_{j+k}^{+} d_{i+1}^{j+k} + (-1)^{\nu} q_{j+k}^{-} d_{i-1}^{j+k}]\},\\
&\qquad \nu = 3, 4, 5, 6, \quad k = 0, 1,
\end{aligned}$$

and

$$\begin{aligned}
T_t^0 &= s_j^{+} + s_j^{0} + s_j^{-} - (q_{j+1}^{+} + q_{j+1}^{0} + q_{j+1}^{-} + q_j^{+} + q_j^{0} + q_j^{-}),\\
T_t^1 &= h[s_j^{+} - s_j^{-} - (q_{j+1}^{+} - q_{j+1}^{-} + q_j^{+} - q_j^{-})],\\
T_t^2 &= \frac{h^2}{2}[s_j^{+} + s_j^{-} - (q_{j+1}^{+} + q_{j+1}^{-} + q_j^{+} + q_j^{-})].
\end{aligned} \tag{2.11}$$

We derive the present schemes by the following conditions

$$T_x^{\nu,k} = 0, \qquad \nu = 0, 1, 2, \quad k = 0, 1, \tag{2.12a}$$

$$T_x^{\nu,k} = O(h^4), \quad \nu = 3, 4, \qquad k = 0, 1, \tag{2.12b}$$

and

$$T_t^\nu = 0, \quad \nu = 0, 1, 2. \tag{2.12c}$$

From the conditions (2.12a) $r_j^{-,0,+}$ are determined in terms of $q_j^{-,0,+}$ as follows:

(2.13)

$$\begin{aligned} r_j^0 &= -r_j^+ - r_j^- - hz(q_j^+ d_{i+1}^j + q_j^0 d_i^j + q_j^- d_{i-1}^j),\\ r_j^- &= \ q_j^+ + q_j^0 + q_j^- + \frac{z}{2}(q_j^+ b_{i+1}^j - q_j^0 b_i^j - 3q_j^- b_{i-1}^j) - hzq_j^- d_{i-1}^j,\\ r_j^+ &= \ q_j^+ + q_j^0 + q_j^- + \frac{z}{2}(3q_j^+ b_{i+1}^j + q_j^0 b_i^j - q_j^- b_{i-1}^j) - hzq_j^+ d_{i+1}^j. \end{aligned}$$

Also, the coefficients $r_{j+1}^{-,0,+}$ are of similar form at the time level $j+1$. In the same manner, from the conditions (2.12c), $s_j^{-,0,+}$ are determined in the form

$$s_j^- = q_{j+1}^- + q_j^-, \quad s_j^0 = q_{j+1}^0 + q_j^0, \quad s_j^+ = q_{j+1}^+ + q_j^+. \tag{2.14}$$

Now, we determine the coefficients $q_j^{-,0,+}$. It follows from (2.13) and (2.12b) that $T_x^{3,0}$ and $T_x^{4,0}$ can be written in terms of $q_j^{-,0,+}$ as

$$\begin{aligned} T_x^{3,0} &= -\frac{\varepsilon h}{3!}\big[-6q_j^+ + 6q_j^- - z(2q_j^+ b_{i+1}^j - q_j^0 b_i^j + 2q_j^- b_{i-1}^j)\big],\\ T_x^{4,0} &= -\frac{\varepsilon h^2}{4!}\big[-10q_j^+ + 2q_j^0 - 10q_j^- - 2z(q_j^+ b_{i+1}^j - q_j^- b_{i-1}^j)\big]. \end{aligned} \tag{2.15}$$

Throughout this section, the j index notation on $q_j^{-,0,+}$ will be dropped in order to simplify the notation. We define $q^{-,0,+}$ as polynomials in z at each mesh point x_i in the form

$$q^{-,0,+} = \sum_{\nu=0}^{2} q_\nu^{-,0,+} z^\nu, \tag{2.16}$$

where the coefficients $q_\nu^{-,0,+}$, $\nu = 0, 1, 2$, are independent of ε. To examine the implications of (2.12b), substitute (2.16) into (2.15) and impose (2.12b). The result is the following asymptotic relations as $h \to 0$ (ε fixed):

$$T_x^{3,0} = -\frac{\varepsilon h}{3!}\,[t_0^3 + t_1^3 z + t_2^3 z^2 + O(z^3)] = O(h^4),$$
$$T_x^{4,0} = -\frac{\varepsilon h^2}{4!}\,[t_0^4 + t_1^4 z + O(z^2)] = O(h^4),$$

where

$$\begin{aligned}
t_0^3 &= -6q_0^+ + 6q_0^- = O(h^3),\\
t_0^4 &= -10q_0^+ + 2q_0^0 - 10q_0^- = O(h^2),\\
t_1^3 &= -6q_1^+ + 6q_1^- - (2q_0^+ b_{i+1}^j - q_0^0 b_i^j + 2q_0^- b_{i-1}^j) = O(h^2),\\
t_1^4 &= -10q_1^+ + 2q_1^0 - 10q_1^- - 2(q_0^+ b_{i+1}^j - q_0^- b_{i-1}^j) = O(h),\\
t_2^3 &= -6q_2^+ + 6q_2^- - (2q_1^+ b_{i+1}^j - q_1^0 b_i^j + 2q_1^- b_{i-1}^j) = O(h).
\end{aligned}$$

We use the approach in [**3**], the coefficients $q_j^{-,0,+}$ are given in the form

$$q_j^- = 2 - zb_i^j, \quad q_j^0 = 20, \quad q_j^+ = 2 + zb_i^j. \tag{2.17}$$

Also, the coefficients $q_{j+1}^{-,0,+}$ are of similar form at the time level $j+1$.

3. Stability analysis and local truncation error. In this section we study the stability of (2.1) by von Neumann analysis. We consider equation (1.1) with constant coefficient b and $d = f = 0$ and perform the standard Fourier stability analysis. In this case the operators R, Q and S are independent of j, thus (2.1) can be written as

$$(\kappa\, Q - R)\, U^{j+1} = (\kappa\, Q + R)\, U^j, \tag{3.1}$$

where U^j is the vector of components $U_1, U_2, \dots, U_{N-1}$ at the time level j. Substitution of $U^j = \chi^j(e^{i\theta}, e^{2i\theta}, \dots, e^{(N-1)i\theta})$ into (3.1) yields

$$\chi = \frac{2\kappa + l(\theta)}{2\kappa - l(\theta)},$$

where

$$l(\theta) = \frac{(\cos\theta - 1)(48 + 4b^2z^2) - 24ibz\sin\theta}{10 + 2\cos\theta - ibz\sin\theta},$$

and $i = \sqrt{-1}$. For stability it is sufficient that $|\chi| \le 1$. Since $\kappa > 0$, a necessary and sufficient condition for $|\chi| \le 1$ is $\operatorname{Re} l(\theta) \le 0$. Direct computation of $\operatorname{Re} l(\theta)$ yields

$$\begin{aligned} \operatorname{Re} l(\theta) &= (\cos\theta - 1)(48 + 4b^2z^2)(10 + 2\cos\theta) + 24b^2z^2\sin^2\theta \\ &= 16(\cos\theta - 1)[30 + 6\cos\theta + b^2z^2(1-\cos\theta)]. \end{aligned} \tag{3.2}$$

From (3.2), it is clear that $\operatorname{Re} l(\theta) \le 0$, then (3.1) is unconditionally stable.

Now, we compute the local truncation error for the scheme described in Section 2. Substituting from (2.13), (2.14) and (2.17) into (2.6) we have

$$T^{\nu,0} = 0, \qquad \nu = 0, 1, 2,$$

and

$$T^{\nu,0} = O(h^4), \quad \nu = 3, \dots, 6. \tag{3.3}$$

Also, substituting from (2.13), (2.14) and (2.17) into (2.7) we have

$$\begin{aligned} T^{\nu,1} &= 0, & \nu &= 0, 1, 2, \\ T^{\nu,1} &= O((\Delta t)^2 + h^4), & \nu &= 3, \dots, 6 \end{aligned}$$

and

$$T^{\nu,2} = O((\Delta t)^2 + h^4), \quad \nu = 0, \dots, 6. \tag{3.4}$$

Substituting from (3.3) and (3.4) into (2.5), it follows that the order of our scheme as described in the above section is $O((\Delta t)^2 + h^4)$ for ε fixed.

4. The exponential fourth-order scheme. In this section we develop the above scheme as described in Section 2. The exponential scheme given here is defined by choosing R, Q such that

$$T_x^{0,k} = 0, \quad k = 0, 1, \tag{4.1a}$$

$$T_x^{\nu,k} = O(h^4), \quad \nu = 1, 2, \dots, 4, \quad k = 0, 1. \tag{4.1b}$$

We define $q^{-,0,+}$ and $r_j^{-,+}$ as polynomials in z at each mesh point x_i in the form

$$q^{-,0,+} = \sum_{\nu=0}^{2} q_\nu^{-,0,+} z^\nu; \tag{4.2a}$$

$$r^{-,+} = \sum_{\nu=0}^{2} r_\nu^{-,+} z^\nu, \tag{4.2b}$$

where the coefficients $q^{-,0,+}$ and $r_\nu^{-,+}$ are independent of ε. We determine these coefficients by substituting from (4.2) into (2.10) and imposing (4.1).

In the case of $d(x,t) = 0$, the polynomials (4.2) are determined in the form

(4.3a)

$$r_j^- = 1 - \frac{z}{2}\, b_{i,j}^- + \frac{z^2}{12}\, b_{i,j}^{-^2}, \quad r_j^+ = 1 + \frac{z}{2}\, b_{i,j}^+ + \frac{z^2}{12}\, b_{i,j}^{+^2}, \quad r_j^0 = -r_j^- - r_j^+;$$

(4.3b)

$$q_j^- = \frac{1}{12} - \frac{z}{24}\, b_{i,j}^-, \quad q_j^+ = \frac{1}{12} + \frac{z}{24}\, b_{i,j}^+, \quad q_j^0 = \frac{2}{3} + q_j^- + q_j^+,$$

where

$$b_{i,j}^- = \frac{1}{6}\,(2b_{i-1}^j + 5b_i^j - b_{i+1}^j), \quad b_{i,j}^+ = \frac{1}{6}\,(-b_{i-1}^j + 5b_i^j + 2b_{i+1}^j). \tag{4.4}$$

Also, the coefficients $r_{j+1}^{-,0,+}$ and $q_{j+1}^{-,0,+}$ are of similar form at the time level $j+1$. Using the following power series

$$\frac{x}{1-\exp(-x)} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots,$$
$$\frac{x\exp(-x)}{1-\exp(-x)} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} + \cdots,$$

we can write the coefficients in (4.3) as follows:

(4.5a)

$$r_j^- = \frac{zb_{i,j}^- \exp(-zb_{i,j}^-)}{1-\exp(-zb_{i,j}^-)}, \quad r_j^+ = \frac{zb_{i,j}^+}{1-\exp(-zb_{i,j}^+)}, \quad r^0 = -r_j^- - r_j^+;$$

(4.5b)

$$q_j^- = \frac{1-r_j^-}{2zb_{i,j}^-} - \frac{1}{6}, \quad q_j^+ = \frac{r_j^+ - 1}{2zb_{i,j}^+} - \frac{1}{6}, \quad q_j^0 = \frac{2}{3} + q_j^- + q_j^+,$$

where $b_{i,j}^-$ and $b_{i,j}^+$ are defined by (4.4). One can verify that the coefficients (4.5) satisfy the conditions (4.1).

In the general form, the exponential scheme for the problem (1.1) is defined by (2.1), (2.14) and

$$r_j^- = \frac{(n_j^- - n_j^+)\exp(n_j^-)}{\exp(n_j^- - n_j^+) - 1}, \quad r_j^+ = \frac{(k_j^- - k_j^+)\exp(-k_j^+)}{\exp(k_j^- - k_j^+) - 1}; \tag{4.6a}$$

(4.6b)

$$\begin{aligned} q_j^- &= \frac{1}{2(1-\exp(n_j^- - n_j^+))} \\ &\quad \times \left[\frac{\exp(n_j^-) - 1}{n_j^-} + \frac{\exp(-n_j^-)(\exp(-n_j^+) - 1)}{n_j^+}\right] - \frac{1}{6}, \\ q_j^+ &= \frac{1}{2(1-\exp(k_j^- - k_j^+))} \\ &\quad \times \left[\frac{1-\exp(-k_j^+)}{k_j^+} - \frac{\exp(-k_j^+)(\exp(k_j^-) - 1)}{k_j^-}\right] - \frac{1}{6}, \\ q_j^0 &= \frac{2}{3} + q_j^- + q_j^+; \end{aligned}$$

$$r_j^0 = -r_j^- - r_j^+ - hz(q_j^+ d_{j+1} + q_j^0 d_j + q_j^- d_{j-1}), \tag{4.6c}$$

where

$$n_j^{\mp} = \frac{z}{2}\left(-b_{i,j}^- \mp \sqrt{b_{i,j}^{-2} + 4\varepsilon d_{i,j}^-}\right), \; k_j^{\mp} = \frac{z}{2}\left(-b_{i,j}^+ \mp \sqrt{b_{i,j}^{+2} + 4\varepsilon d_{i,j}^+}\right),$$

$b_{i,j}^-$ and $b_{i,j}^+$ are defined by (4.4) and $d_{i,j}^-$ and $d_{i,j}^+$ have similar forms. Also, the coefficients $r_{j+1}^{-,0,+}$ and $q_{j+1}^{-,0,+}$ are of similar form at the time level $j+1$. This scheme gives the exact solution for solving (1.1) in the case $b(x,t)$ and $d(x,t)$ are constants and $f(x,t) = 0$.

5. Numerical experiments. In this section we present some numerical results for the proposed schemes in Sections 2 and 4. The numerical examples are solved on $\Omega \equiv (0,1) \times (0,1]$.

Example 1. We consider the linear equation

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial x}, \quad \text{on} \quad \Omega,$$

with steady-state solution $u(x,t) = (e^{bx/\varepsilon} - 1)/(e^{b/\varepsilon} - 1)$. The function $u(x,t)$ determines the initial and boundary conditions for this problem. We compare the present fourth-order scheme (CS4) and the present exponential fourth-order scheme (ES4) with Crank-Nicolson scheme (CNS), see [**9**]; Gears scheme (GS), see [**18**]; extrapolated Crank-Nicolson scheme (ECNS), see [**23**]; the method (AGE), see [**15**]; and scheme (PQI), see [**13**]. The absolute error between the approximate solution and steady state solution is shown in Table I at $\varepsilon = 1$, $b = 1$, $\Delta t = 0.01$, $h = 0.1$ and $t = 1$, and errors that are less than 10^{-15} are recorded as zero in the table.

Example 2. We consider the linear parabolic partial differential equation

$$-\varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + u + \frac{\partial u}{\partial t} = f(x,t), \quad \text{on} \quad \Omega,$$

with analytical solution

$$u(x,t) = t \exp(-(1-x)/\varepsilon) + 1 - x^2 + t^2.$$

The function $f(x,t)$ and the initial-boundary values on Ω are chosen to fit this data. We solve this problem for each $h = 1/N$, $N = 8, 16, 32, 64$ and 128, with $\varepsilon = 1/64$, 1/256, 1/1024 and 1/4096. The maximum error, E^N, and the order of convergence, Rate, are defined, respectively, in the form

$$E^N = \max_{i,j} |U_i^j - u(x_i, t_j)|,$$

$$\text{Rate} = \frac{\log(E^N/E^{2N})}{\log 2},$$

$$i = 1, 2, \ldots, (N-1), \quad j = 1, 2, \ldots .$$

TABLE I.

The Scheme	x	0.2	0.3	0.4	0.5	0.6	0.7	0.8
CNS		1.10E-06	8.30E-06	1.70E-06	4.10E-06	9.50E-07	2.40E-06	1.60E-06
GS		2.20E-06	9.90E-06	4.10E-06	7.50E-06	5.00E-06	7.30E-06	5.70E-06
ECNS		7.20E-07	7.50E-06	6.60E-07	2.90E-06	1.20E-07	1.40E-06	7.70E-07
AGE		5.20E-05	7.20E-05	8.80E-05	9.80E-05	1.00E-04	9.40E-05	7.60E-05
PQI		1.30E-09	1.90E-09	2.40E-09	2.60E-09	2.60E-09	2.30E-09	1.80E-09
CS4		9.36E-10	1.07E-09	1.20E-09	1.33E-09	1.46E-09	1.60E-09	1.68E-09
ES4		0.0	0.0	0.0	0.0	0.0	0.0	0.0

We compare the present fourth-order scheme (CS4) and the present exponential fourth-order scheme (ES4) with the streamline diffusion scheme (SD), see [**19**]. The results are shown in Table II, and errors that are less than 10^{-15} are recorded as zero in the table.

Example 3. We consider the linear parabolic partial differential equation with variable coefficients

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x}, \quad \text{on} \quad \Omega,$$

where

$$b(x,t) = -u(x,t),$$

with the exact solution

$$u(x,t) = \frac{0.1e^{-A} + 0.5e^{-B} + e^{-C}}{e^{-A} + e^{-B} + e^{-C}},$$

where

$$A = \frac{0.05}{\varepsilon}(x - 0.5 + 4.95t),$$
$$B = \frac{0.25}{\varepsilon}(x - 0.5 + 0.75t)$$

and

$$C = \frac{0.5}{\varepsilon}(x - 0.375).$$

The initial-boundary values on Ω are chosen to fit this data. We solve this problem for each $h = 1/N$, $N = 10, 20, 40, 80$, with $\varepsilon = 0.1$ and 0.01 and $t = 0.4$ and 1.0. The maximum error and the order of convergence are given. We compare the present fourth-order scheme (CS4) with the present exponential fourth-order scheme (ES4). The values of E^N and Rate are given in Table III.

TABLE II.

ε		1/64		1/256		1/1024		1/4096	
The scheme	N	E^N	(Rate)	E^N	(Rate)	E^N	(Rate)	E^N	(Rate)
SD	8	1.64E-01		1.99E-01		2.14E-01		2.19E-01	
			(0.99)		(0.92)		(0.85)		(0.81)
	16	8.27E-02		1.05E-01		1.19E-01		1.24E-01	
			(1.31)		(1.15)		(0.95)		(0.85)
	32	3.33E-02		4.76E-02		6.18E-02		6.95E-02	
			(1.32)		(1.18)		(1.03)		(0.91)
	64	1.33E-02		2.09E-02		3.03E-02		3.68E-02	
			(1.45)		(1.26)		(1.11)		(0.96)
	128	4.88E-03		8.76E-03		1.40E-02		1.89E-02	
CS4	8	1.91E-01		5.98E-01		7.91E-01		8.43E-01	
			(1.84)		(0.44)		(0.02)		(-0.08)
	16	5.32E-02		4.38E-01		7.76E-01		8.92E-01	
			(2.91)		(1.02)		(0.22)		(0.01)
	32	7.03E-03		2.16E-01		6.64E-01		8.81E-01	
			(3.77)		(1.91)		(0.51)		(0.11)
	64	5.17E-04		5.72E-02		4.64E-01		8.16E-01	
			(4.06)		(2.94)		(1.05)		(0.25)
	128	3.09E-05		7.41E-03		2.23E-01		6.82E-01	
ES4	8	0.0		0.0		0.0		0.0	

TABLE III.

ε				0.1		0.01	
The scheme	t	N	Δt	E^N	(Rate)	E^N	(Rate)
CS4	0.4	10	0.05	4.47E-05		2.27E-02	
					(4.01)		(2.68)
		20	0.0125	2.77E-06		3.53E-03	
					(4.00)		(3.73)
		40	0.003125	1.73E-07		2.66E-04	
					(4.00)		(4.02)
		80	0.00078125	1.08E-08		1.63E-05	
	1.0	10	0.05	8.36E-05		5.79E-02	
					(3.98)		(2.27)
		20	0.0125	5.29E-06		1.20E-02	
					(4.00)		(3.34)
		40	0.003125	3.30E-07		1.18E-03	
					(4.00)		(3.96)
		80	0.00078125	2.06E-08		7.56E-05	
ES4	0.4	10	0.05	4.76E-05		3.87E-02	
					(4.00)		(2.56)
		20	0.0125	2.96E-06		6.56E-03	
					(4.00)		(3.43)
		40	0.003125	1.85E-07		6.02E-04	
					(4.00)		(3.92)
		80	0.00078125	1.15E-08		3.97E-05	
	1.0	10	0.05	9.34E-05		9.47E-02	
					(3.93)		(2.27)
		20	0.0125	5.87E-06		1.96E-02	
					(3.99)		(3.40)
		40	0.003125	3.67E-07		1.85E-03	
					(4.00)		(3.88)
		80	0.00078125	2.29E-08		1.25E-04	

Example 4. We consider the Burgers equation in the form

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - u \frac{\partial u}{\partial x}, \quad \text{on} \quad \Omega,$$

with the exact solution

$$u(x,t) = \frac{0.1e^{-A} + 0.5e^{-B} + e^{-C}}{e^{-A} + e^{-B} + e^{-C}},$$

where

$$A = \frac{0.05}{\varepsilon}(x - 0.5 + 4.95t),$$
$$B = \frac{0.25}{\varepsilon}(x - 0.5 + 0.75t),$$

and

$$C = \frac{0.5}{\varepsilon}(x - 0.375).$$

The initial-boundary values on Ω are chosen to fit this data. We solve this problem for each $h = 1/N$, $N = 10$, 20, 40, 80, with $\varepsilon = 0.1$ and 0.01 and $t = 0.4$ and 1.0. We compare the present fourth-order scheme (CS4) with the present exponential fourth-order scheme (ES4). The values of E^N and Rate are given in Table IV.

Conclusions. We have described numerical schemes for solving parabolic singular perturbation problems using tridiagonal schemes of exponential type. The schemes have been analyzed for convergence. Test examples have been solved to demonstrate the efficiency of the proposed schemes.

Tables I–III described the linear problems and gave a comparison of the numerical solution of the various schemes for different values of h and ε. The present schemes are the most accurate schemes of those tested in these tables.

TABLE IV.

ε				0.1		0.01	
The scheme	t	N	Δt	E^N	(Rate)	E^N	(Rate)
CS4	0.4	10	0.05	4.94E-05		2.87E-02	
					(4.01)		(2.67)
		20	0.0125	3.06E-06		4.49E-03	
					(3.99)		(3.71)
		40	0.003125	1.92E-07		4.41E-04	
					(4.00)		(4.03)
		80	0.00078125	1.20E-08		2.08E-05	
	1.0	10	0.05	1.04E-04		8.52E-02	
					(3.96)		(2.30)
		20	0.0125	6.65E-06		1.73E-02	
					(4.00)		(3.38)
		40	0.003125	4.14E-07		1.66E-03	
					(4.00)		(3.81)
		80	0.00078125	2.58E-08		1.18E-04	
ES4	0.4	10	0.05	5.26E-05		4.64E-02	
					(4.00)		(2.40)
		20	0.0125	3.27E-06		8.77E-03	
					(4.00)		(3.34)
		40	0.003125	2.04E-07		4.64E-04	
					(4.00)		(3.90)
		80	0.00078125	1.87E-08		5.76E-05	
	1.0	10	0.05	1.16E-04		1.50E-01	
					(3.98)		(1.76)
		20	0.0125	7.35E-06		4.42E-02	
					(4.00)		(3.25)
		40	0.003125	4.59E-07		4.63E-03	
					(3.99)		(3.89)
		80	0.00078125	2.87E-08		3.12E-04	

Table IV gives the numerical solution for different values of h and ε for solving the Burgers equation. The present schemes are second-order accurate in Δt and fourth order accurate in h. They are unconditionally stable.

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