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CHARACTERIZATIONS OF GENERALIZED NEVAI'S CLASS AT THE BOUNDARY POINTS OF CONTRACTED ZEROS

D.W. LEE

ABSTRACT. For generalized Nevai's class, the ratio asymptotics of orthogonal polynomials are obtained by many authors outside the contracted zero interval. We prove that the asymptotic properties can be extended to the point which is essentially a boundary point of the contracted zeros by the coefficients of three term recurrence relation and then several characterizations are found. Lastly we give several applications for such characterizations.

1. Introduction. Let μ be a positive measure with infinitely many points of increase and all moments are finite. The corresponding orthonormal polynomials $\{P_n(x)\}_{n=0}^{\infty}$ such that

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\, d\mu(x) = \delta_{mn}, \quad m, n \geq 0,$$

where δ_{mn} is the Kronecker delta, satisfy a three term recurrence relation

$$\begin{gathered} xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \\ P_0(x) = 1, \quad P_{-1} = 0, \\ n = 0, 1, 2, \dots, \end{gathered}$$

where the coefficients are uniquely determined by

$$a_n = \int_{-\infty}^{\infty} xP_n(x)P_{n-1}(x)\, d\mu(x) > 0, \quad n = 1, 2, \dots,$$

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and

$$b_n = \int_{-\infty}^{\infty} x P_n^2(x)\, d\mu(x), \quad n = 0, 1, 2, \dots .$$

For any regularly varying function φ, that is, $\varphi(x) > 0$ and

$$\lim_{t\to\infty} \frac{\varphi(x+t)}{\varphi(t)} = 1$$

for every real x, if there exist $a \ge 0$ and $b \in \mathbf{R}$ such that

$$\lim_{n\to\infty} \frac{a_n}{\varphi(n)} = \frac{a}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{\varphi(n)} = b,$$

then μ is called a generalized Nevai class (relative to φ) and denoted by $\mu \in \mathcal{M}_\varphi(a, b)$. It is well known that all the zeros of $P_n(x)$ are real and simple. We denote them by x_{jn}, $j = 1, 2, \dots, n$, with $x_{nn} < x_{n-1,n} < \cdots < x_{1n}$. The Christoffel function is a meromorphic function defined by

$$\lambda_n^{-1}(z) = \sum_{k=0}^{n-1} P_k^2(z)$$

and the Christoffel numbers by $\lambda_{jn} = \lambda_n(x_{jn})$. We also define the contracted zero interval $\Delta_\varphi(d\mu)$ of $\{P_n(x)\}_{n=0}^{\infty}$ by

$$\Delta_\varphi(d\mu) = \left[\liminf_{n\to\infty} \frac{x_{nn}}{\varphi(n)},\ \limsup_{n\to\infty} \frac{x_{1n}}{\varphi(n)}\right].$$

Going back to Blumental's work [**1**], there are many results on the asymptotic properties of orthogonal polynomials and the Christoffel functions by the coefficients of the three term recurrence relation for generalized Nevai's class. We refer to [**3, 6, 9, 10, 13**] and therein. More precisely, the following were proved by many authors.

Theorem A. *Let $a > 0$, $b \in \mathbf{R}$ and φ be a regularly varying function. Then the following are all equivalent.*

(a) $\mu \in \mathcal{M}_\varphi(a, b)$.

(b) *If f is bounded on $\Delta_\varphi(d\mu)$ and Riemann integrable on $[b-a, b+a]$, then*

(1.1)

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi(n)}\right)P_{n-1}^2(x_{jn}) = \frac{2}{\pi a^2}\int_{b-a}^{b+a} f(x)\sqrt{a^2-(x-b)^2}\,dx.$$

(c) *For every $z \in \mathbf{C}\setminus\Delta_\varphi(d\mu)$,*

$$\lim_{n\to\infty}\frac{P_{n-1}(\varphi(n)z)}{P_n(\varphi(n)z)} = \frac{a}{z-b+\sqrt{(z-b)^2-a^2}}\,,$$

where $\sqrt{(z-b)^2-a^2} > 0$ for $|z-b| > a$.

If $\mu \in \mathcal{M}_\varphi(a,b)$, $a > 0$, $b \in \mathbf{R}$, then by Theorem A (b) with $f = \chi_{[b-a,b+a]\setminus\Delta_\varphi(d\mu)}$ in (1.1), we obtain

$$\begin{aligned} 0 &= \frac{2}{\pi a^2}\int_{b-a}^{b+a} f(x)\sqrt{a^2-(x-b)^2}\,dx \\ &= \frac{2}{\pi a^2}\int_{[b-a,b+a]\setminus\Delta_\varphi(d\mu)}\sqrt{a^2-(x-b)^2}\,dx \end{aligned}$$

so that $[b-a, b+a]\setminus\Delta_\varphi(d\mu) = \varnothing$, that is, $[b-a, b+a] \subset \Delta_\varphi(d\mu)$. More precisely, it is known [**6, 13**] that if $\varphi(x)$ is nondecreasing, then

$$[b-a, b+a] \subset \Delta_\varphi(d\mu) \subset [\min\{b-a, 0\}\,, \max\{0, b+a\}]$$

and there are no limit points of $\Delta_\varphi(d\mu)$ outside $[b-a, b+a]$. Hence, the points $b-a$ and $b+a$ are essentially the boundary points of $\Delta_\varphi(d\mu)$.

In this paper, we prove that Theorem A can be extended to the points $b-a$ and $b+a$, which is already proved in [**11**] for the case of $\varphi = 1$. Throughout the paper, we will use the notations by $\varphi_n = \varphi(n)$ for regularly varying function φ, $U_n(x)$ for the Tchebychev polynomials of second kind with

$$U_n^{a,b}(x) := U_n((x-b)/a),$$

and

$$I_k^{a,b}(d\nu; f) := -\operatorname{sgn}(k) \int_{b-a}^{b+a} f(x) U_{|k|-1}^{a,b}(x)\, d\nu(x),$$
$$U_{-1} = 0,\ k \in \mathbf{Z}$$

for the measure $d\nu$.

2. Main results.

We start with a lemma whose proof is simple.

Lemma 2.1. *Let $\{e_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. If*

$$\lim_{n\to\infty} \left(\frac{1}{e_{n+1}} + d_n e_n \right) = a, \tag{2.1}$$

where $a > 0$ and $\lim_{n\to\infty} d_n = a^2/4$, then $\lim_{n\to\infty} e_n = 2/a$.

Proof. Without loss of generality we may assume that $a = 1$ and $d_n > 0$ for all n. Let $\alpha = \liminf_{n\to\infty} e_n$ and $\beta = \limsup_{n\to\infty} e_n$. From equation (2.1), we have that $\beta \leq 4$ and clearly $\alpha \geq 0$. It's enough to show that $\alpha = \beta = 2$. Let $\{e_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{e_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} e_{n_k} = \alpha$. Then

$$\lim_{k\to\infty} \left(\frac{1}{e_{n_k}} + d_{n_k-1} e_{n_k-1} \right) = 1,$$

from which $\alpha > 0$. Moreover,

$$\lim_{k\to\infty} e_{n_k-1} = 4\left(1 - \frac{1}{\alpha}\right) = \frac{4\alpha - 4}{\alpha}.$$

By the definition of $\liminf$, we have $\alpha \leq (4\alpha - 4)/\alpha$ so that $\alpha = 2$. By a similar process, we have

$$\beta \geq \lim_{k\to\infty} e_{n_k+1} = \frac{4}{4-\beta}$$

which also implies $\beta = 2$. $\square$

Theorem 2.2. *Let $\mu \in \mathcal{M}_\varphi(a,b)$ for some $a > 0$ and $b \in \mathbf{R}$. If $P_{n-1}((b+a)\varphi_n)P_{n+1}((b+a)\varphi_n) > 0$ for large n, then*

$$\lim_{n\to\infty} \frac{P_{n-1}((b+a)\varphi_n)}{P_n((b+a)\varphi_n)} = \lim_{n\to\infty} \frac{a_n}{\varphi_n} \sum_{j=1}^{n} \frac{\lambda_{jn} P_{n-1}^2(x_{jn})}{b+a-(x_{jn}/\varphi_n)} = 1 \tag{2.2}$$

and if $P_{n-1}((b-a)\varphi_n)P_{n+1}((b-a)\varphi_n) > 0$ for large n, then

$$\lim_{n\to\infty} \frac{P_{n-1}((b-a)\varphi_n)}{P_n((b-a)\varphi_n)} = \lim_{n\to\infty} \frac{a_n}{\varphi_n} \sum_{j=1}^{n} \frac{\lambda_{jn} P_{n-1}^2(x_{jn})}{b-a-(x_{jn}/\varphi_n)} = -1. \tag{2.3}$$

In this case,

(2.4)

$$\lim_{n\to\infty} \sum_{j=1}^{n} \frac{\lambda_{jn} P_{n-1}^2(x_{jn})}{a^2 - ((x_{jn}/\varphi_n) - b)^2} = \frac{2}{a^2} = \frac{2}{\pi a^2} \int_{b-a}^{b+a} \frac{dx}{\sqrt{a^2-(x-b)^2}}.$$

Proof. From the three term recurrence relation, we have $P_n((b+a)\varphi_n) \neq 0$ and

$$\frac{\varphi_n}{\varphi_{n+1}}\left(b+a-\frac{b_n}{\varphi_n}\right) = \frac{a_{n+1}P_{n+1}((b+a)\varphi_n)}{\varphi_{n+1}P_n((b+a)\varphi_n)} + \frac{a_n^2}{\varphi_{n+1}\varphi_n} \frac{\varphi_n P_{n-1}((b+a)\varphi_n)}{a_n P_n((b+a)\varphi_n)}.$$

Note that the coefficients a_n of the three term recurrence relation are positive for $n \geq 1$. Hence, if $P_{n+1}((b+a)\varphi_n) > 0$ and so $P_{n-1}((b+a)\varphi_n) > 0$, then

$$\frac{\varphi_n}{\varphi_{n+1}}\left(b+a-\frac{b_n}{\varphi_n}\right)P_n((b+a)\varphi_n) > 0.$$

Since $\lim_{n\to\infty} b_n/\varphi_n = b$, we have $P_n((b+a)\varphi_n) > 0$ for sufficiently large n. If $P_{n+1}((b+a)\varphi_n) < 0$ and so $P_{n-1}((b+a)\varphi_n) < 0$, then by the same process $P_n((b+a)\varphi_n) < 0$ for sufficiently large n. Hence, $(P_{n-1}((b+a)\varphi_n))/(P_n((b+a)\varphi_n)) > 0$ and so we have by Lemma 2.1,

$$\lim_{n\to\infty} \frac{\varphi_n}{a_n} \frac{P_{n-1}((b+a)\varphi_n)}{P_n((b+a)\varphi_n)} = \frac{2}{a}. \tag{2.5}$$

The first equality of (2.2) immediately follows from the identity (cf. [**13**, p. 9])

$$\frac{\widehat{P}_{n-1}(x)}{\widehat{P}_n(x)} = \sum_{j=1}^{n} \frac{\lambda_{jn} P_{n-1}^2(x_{jn})}{x - x_{jn}} \tag{2.6}$$

where $\widehat{P}_n(x) = a_n a_{n-1} \cdots a_1 P_n(x)$ is the monic polynomial. The equation (2.3) can be proved by the same way. The left equality in (2.4) is an immediate consequence of the following identity

$$\begin{aligned}\lim_{n\to\infty} \sum_{j=1}^{n} &\frac{\lambda_{jn} P_{n-1}^2(x_{jn})}{a^2 - ((x_{jn})/(\varphi_n) - b)^2} \\ &= \frac{1}{2a}\left(\frac{\varphi_n}{a_n}\,\frac{P_{n-1}((b+a)\varphi_n)}{P_n((b+a)\varphi_n)} - \frac{\varphi_n}{a_n}\,\frac{P_{n-1}((b-a)\varphi_n)}{P_n((b-a)\varphi_n)}\right),\end{aligned}$$

and the right equality can be shown by a simple calculation. □

By three term recurrence relations of $\{P_n(x)\}_{n=0}^{\infty}$ and $\{U_n(x)\}_{n=0}^{\infty}$, we can easily see that condition (b) in Theorem A can be generalized as following.

Theorem 2.3. *Let the conditions be the same as in Theorem* A. *Then any one of* (a)$\sim$(c) *in Theorem* A *is also equivalent to*

(b$'$) *If* f *is bounded on* $\Delta_\varphi(d\mu)$ *and Riemann integrable on* $[b-a, b+a]$, *then for any integer* k,

$$\begin{aligned}\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) &P_{n-1}(x_{jn}) P_{n+k}(x_{jn}) \\ &= I_k^{a,b}\left(\frac{2}{\pi a^2}\sqrt{a^2 - (x - b^2)}\,dx; f\right).\end{aligned}$$

Proof. It suffices to prove that (b) implies (b$'$) where the converse is trivial. Induction will be used. If $k = 0$, then it is trivial. Let $k = 1$. Since $P_{n+1}(x_{jn}) = -(a_n/a_{n+1})P_{n-1}(x_{jn})$ by the three term recurrence relation, we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)&P_{n-1}(x_{jn})P_{n+1}(x_{jn})\\ &= -\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)P_{n-1}^2(x_{jn})\\ &= -\frac{2}{\pi a^2}\int_{b-a}^{b+a} f(t)\sqrt{a^2-(t-b)^2}\,dt\\ &= I_1^{a,b}\left(\frac{2}{\pi a^2}\sqrt{a^2-(x-b)^2}\,dx; f\right).\end{aligned}$$

Now assume that the theorem is true for $0, 1, 2, \dots, k-1$, $k \geq 2$. By the three term recurrence relation again, we have

$$P_{n+k}(x_{jn}) = \frac{\varphi_n}{a_{n+k}}\left(\frac{x_{j,n}}{\varphi_n}-\frac{b_{n+k-1}}{\varphi_n}\right)P_{n+k-1}(x_{jn})-\frac{a_{n+k-1}}{a_{n+k}}\,P_{n+k-2}(x_{jn})$$

and so

$$\begin{aligned}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)&P_{n-1}(x_{jn})P_{n+k}(x_{jn})\\ &= \frac{\varphi_n}{a_{n+k}}\sum_{j=1}^{n}\lambda_{jn}\frac{x_{jn}}{\varphi_n}\,f\left(\frac{x_{jn}}{\varphi_n}\right)P_{n-1}(x_{jn})P_{n+k-1}(x_{jn})\\ &\quad -\frac{b_{n+k-1}}{a_{n+k}}\sum_{j=1}^{n}\lambda_{j,n}f\left(\frac{x_{j,n}}{\varphi_n}\right)P_{n-1}(x_{jn})P_{n+k-1}(x_{jn})\\ &\quad -\frac{a_{n+k-1}}{a_{n+k}}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)P_{n-1}(x_{jn})P_{n+k-2}(x_{jn}).\end{aligned}$$

By induction hypothesis, the regularity of φ, and the three term recurrence relation

$$\frac{x-b}{a}\,U_n^{a,b}(x) = \frac{1}{2}\,U_{n+1}^{a,b}(x)+\frac{1}{2}\,U_{n-1}^{a,b}(x), \quad n \geq 0,$$

taking a limit, we have for $k \geq 2$,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) P_{n-1}(x_{jn}) P_{n+k}(x_{jn}) \\
&= -\frac{2}{a}\int_{b-a}^{b+a} t f(t)\, U_{k-2}^{a,b}(t)\, \frac{2}{\pi a^2}\, \sqrt{a^2-(t-b)^2}\, dt \\
&\quad + \frac{2b}{a}\int_{b-a}^{b+a} f(t)\, U_{k-2}^{a,b}(t)\, \frac{2}{\pi a^2}\, \sqrt{a^2-(t-b)^2}\, dt \\
&\quad + \int_{b-a}^{b+a} f(t)\, U_{k-3}^{a,b}(t)\, \frac{2}{\pi a^2}\, \sqrt{a^2-(t-b)^2}\, dt \\
&= -\frac{2}{\pi a^2}\int_{b-a}^{b+a} f(t)\, U_{k-1}^{a,b}(t) \sqrt{a^2-(t-b)^2}\, dt \\
&= I_k^{a,b}\left(\frac{2}{\pi a^2}\, \sqrt{a^2-(x-b)^2}\, dx; f\right).
\end{aligned}$$

Hence, the theorem is true for the case of $k \geq -1$. For $k \leq -2$, it can be proved by the same method as above. $\square$

Theorem 2.4. *Let $a > 0$, $b \in \mathbf{R}$, and let φ be a regularly varying function. If $P_{n+1}((b+a)\varphi_n)P_{n-1}((b+a)\varphi_n) > 0$ and $P_{n+1}((b-a) \times \varphi_n)P_{n-1}((b-a)\varphi_n) > 0$ for all $n \geq 1$, then the following are all equivalent.*

(a) $\mu \in \mathcal{M}_\varphi(a,b)$.

(b) *If f is bounded on $\Delta_\varphi(d\mu)$ and Riemann integrable on $[b-a, b+a]$, then for any integer k,*

$$\begin{aligned}
\lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}(x_{jn})P_{n+k}(x_{jn})}{(b+a)-(x_{jn}/\varphi_n)} \\
&= I_k^{a,b}\left(\frac{2}{\pi a^2}\, \sqrt{\frac{a+(x-b)}{a-(x-b)}}\, dx; f\right).
\end{aligned}$$

(c) *If f is bounded on $\Delta_\varphi(d\mu)$ and Riemann integrable on $[b-a, b+a]$,* *then for any integer k,*

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)\frac{P_{n-1}(x_{jn})P_{n+k}(x_{jn})}{(b-a)-(x_{jn}/\varphi_n)} = -I_k^{a,b}\left(\frac{2}{\pi a^2}\sqrt{\frac{a-(x-b)}{a+(x-b)}}\,dx;f\right).$$

(d) *If f is bounded on $\Delta_\varphi(d\mu)$ and Riemann integrable on $[b-a,b+a]$, then for any integer k,*

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}f\left(\frac{x_{jn}}{\varphi_n}\right)\frac{P_{n-1}(x_{jn})P_{n+k}(x_{jn})}{a^2-(b-(x_{jn}/\varphi_n))^2} = I_k^{a,b}\left(\frac{2}{\pi a^2}\,\frac{dx}{\pi\sqrt{a^2-(x-b)^2}};f\right). \tag{2.7}$$

(e) *For every $z\in\{b\pm a\}\cup\mathbf{C}\setminus\Delta_\varphi(d\mu)$,*

$$\lim_{n\to\infty}\frac{P_{n-1}(\varphi_n z)}{P_n(\varphi_n z)} = \begin{cases} 1 & \text{if } z=b+a\\ -1 & \text{if } z=b-a\\ a/(z-b+\sqrt{(z-b)^2-a^2}) & \text{otherwise,}\end{cases}$$

where $\sqrt{(z-b)^2-a^2}>0$ for $|z-b|>a$.

Proof. Without loss of generality, we may assume that $a=1$ and $b=0$. The equivalence (a) and (e) was proved by Theorem A and Theorem 2.2. Note that (b) and (c) can be easily proved from (d) by taking $(1-x)f(x)$ and $(1+x)f(x)$, respectively. It is also easy to see that (b), respectively (c), implies (a) by Theorem A with $(1-x)f(x)$, respectively $(1+x)f(x)$. Hence, it suffices to show that (a) implies (d). Let $0<\varepsilon<1$ and $g(x)=1/(1-x^2)\chi_{[-1+\varepsilon,1-\varepsilon]}(x)$. By Theorem A with $k=-1$, we have

$$\lim_{n\to\infty}\sum_{|x_{jn}/\varphi_n|\le 1-\varepsilon}\frac{\lambda_{jn}P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} = \lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn}\,g\left(\frac{x_{jn}}{\varphi_n}\right)P_{n-1}^2(x_{jn}) = \frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon}\frac{dx}{\sqrt{1-x^2}}$$

so that by Theorem 2.2,

$$\begin{aligned}
\lim_{n\to\infty} &\sum_{|x_{jn}/\varphi_n|>1-\varepsilon} \frac{\lambda_{jn}P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} \\
&= \lim_{n\to\infty} \left(\sum_{j=1}^{n} \frac{\lambda_{jn}P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} - \sum_{|x_{jn}/\varphi_n|\le 1-\varepsilon} \frac{\lambda_{jn}P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} \right) \\
&= \frac{2}{\pi}\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} - \frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} \\
&= \frac{4}{\pi}\int_{1-\varepsilon}^{1} \frac{dx}{\sqrt{1-x^2}}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
(2.8)\quad &\left| \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} - \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \right| \\
&= \Bigg| \Bigg[\sum_{|x_{jn}/\varphi_n|\le 1-\varepsilon} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} - \frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon} \frac{f(x)}{\sqrt{1-x^2}}\,dx \Bigg] \\
&\quad + \sum_{|x_{jn}/\varphi_n|>1-\varepsilon} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} \\
&\quad - \frac{2}{\pi}\left[\int_{-1}^{-1+\varepsilon} + \int_{1-\varepsilon}^{1}\right] \frac{f(x)}{\sqrt{1-x^2}}\,dx \Bigg| \\
&\le \left| \sum_{|x_{jn}/\varphi_n|\le 1-\varepsilon} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} - \frac{2}{\pi}\int_{-1+\varepsilon}^{1-\varepsilon} \frac{f(x)}{\sqrt{1-x^2}}\,dx \right| \\
&\quad + \sup_{-1\le x\le 1} |f(x)| \left[\sum_{|x_{jn}/\varphi_n|>1-\varepsilon} \frac{\lambda_{jn}P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} + \frac{4}{\pi}\int_{1-\varepsilon}^{1} \frac{dx}{\sqrt{1-x^2}} \right].
\end{aligned}$$

Since the first term of the right-hand side of (2.8) tends to 0 as $n\to\infty$ by Theorem A with the function $f(x)=1/(1-x^2)\chi_{[-1+\varepsilon,1-\varepsilon]}$, taking limsup on both sides, we have

$$\begin{aligned}
\limsup_{n\to\infty} &\left| \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}^2(x_{jn})}{1-(x_{jn}/\varphi_n)^2} - \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\,dx \right| \\
&\le \sup_{-1\le x\le 1} |f(x)| \frac{8}{\pi}\int_{1-\varepsilon}^{1} \frac{dx}{\sqrt{1-x^2}}.
\end{aligned}$$

Letting $\varepsilon \to 0$, the relation (2.7) holds for $k = -1$. For $k = 0$, it is trivial. The proof for every integer k is just the same as that of Theorem 2.3 by induction and three term recurrence relations of $\{P_n(x)\}_{n=0}^\infty$ and $\{U_n^{a,b}(x)\}_{n=0}^\infty$. $\square$

Example 2.1. Consider Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^\infty$, $\alpha > -1$, satisfying (cf. [**2, 12**])

$$\begin{aligned} xL_n^{(\alpha)}(x) &= \sqrt{(n+1)(n+\alpha+1)}\, L_{n+1}^{(\alpha)}(x) \\ &\quad + (2n+\alpha+1)L_n^{(\alpha)}(x) + \sqrt{n(n+\alpha)}\, L_{n-1}^{(\alpha)}(x), \quad n \ge 0, \end{aligned}$$

which are orthogonal with respect to $d\mu(x) = x^\alpha e^{-x} H(x)\, dx$, where $H(x)$ is the Heaviside function. Clearly, $\mu \in \mathcal{M}_\varphi(2,2)$, where $\varphi_n = n+\varepsilon_n$ with $\lim_{n\to\infty}(\varepsilon_n/n) = 0$. Moreover it is well known (cf. [**5**]) that $\lim_{n\to\infty}(x_{1n}/n) = 4$ and $\lim_{n\to\infty}(x_{nn}/n) = 0$ so that $\Delta_\varphi(d\mu) = [0,4]$. Let ε_n be a sequence such that $4\varphi_n > x_{1,n+1}$. Then by Theorem 2.4,

$$\lim_{n\to\infty} \sum_{j=1}^n \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{L_{n-1}^{(\alpha)}(x_{jn}) L_{n+k}^{(\alpha)}(x_{jn})}{4-(x_{jn}/\varphi_n)} = I_k^{2,2}\left(\frac{1}{2\pi}\sqrt{\frac{x}{4-x}}\, dx; f\right)$$

and

$$\lim_{n\to\infty} \sum_{j=1}^n \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{L_{n-1}^{(\alpha)}(x_{jn}) L_{n+k}^{(\alpha)}(x_{jn})}{x_{jn}/\varphi_n} = -I_k^{2,2}\left(\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}\, dx; f\right).$$

In particular,

$$\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}(4\varphi_n)}{L_n^{(\alpha)}(4\varphi_n)} = -\lim_{n\to\infty} \frac{L_{n-1}^{(\alpha)}(0)}{L_n^{(\alpha)}(0)} = 1.$$

Example 2.2. Consider the Meixner-Pollaczek polynomials

$$\{P_n^{(\lambda,\phi)}(x)\}_{n=0}^\infty$$

satisfying (see [**4**, p. 32])

$$\begin{aligned} xP_n^{(\lambda,\phi)}(x) &= \frac{\sqrt{(n+1)(n+2\lambda)}}{2\sin\phi}\, P_{n+1}^{(\lambda,\phi)}(x) - \frac{(n+\lambda)\cos\phi}{\sin\phi}\, P_n^{(\lambda,\phi)}(x) \\ &\quad + \frac{\sqrt{n(n+2\lambda-1)}}{2\sin\phi}\, P_{n-1}^{(\lambda,\phi)}(x), \quad n \ge 0, \end{aligned}$$

which are orthogonal with respect to $d\mu(x) = W^{\lambda,\phi}(x)\,dx$, where

$$W^{\lambda,\phi}(x) = e^{(2\phi-\pi)x}\left|\Gamma(\lambda+ix)\right|^2,$$
$$0<\phi<\pi,\ \lambda>0, \quad -\infty<x<\infty.$$

Here $\Gamma(x)$ is the Gamma function on the complex plane. Clearly, $\mu \in \mathcal{M}_\varphi((1/\sin\phi), -(\cos\phi/\sin\phi))$, where $\varphi_n = n+\varepsilon_n$ with $\lim_{n\to\infty}\varepsilon_n/n = 0$. It is well known (see [**5, 13**]) that

$$\Delta_\varphi(d\mu) = \big[-(1+\cos\phi/\sin\phi),\ (1-\cos\phi/\sin\phi)\big].$$

If ε_n is taken by $(1-\cos\phi/\sin\phi)\varphi_n > x_{1,n+1}$, then by Theorem 2.4,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right)\frac{P_{n-1}^{(\lambda,\phi)}(x_{jn})P_{n+k}^{(\lambda,\phi)}(x_{jn})}{(1-\cos\phi/\sin\phi)-(x_{jn}/\varphi_n)} = I_k^{a,b}(w dx; f)$$

and if ε_n is taken by $-(1+\cos\phi/\sin\phi)\varphi_n < x_{n+1,n+1}$, then by Theorem 2.4,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right)\frac{P_{n-1}^{(\lambda,\phi)}(x_{jn})P_{n+k}^{(\lambda,\phi)}(x_{jn})}{(1+\cos\phi/\sin\phi)+(x_{jn}/\varphi_n)} = I_k^{a,b}(\widetilde{w} dx; f),$$

where

$$\begin{aligned} a &= 1/(\sin\phi),\\ b &= -(\cos\phi/\sin\phi),\\ w(x) &= (2\sin^2\phi/\pi)\sqrt{(1-\cos\phi+\sin\phi x)/(1+\cos\phi-\sin\phi x)}, \end{aligned}$$

and

$$\widetilde{w}(x) = (2\sin^2\phi/\pi)\sqrt{(1+\cos\phi-\sin\phi x)/(1-\cos\phi+\sin\phi x)}.$$

Moreover,

$$\begin{aligned} &\lim_{n\to\infty}\frac{P_{n-1}^{(\lambda,\phi)}(((1-\cos\phi)\varphi_n)/(\sin\phi))}{P_{n}^{(\lambda,\phi)}(((1-\cos\phi)\varphi_n)/(\sin\phi))}\\ &\qquad = -\lim_{n\to\infty}\frac{P_{n-1}^{(\lambda,\phi)}(-((1+\cos\phi)\varphi_n)/(\sin\phi))}{P_{n}^{(\lambda,\phi)}(-((1+\cos\phi)\varphi_n)/(\sin\phi))} = 1. \end{aligned}$$

Example 2.3. Let $\{P_n(x)\}_{n=0}^{\infty}$ be Freud polynomials which are orthogonal with respect to $d\mu(x) = e^{-|x|^\alpha}\,dx$, $\alpha > -1$. Since the measure $d\mu$ is symmetric, $b_n = 0$ for all $n \geq 1$. It was proved by Lubinsky et al. [**7, 8**] that

$$\lim_{n\to\infty} n^{-1/\alpha} a_n = \frac{\gamma_\alpha}{2} := \left[\frac{\Gamma(\alpha/2)\Gamma((\alpha/2)+1)}{\Gamma(\alpha+1)}\right]^{1/\alpha}$$

and

$$\lim_{n\to\infty} n^{-1/\alpha} x_{1n} = -\lim_{n\to\infty} n^{-1/\alpha} x_{nn} = \gamma_\alpha.$$

It is also well known (see [**5, 13**]) that $\mu \in \mathcal{M}_\varphi(\gamma_\alpha, 0)$ and $\Delta_\varphi(d\mu) = [-\gamma_\alpha,\, \gamma_\alpha]$, where $\varphi_n = n^{1/\alpha} + \varepsilon_n$ with $\lim_{n\to\infty} \varepsilon_n n^{-1/\alpha} = 0$. Let ε_n be a sequence such that $\gamma_\alpha \varphi_n > x_{1,n+1}$. Then by Theorem 2.4,

$$\text{(2.9)}\quad \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}(x_{jn})P_{n+k}(x_{jn})}{\gamma_\alpha - (x_{jn}/\varphi_n)} = I_k^{\gamma_\alpha,0}\left(\frac{2}{\pi\gamma_\alpha^2}\sqrt{\frac{\gamma_\alpha + x}{\gamma_\alpha - x}}\,dx; f\right)$$

and by the symmetry of $\{P_n(x)\}_{n=0}^{\infty}$,

$$\text{(2.10)}\quad \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_{jn} f\left(\frac{x_{jn}}{\varphi_n}\right) \frac{P_{n-1}(x_{jn})P_{n+k}(x_{jn})}{\gamma_\alpha + (x_{jn}/\varphi_n)} = I_k^{\gamma_\alpha,0}\left(\frac{2}{\pi\gamma_\alpha^2}\sqrt{\frac{\gamma_\alpha + x}{\gamma_\alpha - x}}\,dx; f\right).$$

In particular,

$$\lim_{n\to\infty} \frac{P_{n-1}(\gamma_\alpha\varphi_n)}{P_n(\gamma_\alpha\varphi_n)} = -\lim_{n\to\infty} \frac{P_{n-1}(-\gamma_\alpha\varphi_n)}{P_n(-\gamma_\alpha\varphi_n)} = 1.$$

Let $d\nu$ be a measure defined by

$$d\nu(x) = |x|^\rho\, d\mu(x) = |x|^\rho e^{-|x|^\alpha}\, dx, \quad \rho > -1$$

and the corresponding orthonormal polynomials by $\{Q_n(x)\}_{n=0}^{\infty}$ with zeros by y_{jn}, $j = 1, 2, \ldots, n$. Since the largest and smallest zeros of

$P_n(x)$ and $Q_n(x)$ have the same asymptotic behaviors, we obtain the same equation (2.9) and (2.10) for $d\nu(x)$, i.e., the limit does not change even if we replace $P_n(x)$ and x_{jn} by $Q_n(x)$ and y_{jn}, respectively.

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Department of Mathematics, Teachers College, Kyungpook National University, Taegu 702-701, Korea
E-mail address: dongwon@knu.ac.kr