ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 3, 2006

MIXED ORDER SYSTEMS OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

SIAKA KONÉ AND MANFRED MÖLLER

ABSTRACT. Expansions into eigenfunctions and associated functions of nth order ordinary scalar differential equations and first order systems of ordinary linear differential equations have been extensively investigated. Here we consider systems of higher order differential equations. In this case, it is not always possible to obtain an associated first order systems. Two mixed order systems which are equivalent to first order systems are considered. Results on eigenfunction expansion are established for each of them.

1. Introduction. Expansions into eigenfunctions and associated functions of nth order ordinary scalar differential equations and first order systems of ordinary linear differential equations have been extensively investigated. However, it seems that there are no such results for mixed order systems of differential equations. When dealing with nth order differential equations or systems of first order equations on a bounded interval, the assumption of suitably regular coefficients guarantees that the spectrum is either all of $\mathbf{C}$ or consists of isolated eigenvalues. One obstacle to eigenfunction expansion for mixed order systems is that this alternative need not be true for mixed order systems, see e.g., [**3**]. The problem here is that a mixed order system of differential equations cannot always be transformed into a first order system of differential equations. We will henceforth focus our attention on the case that the mixed order system has constant coefficients and is equivalent to a first order system

$$y' = A(\lambda)y, \tag{1.1}$$

where λ is the eigenvalue parameter.

We are going to use estimates of the resolvent of the operator associated with (1.1) and suitable boundary conditions as obtained

This paper is based on the first author's Ph.D. Thesis [**5**].
2000 AMS *Mathematics Subject Classification.* Primary 34L10, 34B05.
Key words and phrases. Mixed order system, eigenfunction expansion, coil spring.
Received by the editors on November 28, 2003.

in [**6, 7**]. For this a transformation $\lambda = \rho^\alpha$, $\alpha > 0$, is needed such that (1.1) is equivalent to an asymptotically ρ-linear system with a diagonalizable leading coefficient. Looking at the zeros μ of $\det(A(\rho^\alpha) - \mu)$, it is clear that a necessary condition for such an equivalence is that $\mu = O(\rho)$ as $\rho \to \infty$. So one should take α to be the largest number such that $\mu = O(\rho)$. Otherwise, $\mu = o(\rho)$, and the coefficient of ρ would be the zero matrix, in which case there is no suitable estimate of the resolvent. In general, the different zeros of $\det(A(\rho^\alpha) - \mu)$ have different ρ-asymptotics, so that the leading coefficient of the ρ-linear system is a non-invertible matrix. This will result in restrictions on the functions which can be expanded. As for nth order differential equations in [**7**], these estimates of the resolvent of the first order system are used to obtain expansion theorems.

This paper aims to give first results and to show the possibilities and limitations of this approach. Therefore, in Section 3 we start with a very simple example, which nevertheless will give some insight into the problem and which justifies the consideration of mixed order systems in their own right. The example in Section 4 comes from a concrete problem in mathematical physics; however, we modified it slightly in order to stay within the framework of Birkhoff regularity, in accordance with our intention to give an introductory account on mixed order systems. The original problem requires the consideration of Stone regularity and will be dealt with in another paper.

This paper is organized as follows. In Section 2 we recall some basic definitions and facts on bounded operator functions. In Section 3 we consider systems of a second order and a first order differential equation. In this case it turns out that the best choice for α is the non-integer $3/2$. In Theorem 3.8 it is shown that any vector function in $(L_2(0,1))^2$ has an expansion in terms of eigenfunctions and associated functions of the given problem. In Section 4 we deal with a problem in mathematical physics which arises in elasticity theory when considering the equations of motion for a coil spring, see [**2**]. This is technically much more complicated than the problem considered in Section 3. These are systems of a second order, a fourth order and a sixth order differential equation. Here the leading coefficient after the transformation $\lambda = \rho^2$ is a 12×12 matrix of rank 6. This has the consequence that we cannot prove that the vector functions in $(L_2(0,1))^3$ are expandable in terms of eigenfunctions and associated functions of the given problem; for the second and third components, we have to take certain derivatives, see Theorem 4.8.

2. Preliminaries. Throughout this paper let $1 < p < \infty$. For $k \in \mathbf{N}$, $W_p^k(0,1)$ denotes the Sobolev space of order k on the interval $(0,1)$,

$$W_p^k(0,1) = \{f \in L_p(0,1) : f^{(j)} \in L_p(0,1) \text{ for } j = 1, \dots, k\},$$

see, e.g., [**1**, Chapter III]. For a multi-index $k = (k_1, \dots, k_n) \in \mathbf{N}^n$, we denote

$$\mathcal{W}_p^k(0,1) = W_p^{k_1}(0,1) \times \cdots \times W_p^{k_n}(0,1).$$

Clearly, $\mathcal{W}_p^k(0,1)$ is a Banach space, and even a Hilbert space in case $p = 2$.

Throughout this paper, operator stands for bounded linear operator in Banach spaces. For an operator function $S(\lambda)$, $\lambda \in \mathbf{C}$, the resolvent set $\rho(S)$ denotes those $\lambda \in \mathbf{C}$ such that $S(\lambda)$ is invertible, and $\sigma(S) = \mathbf{C} \setminus \rho(S)$ is its spectrum. $S(\lambda)$ is called a Fredholm operator with index zero if $\operatorname{codim} R(S(\lambda)) = \dim N(S(\lambda)) < \infty$, where R and N denote range and null space.

For the convenience of the reader, we recall the definition of biorthogonal canonical systems of eigenvectors and associated vectors (CSEAVs), see e.g., [**7**, Corollary 1.6.6].

Let $S(\lambda) : E \to F$ be an operator function in Banach spaces E and F which depends analytically on $\lambda \in \mathbf{C}$ and such that $S(\lambda)$ is Fredholm valued for all $\lambda \in \mathbf{C}$. We will also assume that $\rho(S) \neq \varnothing$. Indeed, in all our applications S will depend linearly on λ, i.e., $S(\lambda) = S_0 + \lambda S_1$. Then S has discrete spectrum and each $\mu \in \sigma(S)$ is an eigenvalue of finite multiplicity, see e.g., [**7**, Theorem 1.3.1].

Definition 2.1. Let $\mu \in \sigma(S)$. i) An ordered set $\{y_0, y_1, \dots, y_h\}$ in E is called a chain of an eigenvector and associated vectors (CEAV) of S at μ if $y_0 \neq 0$ and

$$y(\lambda) := \sum_{l=0}^{h} (\lambda - \mu)^l y_l$$

satisfies $S(\lambda)y(\lambda) = O((\lambda-\mu)^{h+1})$ at μ.

ii) Let $y_0 \in N(S(\mu)) \setminus \{0\}$. Then $\nu(y_0)$ denotes the maximum of all h such that there is a CEAV $\{y_0, y_1, \dots, y_h\}$ of S at μ.

iii) A system $\{y_l^{(j)} : 1 \le j \le r,\ 0 \le l \le m_j - 1\}$ is called a canonical system of eigenvectors and associated vectors (CSEAV) of S at μ if

$$\begin{gathered}\{y_0^{(1)}, \dots, y_0^{(r)}\} \text{ is a basis of } N(S(\mu)),\\ \{y_0^{(j)}, y_1^{(j)}, \dots, y_{m_j-1}^{(j)}\} \text{ is a CEAV of } S \text{ at } \mu, \quad j = 1, \dots, r,\\ m_j = \max\{\nu(y) : y \in N(S(\mu)) \setminus \operatorname{span}\{y_0^{(k)} : k < j\}\}, \quad j = 1, \dots, r.\end{gathered}$$

The numbers m_j in Definition 2.1 iii) are independent of the CSEAV and are called the partial multiplicities of S at μ.

Note that the adjoint operator function $S(\lambda)^* : F' \to E'$ satisfies the same properties as $S(\lambda)$. Then we have, see [**7**, Corollary 1.6.6],

Proposition 2.2. *Let $\mu \in \sigma(S)$, $r = \dim N(S(\mu))$, and m_j, $j = 1, \dots, r$, be the partial multiplicities of S at μ. Then there are* CEAV*s $y_0^{(j)}, \dots, y_{m_j-1}^{(j)}$, $j = 1, \dots, r$, of S at μ and $v_0^{(j)}, \dots, v_{m_j-1}^{(j)}$, $j = 1, \dots, r$, of S^* at μ such that the following properties hold*:

$$\nu(y_0^{(j)}) = \nu(v_0^{(j)}) = m_j, \quad j = 1, \dots, r,$$

$$(2.1)\qquad \begin{gathered}\sum_{k=0}^{l}\sum_{q=1}^{m_i} \frac{1}{(k+q)!}\left\langle \frac{\mathrm{d}^{k+q}S}{\mathrm{d}\lambda^{k+q}}(\mu) y_{m_i-q}^{(i)},\, v_{l-k}^{(j)}\right\rangle = \delta_{ij}\delta_{0l}\\ 1 \le i \le r,\ 1 \le j \le r,\ 0 \le l \le m_j - 1,\end{gathered}$$

and the operator function

$$(2.2)\qquad \widetilde{D} := S^{-1} - \sum_{j=1}^{r}\sum_{l=1}^{m_j} (\cdot - \mu)^{-l} \sum_{h=0}^{m_j-l} y_h^{(j)} \otimes v_{m_j-l-h}^{(j)}$$

is holomorphic at μ. We call the systems $\{y_l^{(j)} : 1 \le j \le r, 0 \le l \le m_j - 1\}$ and $\{v_l^{(j)} : 1 \le j \le r, 0 \le l \le m_j - 1\}$ biorthogonal CSEAVs *of S and S^* at μ.*

3. A mixed order 2×2 system.

3.1 *The system of differential equations and the associated operators.*

For $\lambda \in \mathbf{C}$, we consider the mixed order system of differential equations

$$\eta_2^{''} - \lambda\eta_1 + a\eta_2 = 0, \tag{3.1}$$

$$\eta_1^{'} - \lambda\eta_2 = 0, \tag{3.2}$$

where $a \in C[0,1]$, subject to periodic boundary conditions

$$\eta_1(0) = \eta_1(1), \quad \eta_2(0) = \eta_2(1), \quad \eta_2^{'}(0) = \eta_2^{'}(1).$$

We associate an operator with this problem as follows. If $\eta \in \mathcal{W}_p^{(1,2)}(0,1)$, let

$$L^D(\lambda)\eta = \begin{pmatrix} \eta_2'' - \lambda\eta_1 + a\eta_2 \\ \eta_1' - \lambda\eta_2 \end{pmatrix}, \tag{3.3}$$

$$L^R\eta = \begin{pmatrix} \eta_1(0) - \eta_1(1) \\ \eta_2(0) - \eta_2(1) \\ \eta_2'(0) - \eta_2'(1) \end{pmatrix}, \tag{3.4}$$

and

$$L(\lambda) = \left(L^D(\lambda), L^R\right). \tag{3.5}$$

Clearly,

Remark 3.1. $L(\lambda) \in L\big(\mathcal{W}_p^{(1,2)}(0,1), (L_p(0,1))^2 \times \mathbf{C}^3\big)$.

3.2 *The associated first order system.* Here we consider the transformation to a first order system, that is, we introduce the operator $T^D(\lambda)$ given by

$$T^D(\lambda)y = y' - A(\lambda)y, \tag{3.6}$$

where

$$A(\lambda) = \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 1 \\ \lambda & -a & 0 \end{pmatrix}. \tag{3.7}$$

Proposition 3.2. *Let* $y = (y_1, y_2, y_3)^t \in \left(W_p^1(0,1)\right)^3$ *be a solution of the equation* $T^D(\lambda)\, y = 0$, *and put*

$$\eta = (y_1, y_2)^t. \tag{3.8}$$

Then $\eta \in \mathcal{W}_p^{(1,2)}(0,1)$ *and* $L^D(\lambda)\,\eta = 0$.

Proof. Setting $\eta =: (\eta_1, \eta_2)^t$, we get from (3.8) that $\eta_1 = y_1 \in W_p^1(0,1)$ and $\eta_2 = y_2 \in W_p^1(0,1)$. Writing $T^D(\lambda)\, y = 0$ in the form

$$y_1' = \lambda y_2, \tag{3.9}$$

$$y_2' = y_3, \tag{3.10}$$

$$y_3' = \lambda y_1 - a y_2, \tag{3.11}$$

it follows from (3.10) that

$$\eta_2' = y_2' = y_3 \in W_p^1(0,1).$$

Thus $\eta_2 \in W_p^2(0,1)$, and hence $\eta \in \mathcal{W}_p^{(1,2)}(0,1)$.

Now we shall prove that $L^D(\lambda)\,\eta = 0$. Differentiating in (3.10) we have $y_2'' = y_3'$. Thus (3.11) gives

$$y_2'' = \lambda y_1 - a y_2. \tag{3.12}$$

Observing that $\eta_1 = y_1$ and $\eta_2 = y_2$, we get from (3.9) and (3.12) that

$$L^D(\lambda)\,\eta = \begin{pmatrix} \eta_2'' - \lambda\eta_1 + a\eta_2 \\ \eta_1' - \lambda\eta_2 \end{pmatrix} = 0. \qquad \square$$

Proposition 3.3. *Let* $\eta = (\eta_1, \eta_2)^t \in \mathcal{W}_p^{(1,2)}(0,1)$ *be a solution of the equation* $L^D(\lambda)\,\eta = 0$, *and put*

$$y = (\eta_1, \eta_2, \eta_2')^t.$$

Then $y \in \left(W_p^1(0,1)\right)^3$ *and* $T^D(\lambda)\, y = 0$.

Proof. By assumption on η,

$$y = (\eta_1, \eta_2, \eta_2')^t \in \mathcal{W}_p^{(1,2,1)}(0,1) \subset \left(W_p^1(0,1)\right)^3.$$

Now we shall prove that $T^D(\lambda)\,y = 0$. Writing $y = (y_1, y_2, y_3)^t$, the definition of y and $L^D(\lambda)\,\eta = 0$ give

$$y_1' = \lambda y_2, \quad y_2' = y_3, \quad y_3' = \lambda y_1 - a y_2.$$

This can be written as $y' = A(\lambda)\,y$, and $T^D(\lambda)\,y = 0$ follows. □

For $y \in \left(W_p^1(0,1)\right)^3$, we put

$$T^R y = y(0) - y(1) \tag{3.13}$$

and introduce the operator function

$$T(\lambda) = \left(T^D(\lambda), T^R\right), \tag{3.14}$$

where $T^D(\lambda)$ is given by (3.6).

Remark 3.4. $T(\lambda) \in L\big(\left(W_p^1(0,1)\right)^3, \left(L_p(0,1)\right)^3 \times \mathbf{C}^3\big)$.

Proposition 3.5. $\rho(L) = \rho(T)$.

Proof. It is well known that $T(\lambda)$ is Fredholm operator valued with index zero for all $\lambda \in \mathbf{C}$, see e.g., [**7**, Corollary 3.1.3]. In a similar way, see also [**7**, Section 6.3], it can be shown that $L(\lambda)$ is a Fredholm operator. In particular, $\lambda \in \rho(L)[\lambda \in \rho(T)]$ if and only if $L(\lambda)$ $[T(\lambda)]$ is injective.

Now let $\lambda \in \rho(L)$ and $y \in (W_p^1(0,1))^3$ be such that $T(\lambda) = 0$. Putting $\eta = (y_1, y_2)^t$, it follows from $T^D(\lambda)y = 0$ that $y_3 = \eta_2'$, $L^D(\lambda)\eta = 0$ by Proposition 3.2, and $L^R\eta = T^R y = 0$ by (3.4) and (3.13). Hence $\eta = 0$ since $\lambda \in \rho(L)$, and $y = (\eta_1, \eta_2, \eta_2')^t = 0$ follows. Thus $T(\lambda)$ is injective, and $\rho(L) \subset \rho(T)$ has been shown.

Using Proposition 3.3 instead of Proposition 3.2, $\rho(T) \subset \rho(L)$ can be shown in a similar manner. □

Proposition 3.6. *Let* $(f_1,f_2) \in (L_p(0,1))^2$ *and* $f_3 \in \mathbf{C}^3$. *Then, for* $\lambda \in \rho(L)$,

$$L^{-1}(\lambda)(f_1,f_2,f_3)^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T^{-1}(\lambda)(f_2,0,f_1,f_3)^t.$$

Proof. Let $\eta = L^{-1}(\lambda)(f_1,f_2,f_3)^t$ and $y = (\eta_1,\eta_2,\eta_2')^t$. We calculate

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = L^D(\lambda)\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \eta_2'' - \lambda\eta_1 + a\eta_2 \\ \eta_1' - \lambda\eta_2 \end{pmatrix} = \begin{pmatrix} y_3' - \lambda y_1 + a y_2 \\ y_1' - \lambda y_2 \end{pmatrix}.$$

By definition of y, this implies that

$$y_1' - \lambda y_2 = f_2, \quad y_2' - y_3 = 0, \quad y_3' - \lambda y_1 + y_2 = f_1.$$

Hence, by definition of $T^D(\lambda)$ in (3.6), $T^D(\lambda)y = (f_2,0,f_1)^t$. Since $T^R y = L^R y$ by (3.13) and (3.4), we have $T(\lambda)y = \left(T^D(\lambda)y, T^R y\right) = (f_2,0,f_1,f_3)^t$, and therefore $y = T^{-1}(\lambda)(f_2,0,f_1,f_3)^t$ as $\lambda \in \rho(T)$ by Proposition 3.5. By definition of y this gives

$$L^{-1}(\lambda)(f_1,f_2,f_3)^t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T^{-1}(\lambda)(f_2,0,f_1,f_3)^t. \qquad \square$$

3.3 *Transformations to third order differential equations.* We note that when dealing with nth order differential equations they are often transformed into first order systems so that one can use, e.g., estimates of the Green's function for systems, see [**7**, Chapters VI, VII]. That is exactly what we have repeated for the case under consideration here. On the other hand, using the special structure of nth order differential equations, one can obtain stronger results for them, see, e.g., [**7**, Chapter VIII]. There are two canonical ways to obtain a third order differential equation from (3.1) and (3.2), namely substituting one of η_1, η_2 from (3.2) into (3.1). Eliminating η_2 we have to take $\eta_2 = (1/\lambda)\eta_1'$, which excludes the value $\lambda = 0$. Although the resulting differential equation can be written as

$$\eta_1''' - \lambda^2\eta_1 + a\eta_1' = 0, \tag{3.15}$$

with no problem for $\lambda = 0$, we still would not have an equivalence of this differential equation to the given problem for $\lambda = 0$.

Eliminating η_1, we would have to differentiate (3.1), which would result in loss of information. For example, taking $a = 1$, if in the resulting differential equation

$$\eta_2''' - \lambda^2 \eta_2 + \eta_2' = 0 \tag{3.16}$$

we put $\lambda = 0$, then any constant η_2 satisfies this differential equation, whereas with η_2 being a nonzero constant, (3.1) is not satisfied.

In the following, the first order system operator T will be used through estimates of its resolvent for $|\lambda|$ large. However, if we would consider the operators associated with the nth order differential equation (3.15) or (3.16), respectively, then we should make use of the corresponding expansion theorems, see [**7**, Chapter VIII]—otherwise we would have to go back to the estimates for the resolvent of the associated first order systems and any possible advantage of introducing the third order differential equation would be lost. Then, of course, all $\lambda \in \mathbf{C}$ had to be taken into account, and $\lambda = 0$ would need to be investigated separately. Although it might be possible to overcome these difficulties, we will not pursue this direction.

3.4 *Asymptotic parameter-linearization.* Let L be the differential operator function defined by (3.5) in subsection 3.1. We want to prove expansion theorems in terms of eigenfunctions and associated functions of L. For this we are going to use the techniques and results in [**7**]. That is, we have to consider the associated first order system function T defined by (3.14). To apply the estimates in [**7**] we would need that T is asymptotically linear in the eigenvalue parameter and that the coefficient matrix of the parameter-linear term in T^D is diagonalizable. Although T is λ-linear, the coefficient matrix of λ is not diagonalizable. Therefore, one introduces $\rho^\alpha = \lambda$, $\alpha > 0$, and looks for an asymptotic ρ-linearization with a diagonalizable ρ-coefficient. We are also trying to make this ρ-coefficient an invertible matrix. The reason for this is that the larger the rank of this matrix, the more functions will be expandable. The asymptotic ρ-linearization is achieved by a matrix

$$C_0(\rho) = \operatorname{diag}\{\rho^{\nu_1}, \rho^{\nu_2}, \rho^{\nu_3}\} \tag{3.17}$$

with real numbers ν_1, ν_2, ν_3 such that

$$C_0^{-1}(\rho)\, A(\rho^\alpha)\, C_0(\rho) =: \tilde{A}(\rho) = \rho \tilde{A}_1 + \tilde{A}_0(\cdot, \rho), \tag{3.18}$$

where A is given by (3.7) and $\tilde{A}_0(\rho)$ is bounded with respect to ρ as $\rho \to \infty$. We have

$$\tilde{A}(\rho) = \begin{pmatrix} 0 & \rho^\alpha \rho^{\nu_2-\nu_1} & 0 \\ 0 & 0 & \rho^{\nu_3-\nu_2} \\ \rho^\alpha \rho^{\nu_1-\nu_3} & -a\rho^{\nu_2-\nu_3} & 0 \end{pmatrix}.$$

The asymptotic behavior required in (3.18) yields

$$\alpha + \nu_2 - \nu_1 \le 1, \quad \nu_3 - \nu_2 \le 1, \quad \alpha + \nu_1 - \nu_3 \le 1, \quad \nu_2 - \nu_3 \le 1,$$

and for the invertibility of the matrix $\tilde{A}_1$ we need

$$\alpha + \nu_2 - \nu_1 = 1, \quad \nu_3 - \nu_2 = 1, \quad \alpha + \nu_1 - \nu_3 = 1$$

since each row and column of $\tilde{A}_1$ must have a nonzero element. This is satisfied if we put

$$\alpha = \frac{3}{2}, \quad \nu_1 = 0, \quad \nu_2 = -\frac{1}{2}, \quad \nu_3 = \frac{1}{2}. \tag{3.19}$$

Then the matrix $\tilde{A}$ has the form

$$\tilde{A}(\rho) = \begin{pmatrix} 0 & \rho & 0 \\ 0 & 0 & \rho \\ \rho & -a\rho^{-1} & 0 \end{pmatrix},$$

and therefore

$$\tilde{A}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

A straightforward calculation shows that $\tilde{A}_1$ can be diagonalized as

$$A_1 = C^{-1} \tilde{A}_1 C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \tag{3.21}$$

where $\omega = e^{2\pi i/3}$ and

$$C = \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}.$$

We introduce the differential operator

(3.22)

$$\widetilde{T}(\rho) = \begin{pmatrix} \widetilde{T}^D(\rho) \\ \widetilde{T}^R(\rho) \end{pmatrix} = \begin{pmatrix} C^{-1}C_0(\rho)^{-1} & 0 \\ 0 & C^{-1}C_0(\rho)^{-1} \end{pmatrix} T\big(\rho^{3/2}\big)C_0(\rho)C,$$

where the above considerations show that

$$\widetilde{T}^D(\rho)\, y = y' - (\rho A_1 + \rho^{-1}A_{-1})\, y, \tag{3.23}$$

where the matrix A_{-1} does not depend on ρ.

3.5 *Birkhoff regularity for* $\widetilde{T}$. In order to prove expansion theorems in terms of eigenfunctions and associated functions of L we are first going to verify Birkhoff regularity for $\widetilde{T}$. From (3.22) we obtain that

$$\widetilde{T}^R(\rho)y = C^{-1}C_0^{-1}(\rho)\, T^R\, (C_0(\rho)\, Cy) = y(0) - y(1).$$

Hence, all matrices in [**7**, (4.1.25)] are of the form $I_3 - \Delta + \Delta = I_3$ and thus invertible, where I_3 is the 3×3 identity matrix. That is, by [**7**, Definition 4.1.2], the boundary eigenvalue problem $\widetilde{T}(\rho)y = 0$ is Birkhoff regular. Hence, by [**7**, Theorem 4.6.9], there are circles $\gamma_\nu = \{\lambda \in \mathbf{C} : |\lambda| = \rho_\nu\}\,(\nu \in \mathbf{N})$ with $\rho_\nu \to \infty$ as $\nu \to \infty$ such that $\gamma_\nu \subset \rho(\widetilde{T})$ and the operators

$$P_\nu h = -\frac{1}{2\pi i}\oint_{\gamma_\nu} \widetilde{J}\widetilde{T}^{-1}(\rho)\,(A_1 h, 0)\; d\rho, \qquad h \in (L_p(0,1))^3,\ \nu \in \mathbf{N}, \tag{3.24}$$

where $\widetilde{J} : \big(W_p^1(0,1)\big)^3 \to (L_p(0,1))^3$ is the canonical embedding, satisfy $P_\nu h \to h$ as $\nu \to \infty$ in $(L_p(0,1))^3$ for all $h \in (L_p(0,1))^3$ and $1 < p < \infty$.

3.6 *Expansion theorems for* $L(\lambda)$.

Theorem 3.7. *Let* $1 < p < \infty$. *There are circles* $\gamma_\nu = \{\lambda \in \mathbf{C} : |\lambda| = \mu_\nu\}$, $\nu \in \mathbf{N}$, *with* $\mu_\nu \to \infty$ *as* $\nu \to \infty$, *such that* $\gamma_\nu \subset \rho(L)$ *and* $Q_\nu^j g \to g$ *in* $L_p(0,1)$ *for* $j = 1,2$, *and* $g \in L_p(0,1)$, *where*

$$Q_\nu^j g = -\frac{1}{2\pi i} \oint_{\gamma_\nu} e_j^t \overline{J} L^{-1}(\lambda)(e_j g, 0)\, d\lambda$$

and $\overline{J} : \mathcal{W}_p^{(1,2)}(0,1) \to (L_p(0,1))^2$ *is the canonical embedding.*

Proof. From (3.22) we infer for $h \in (L_p(0,1))^3$ that

$$\widetilde{T}^{-1}(\rho)(A_1 h, 0) = C^{-1} C_0(\rho)^{-1} T^{-1}(\rho^{3/2})(C_0(\rho) C A_1 h, 0). \tag{3.25}$$

Putting

$$h = A_1^{-1} C^{-1} \begin{pmatrix} f_2 \\ 0 \\ f_1 \end{pmatrix} \tag{3.26}$$

we get in view of (3.17) and (3.19) that

$$C_0(\rho) C A_1 h = \begin{pmatrix} f_2 \\ 0 \\ \rho^{1/2} f_1 \end{pmatrix}. \tag{3.27}$$

In order to express (3.25) in terms of L^{-1}, we also need

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T^{-1}.$$

To this end we set

$$F(\rho) = \begin{pmatrix} 1 & 0 \\ 0 & \rho^{1/2} \end{pmatrix}, \tag{3.28}$$

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} C, \tag{3.29}$$

and obtain

$$EC^{-1}C_0(\rho)^{-1} = F(\rho)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.30}$$

From (3.24), (3.25), (3.30) and (3.27), it follows that

$$\begin{aligned} EP_\nu h &= -\frac{1}{2\pi i}\oint_{\gamma_\nu} \widehat{J}EC^{-1}C_0(\rho)^{-1}\,T^{-1}(\rho^{3/2})\,(C_0(\rho)\,CA_1h, 0)\ d\rho \\ &= -\frac{1}{2\pi i}\oint_{\gamma_\nu} \widehat{J}F(\rho)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T^{-1}(\rho^{3/2})(f_2, 0, \rho^{1/2}f_1, 0)^t\, d\rho \\ &= -\frac{1}{2\pi i}\oint_{\gamma_\nu} \overline{J}\begin{pmatrix} 1 & 0 \\ 0 & \rho^{1/2} \end{pmatrix} L^{-1}(\rho^{3/2})(\rho^{1/2}f_1, f_2, 0)^t\, d\rho, \end{aligned}$$

where $\widehat{J} : (W_p^1(0,1))^2 \to (L_p(0,1))^2$ is the canonical embedding and the last identity follows from Proposition 3.6.

Clearly, we want to express this integral as an integral over $\lambda = \rho^{3/2}$. If ρ runs twice through the circle γ_ν with radius ρ_ν, then λ runs three times through the circle γ_ν with radius $r_\nu = \rho_\nu^{3/2}$, and since $d\lambda = 3\rho^{1/2}/2\, d\rho$, it follows that

$$EP_\nu h = -\frac{1}{2\pi i}\oint_{\gamma_\nu} \overline{J}\begin{pmatrix} 1 & 0 \\ 0 & \rho^{1/2} \end{pmatrix} L^{-1}(\lambda)\,(f_1, \rho^{-1/2}f_2, 0)\, d\lambda.$$

If we now define

$$h_1 = A_1^{-1}C^{-1}\begin{pmatrix} 0 \\ 0 \\ f_1 \end{pmatrix}, \quad h_2 = A_1^{-1}C^{-1}\begin{pmatrix} f_2 \\ 0 \\ 0 \end{pmatrix},$$

then clearly $e_j^tEP_\nu h_j = Q_\nu^j f_j,\ j = 1, 2$. But in subsection 3.5 we have seen that $P_\nu h_j \to h_j$ as $\nu \to \infty$, and therefore $Q_\nu^j f_j \to e_j^t Eh_j$ as $\nu \to \infty$ for $j = 1, 2$.

We note that (3.21) gives $\tilde{A}_1 C = CA_1$, and that

$$\tilde{A}_1\begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix} = \begin{pmatrix} f_2 \\ 0 \\ f_1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tilde{A}_1^{-1} \begin{pmatrix} f_2 \\ 0 \\ f_1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tilde{A}_1^{-1} C A_1 h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} Ch = Eh.$$

Altogether, we have shown that $Q_\nu^j f_j \to f_j$ as $\nu \to \infty$. □

In the proof of Proposition 3.5 we have shown that $L(\lambda)$ is Fredholm valued, and the Birkhoff regularity of $\widetilde{T}$ and Proposition 3.5 give $\rho(L) \neq \varnothing$. Hence L has discrete spectrum.

Theorem 3.8. *Let $1 < p < \infty$. Let $\lambda_0, \lambda_1, \dots,$ be the eigenvalues of L defined by* (3.5), *and let $\{y_{k,l}^{(j)} : j = 1, \dots, r(\lambda_k); l = 0, \dots, m_{k,j} - 1\}$ and $\{(u_{k,l}^{(j)}, d_{k,l}^{(j)}) : j = 1, \dots, r(\lambda_k); l = 0, \dots, m_{k,j} - 1\}$ be biorthogonal* CSEAV*s of L and L^* at λ_k. Choose the curves γ_ν, $\nu \in \mathbf{N}$, according to Theorem* 3.7. *Then*

$$h = -\lim_{\nu\to\infty} \sum_{\substack{k\in\mathbf{N} \\ |\lambda_k|<\mu_\nu}} \sum_{j=1}^{r(\lambda_k)} \sum_{l=0}^{m_{k,j}-1} y_{k,l}^{(j)} \int_0^1 u_{k,m_{k,j}-1-l}^{(j)} (x)^t \, h\,(x)\; dx$$

in $(L_p\,(0,1))^2$ for each $h \in (L_p\,(0,1))^2$.

Proof. The result follows by inserting (2.2) into Theorem 3.7. It is convenient to elaborate on the biorthogonal CSEAVs. Since $L(\lambda) : \mathcal{W}_p^{(1,2)}(0,1) \to (L_p\,(0,1))^2 \times \mathbf{C}^3$, we have that the $y_{k,l}^{(j)}$ belong to $\mathcal{W}_p^{(1,2)}(0,1)$, and, since $L(\lambda)^* : (L_{p'}\,(0,1))^2 \times \mathbf{C}^3 \to (\mathcal{W}_p^{(1,2)}(0,1))'$, $1/p + 1/p' = 1$, we have that $u_{k,l}^{(j)} \in (L_{p'}(0,1))^2$ and $d_{k,l}^{(j)} \in \mathbf{C}^3$. Since L is linear in λ, only $k = 0$ and $q = 1$ occur in (2.1), which thus has the form

$$-\int_0^1 \left(u_{k,l}^{(j)}(x)\right)^t y_{k,m_i-1}^{(i)}(x)\, dx = \delta_{ij}\delta_{0l}.$$

We also note that the $d_{k,l}^{(j)}$ from the CEAV of L^* do not occur in the expansion or the biorthogonal relations because the boundary conditions do not depend on the eigenvalue parameter λ. □

4. A system of differential equations for a coil spring.

4.1 *The system of differential equations and the associated operator* $L(\lambda)$. Goolin and Kartyshov [**4**], see also [**2**], considered the equation of motion for a coil spring. Separation of variables leads to a system of three ordinary differential equations of second, fourth and sixth order respectively. A method to evaluate the eigenvalues was given in [**4**]. Here we will prove eigenfunction expansions for this system under modified boundary conditions which lead to a Birkhoff regular case. The original problem, which would give a Stone regular problem, will be dealt with in a forthcoming paper. We omit the formulation of the problem as a system of differential equations with boundary conditions and immediately proceed to its operator form.

For $\lambda \in \mathbf{C}$ and $\eta \in \mathcal{W}_p^{(2,4,6)}(0,1)$, let

$$L^D(\lambda)\,\eta = \begin{pmatrix} A_{11}\eta_1 + A_{12}\eta_2 + A_{13}\eta_3 - \lambda B_{11}\eta_1 \\ A_{21}\eta_1 + A_{22}\eta_2 + A_{23}\eta_3 - \lambda(B_{22}\eta_2 + B_{23}\eta_3) \\ A_{31}\eta_1 + A_{32}\eta_2 + A_{33}\eta_3 - \lambda(B_{32}\eta_2 + B_{33}\eta_3) \end{pmatrix}, \tag{4.1}$$

where A_{ij} and B_{ij} are differential expressions of the form

$$\begin{aligned} A_{ij} &= a_{ij}\frac{d^2}{ds^2} + b_{ij},\ (i,j) = (1,1),(1,2),(1,3),(2,1),(3,1), \\ A_{ij} &= a_{ij}\frac{d^4}{ds^4} + b_{ij}\frac{d^2}{ds^2} + c_{ij},\ (i,j) = (2,2),(2,3),(3,2), \\ A_{33} &= a_{33}\frac{d^6}{ds^6} + b_{33}\frac{d^4}{ds^4} + c_{33}\frac{d^2}{ds^2} + e_{33}, \\ B_{11} &= f_{11}, \\ B_{ij} &= f_{ij} + g_{ij}\frac{d^2}{ds^2},\ (i,j) = (2,2),(2,3),(3,2), \\ B_{33} &= f_{33} + g_{33}\frac{d^2}{ds^2} + h_{33}\frac{d^4}{ds^4}, \end{aligned} \tag{4.2}$$

and s is the independent variable. Here all coefficients are constant and real. We assume that the maximum order of differentiation is attained, i.e., that a_{11}, a_{22}, a_{33}, f_{11}, g_{22}, and h_{33} are different from 0. We also let

$$L^R\eta = \big(\eta_1(0), \eta_1(1), \eta_2(0), \eta_2(1), \eta_2^{''}(0), \eta_2^{''}(1), \eta_3(0), \eta_3(1), \eta_3^{''}(0), \eta_3^{''}(1), \eta_3^{(4)}(0), \eta_3^{(4)}(1)\big)^t. \tag{4.3}$$

For the boundary eigenvalue operator function

$$L(\lambda) = \big(L^D(\lambda), L^R\big) \tag{4.4}$$

we clearly have

Remark 4.1. $L(\lambda) \in L\big(\mathcal{W}_p^{(2,4,6)}(0,1), (L_p(0,1))^3 \times \mathbf{C}^{12}\big)$.

For $\eta \in \mathcal{W}_p^{(2,4,6)}(0,1)$ we set

$$y = \big(\eta_1, \eta_1', \eta_2, \eta_2', \eta_2'', \eta_2^{(3)}, \eta_3, \eta_3', \eta_3'', \eta_3^{(3)}, \eta_3^{(4)}, \eta_3^{(5)}\big)^t.$$

Then $L^D(\lambda)\eta = 0$ is equivalent to a first order system

$$y' = A(\lambda)y, \tag{4.5}$$

where the first, second and third components of $L^D(\lambda)\eta = 0$ are used to find the second, sixth and twelfth rows of $A(\lambda)$. The components of this 12×12 matrix are obtained by a straightforward but lengthy calculation, see [**5**] for its explicit representation.

We define the operator $T^D(\lambda)$ by

$$T^D(\lambda)y = y' - A(\lambda)y, \quad y \in \big(W_p^1(0,1)\big)^{12}. \tag{4.6}$$

Analogous to Propositions 3.2 and 3.3 we obtain

Proposition 4.2. *Let* $y = (y_j)_{j=1}^{12} \in \big(W_p^1(0,1)\big)^{12}$ *be a solution of the equation* $T^D(\lambda)y = 0$, *and set* $\eta = (y_1, y_3, y_7)^t$. *Then* $\eta \in \mathcal{W}_p^{(2,4,6)}(0,1)$ *and* $L^D(\lambda)\eta = 0$.

Proposition 4.3. *Let* $\eta = (\eta_1, \eta_2, \eta_3)^t \in \mathcal{W}_p^{(2,4,6)}(0,1)$ *be a solution of* $L^D(\lambda)\eta = 0$, *and set*

$$y = \big(\eta_1, \eta_1', \eta_2, \eta_2', \eta_2'', \eta_2^{(3)}, \eta_3, \eta_3', \eta_3'', \eta_3^{(3)}, \eta_3^{(4)}, \eta_3^{(5)}\big)^t.$$

Then $y \in \left(W_p^1(0,1)\right)^{12}$ *and* $T^D(\lambda)y = 0$.

Let T^D be given by (4.6) and

$$T^R y = \big(y_1(0), y_1(1), y_3(0), y_3(1), y_5(0), y_5(1), y_7(0), y_7(1), y_9(0), y_9(1), y_{11}(0), y_{11}(1)\big)^t. \tag{4.7}$$

We associate to it the operator function

$$T(\lambda) = \left(T^D(\lambda), T^R\right). \tag{4.8}$$

Remark 4.4. $T(\lambda) \in L\big((W_p^1(0,1))^{12}, (L_p(0,1))^{12} \times \mathbf{C}^{12}\big)$.

Proposition 4.5. *Let* $(f_1, f_2, f_3) \in (L_p(0,1))^3$, $f_4 \in \mathbf{C}^{12}$ *and set*

$$E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.9}$$

and $U = \operatorname{diag}\left(1, D^2, D^4\right)$, *where* D *is differentiation with respect to the independent variable. Then, for* $\lambda \in \rho(L)$,

$$UL^{-1}(\lambda)(f_1, f_2, f_3, f_4)^t = E_2 T^{-1}(\lambda)\left(0, \frac{1}{a_{11}} f_1, 0, 0, 0, \frac{1}{a_{22}} f_2, 0, 0, 0, 0, 0, \frac{1}{a_{33}} f_3, f_4\right)^t.$$

Proof. Let $\eta = L^{-1}(\lambda)(f_1, f_2, f_3, f_4)^t$ and

$$y = \left(\eta_1, \eta_1', \eta_2, \eta_2', \eta_2'', \eta_2^{(3)}, \eta_3, \eta_3', \eta_3'', \eta_3^{(3)}, \eta_3^{(4)}, \eta_3^{(5)}\right)^t. \tag{4.10}$$

Then

$$(f_1, f_2, f_3)^t = L^D(\lambda)\eta = \begin{pmatrix} A_{11}\eta_1 + A_{12}\eta_2 + A_{13}\eta_3 - \lambda B_{11}\eta_1 \\ A_{21}\eta_1 + A_{22}\eta_2 + A_{23}\eta_3 - \lambda(B_{22}\eta_2 + B_{23}\eta_3) \\ A_{31}\eta_1 + A_{32}\eta_2 + A_{33}\eta_3 - \lambda(B_{32}\eta_2 + B_{33}\eta_3) \end{pmatrix}.$$

By definition of y and $A(\lambda)$, this implies that

$$T^D(\lambda)y = y' - A(\lambda)\, y = \left(0,\ \frac{1}{a_{11}}\, f_1, 0, 0, 0,\ \frac{1}{a_{22}}\, f_2, 0, 0, 0, 0, 0,\ \frac{1}{a_{33}}\, f_3\right)^t;$$

note that, e.g., the first component of $L^D(\lambda)\eta = (f_1, f_2, f_3)^t$ reads $a_{11}\eta_1'' + \cdots = f_1$. This shows that

$$y = T^{-1}(\lambda)\left(0,\ \frac{1}{a_{11}}\, f_1, 0, 0, 0,\ \frac{1}{a_{22}}\, f_2, 0, 0, 0, 0, 0,\ \frac{1}{a_{33}}\, f_3, f_4\right)^t,$$

which completes the proof in view of (4.10). □

4.2 *Asymptotic parameter-linearization.* Let L be the differential operator as defined by (4.4) in subsection 3.1. We want to prove expansion theorems in terms of eigenfunctions and associated functions of L. For this we are going to use the techniques and results in [**7**]. That is, we have to consider the associated first order systems T. To apply the estimates in [**7**] we need that T is asymptotically linear in the eigenvalue parameter and that the coefficient matrix of the parameter-linear term in T^D is diagonalizable. Although T is λ-linear, the coefficient matrix of λ is not diagonalizable. Therefore, we introduce $\rho^2 = \lambda$ and look for an asymptotic ρ-linearization with a diagonalizable ρ-coefficient. The asymptotic ρ-linearization is achieved by a matrix

$$C_0(\rho) = \operatorname{diag}\{\rho^{\nu_1}, \rho^{\nu_2}, \dots, \rho^{\nu_{12}}\},$$

where we require that

$$C_0^{-1}(\rho)\, A\left(\rho^2\right) C_0(\rho) = \tilde{A}(\rho) = \rho\tilde{A}_1 + \tilde{A}_0 + \tilde{A}_1(\rho)$$

with $\tilde{A}_1(\rho) \to 0$ as $\rho \to \infty$. This is satisfied if we put $\nu_1 = 0$, $\nu_6 = \nu_{12} = 2$ and $\nu_j = 1$ for all other j, and then the matrix $\tilde{A}_1 = (a_{i,j})_{i,j=1}^{12}$ is of the form that $a_{1,2} = a_{5,6} = a_{11,12} = 1$, the $a_{i,j}$ with (i,j) having the values $(2,1)$, $(6,3)$, $(6,5)$, $(6,7)$, $(6,9)$, $(12,3)$, $(12,5)$, $(12,7)$, $(12,9)$, $(12,11)$ are possibly nonzero, and all other entries of $\tilde{A}$ are 0. For details we refer to [**5**].

Clearly, $\tilde{A}$ has rank at most 6, and computer algebra calculations show that for certain values of the coefficients in (4.2), $\tilde{A}$ has 6 distinct nonzero eigenvalues. Since the eigenvalues depend algebraically on the coefficients of $\widetilde{A}$, this is the generic case. Therefore we obtain

Proposition 4.6. *For generic coefficients in* (4.2), *the matrix* $\tilde{A}_1$ *has rank* 6 *and is diagonalizable.*

In the sequel we will only consider this generic case in which Proposition 4.6 holds. Let C_1 be a matrix which diagonalizes $\tilde{A}_1$, i.e., C_1 is invertible and

$$A_1 = C_1^{-1} \tilde{A}_1 C_1$$

is a diagonal matrix with zeros in the first 6 diagonal entries. Putting

$$C(\rho) = C_0(\rho) C_1,$$

it follows that

$$C^{-1}(\rho) A\left(\rho^2\right) C(\rho) = \rho A_1 + A_0 + \rho^{-1} A_{-1}(\rho).$$

We introduce the differential operator

$$\widetilde{T}^D(\rho) y = C_1^{-1} C_0(\rho)^{-1} T^D\left(\rho^2\right) C_0(\rho) C_1 y.$$

The above considerations shows that

$$\widetilde{T}^D(\rho) y = y' - \left(\rho A_1 + A_0 + \rho^{-1} A_{-1}(\rho)\right) y. \tag{4.11}$$

We define

$$\widetilde{T}(\rho) = \begin{pmatrix} C(\rho)^{-1} & 0 \\ 0 & C_2(\rho)^{-1} \end{pmatrix} T\left(\rho^2\right) C(\rho),$$

where $C_2(\rho)$ is a 12×12 invertible matrix polynomial which will be determined later.

4.3 *Birkhoff regularity for* $\widetilde{T}$. In order to prove expansion theorems in terms of eigenfunctions and associated functions of L we are going to prove first Birkhoff regularity for $\widetilde{T}$. For this we will use the techniques in [**7**, Theorem 4.1.3]. The boundary term is given by (4.6) and can therefore be written as

$$T^R y = \widehat{W}^{(0)} y\,(0) + \widehat{W}^{(1)} y\,(1)\,.$$

Put $\widetilde{W}^{(j)}\,(\rho) = \widehat{W}^{(j)} C_0\,(\rho)\,C_1,\ j = 0, 1$. We want to find an invertible 12×12 matrix polynomial $C_2\,(\rho)$ such that

$$C_2^{-1}\,(\rho)\left(\widehat{W}^{(0)} C_0\,(\rho)\,C_1, \widehat{W}^{(1)} C_0\,(\rho)\,C_1\right) = \left(W_0^{(0)}, W_0^{(1)}\right) + O\left(\rho^{-1}\right)$$

as $\rho \to \infty$ and such that $\left(W_0^{(0)}, W_0^{(1)}\right)$ is a 12×24 matrix of rank 12. We write

$$A_0 = \begin{pmatrix} A_{00} & * \\ * & * \end{pmatrix}.$$

Then $P^{[0]}$ is defined as $P^{[0]} = \operatorname{diag}\left(P_{00}^{[0]}, P_{11}^{[0]}, \dots, P_{66}^{[0]}\right)$, where $P_{00}^{[0]}$ is uniquely given by

$$\begin{aligned} P_{00}^{[0]\prime} &= A_{00} P_{00}^{[0]}, \\ P_{00}^{[0]}\,(0) &= I_6, \end{aligned}$$

and $P_{jj}^{[0]}$, $j = 1, \dots, 6$, are scalar functions.

For the calculation of the Birkhoff matrices, we need the diagonal matrices $\Delta_0 = \operatorname{diag}(0, \dots, 0, 1, \dots, 1)$ with 6 zeros and 6 ones, $\Lambda = \operatorname{diag}(\delta_1, \delta_2, \dots, \delta_{12})$, where $\delta_j \in \{0, 1\}$, $j = 1, 2, \dots, 12$, and $\widetilde{M}_2 = W_0^{(0)} + W_0^{(1)} P^{[0]}$.

The Birkhoff matrices are $W_0^{(0)}\,(I_n - \Lambda)\,\Delta_0 + W_0^{(1)} \Lambda \Delta_0 + \widetilde{M}_2\,(I_n - \Delta_0)$ and $W_0^{(0)} \Lambda \Delta_0 + W_0^{(1)}\,(I_n - \Lambda)\,\Delta_0 + \widetilde{M}_2\,(I_n - \Delta_0)$. Computer algebra calculations show that, for certain choices of the coefficients in (4.2), these Birkhoff matrices are invertible. Again, this is the generic case. In the following, we shall therefore assume that the coefficients are chosen in such a way that the problem (4.1), (4.3) is Birkhoff regular.

The eigenvalues and Birkhoff matrices can be calculated explicitly. Hence, given any particular choice of coefficients, it can be verified whether the problem is Birkhoff regular in that case. Now we can explain why we changed the original boundary conditions to those given in (4.3). The operator $\widetilde{T}$ with the original boundary conditions as in [**4**] is not Birkhoff regular. With a different choice of $C_0(\rho)$, one might obtain Birkhoff regularity, or one can consider Stone regularity. We will investigate this in a forthcoming paper.

By [**7**, Theorem 4.6.9], there are circles $\gamma_\nu = \{\lambda \in \mathbf{C} : |\lambda| = \rho_\nu\}$, $\nu \in \mathbf{N}$, with $\rho_\nu \to \infty$ as $\nu \to \infty$ such that $\gamma_\nu \subset \rho(\widetilde{T})$ and operators

(4.12)

$$P_\nu h = -\frac{1}{2\pi i}\oint_{\gamma_\nu} \widetilde{J}\widetilde{T}^{-1}(\rho)(A_1 h, 0)\, d\rho, \quad h \in (L_p(0,1))^{12},\ \nu \in \mathbf{N},$$

where $\widetilde{J} : \left(W_p^1(0,1)\right)^{12} \to (L_p(0,1))^{12}$ is the canonical embedding, satisfy $P_\nu h \to h$ as $\nu \to \infty$ in $(L_p(0,1))^{12}$ for all $h \in (L_p(0,1))^{12}$ with $\Delta_0 h = h$ and $1 < p < \infty$.

4.4 *Expansion theorems for $L(\lambda)$.*

Theorem 4.7. *Let $1 < p < \infty$. There are circles $\gamma_\nu = \{\lambda \in \mathbf{C} : |\lambda| = r_\nu\}$, $\nu \in \mathbf{N}$, with $r_\nu \to \infty$ as $\nu \to \infty$ such that $Q_\nu^j g \to g$ as $\nu \to \infty$ for all $g \in L_p(0,1)$, where*

$$Q_\nu^j g = -\frac{1}{2\pi i}\oint_{\gamma_\nu} \overline{J} D^{2j-2} e_j^t L^{-1}(\lambda)(e_j \beta_j g, 0)\, d\lambda$$

for $j = 1, 2, 3$, D denotes differentiation with respect to the independent variable, $\overline{J} : W_p^2(0,1) \to L_p(0,1)$ is the canonical embedding, and $\beta_1 = f_{11}$, $\beta_2 = g_{22}$, $\beta_3 = h_{33}$.

Proof. In order to prove the theorem we express $Q_\nu^j g$ in terms of P_ν, i.e., $L^{-1}(\lambda)$ in terms of $\widetilde{T}^{-1}(\rho)$. Let $f_1, f_2, f_3 \in L_p(0,1)$, and set

$$f = (0, a_{2,1} f_1, 0, 0, 0, a_{6,5} f_2, 0, 0, 0, 0, 0, a_{12,11} f_3)^t.$$

Define $\tilde{f} = \left(\tilde{f}_j\right)_{j=1}^{12}$ by

$$\tilde{f}_1 = f_1, \quad \tilde{f}_5 = f_2, \quad \tilde{f}_{11} = f_3 - \frac{a_{12,5}}{a_{12,11}}\tilde{f}_5,$$

$\tilde{f}_j = 0$ for the remaining indices. Then $f = \tilde{A}_1 \tilde{f}$, where we have used the last of the following properties:

$$a_{2,1} = \frac{f_{11}}{a_{11}} \neq 0, \quad a_{6,5} = \frac{g_{22}}{a_{22}} \neq 0, \quad a_{12,11} = \frac{h_{33}}{a_{33}} \neq 0.$$

Then, putting

$$\hat{f} = \left(0, \tilde{f}_1, 0, 0, 0, , \tilde{f}_5, 0, 0, 0, 0, 0, \tilde{f}_{11}\right)^t$$

it follows that $\tilde{f} = \tilde{A}_1 \widehat{f}$. For $h = C_1^{-1} \tilde{f}$ the identity $\tilde{A}_1 C_1 = C_1 A_1$ implies that

$$C_0(\rho) C_1 A_1 h = C_0(\rho) \tilde{A}_1 C_1 h = C_0(\rho) f.$$

We put

$$F_2(\rho) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^{-1} & 0 \\ 0 & 0 & \rho^{-1} \end{pmatrix} \tag{4.13}$$

and

$$E_1 = F_2(\rho) E_2 C_0(\rho) C_1. \tag{4.14}$$

Observe that

$$E_1 = E_2 C_1.$$

Then we obtain

$$\begin{aligned} E_1 P_\nu h &= -\frac{1}{2\pi i} \oint_{\gamma_\nu} \widehat{J} E_1 C_1^{-1} C_0^{-1}(\rho) T^{-1}\left(\rho^2\right) \left(C_0(\rho) C_1 A_1 h, 0\right) d\rho \\ &= -\frac{1}{2\pi i} \oint_{\gamma_\nu} \widehat{J} F_2(\rho) E_2 T^{-1}\left(\rho^2\right) \left(C_0(\rho) f, 0\right) d\rho \\ &= -\frac{1}{2\pi i} \oint_{\gamma_\nu} \widehat{J} F_2(\rho) U L^{-1}\left(\rho^2\right) \left(\rho\beta_1 f_1, \rho^2 \beta_2 f_2, \rho^2 \beta_3 f_3, 0\right) d\rho, \end{aligned}$$

where $\widehat{J} : (W_p^1(0,1))^3 \to (L_p(0,1))^3$ is the canonical embedding and the last identity follows from Proposition 4.5. From

$$h = C_1^{-1} \tilde{f} = C_1^{-1} \tilde{A}_1 \hat{f} = A_1 C_1^{-1} \hat{f}$$

and $\Delta_0 A_1 = A_1$ we infer $\Delta_0 h = h$. The considerations in subsection 4.3 show that $E_1 P_\nu h \to E_1 h$ as $\nu \to \infty$. If $f_2 = f_3 = 0$, we write h_1 for h; h_2 and h_3 are defined correspondingly. Then, for $j = 1, 2, 3$,

$$e_j^t E_1 P_\nu h_j = -\frac{1}{2\pi i} \oint_{\gamma_\nu} \rho \overline{J} e_j^t U L^{-1} \left(\rho^2\right) \left(e_j \beta_j f_j, 0\right) \, d\rho.$$

Clearly, we want to express this integral as an integral over $\lambda = \rho^2$. If ρ runs once through the circle γ_ν with radius ρ_ν, λ runs two times through the circle γ_ν with radius $r_\nu = \rho_\nu^2$, and since $d\lambda = 2\rho d\rho$, it follows that

$$e_j^t E_1 P_\nu h_j = -\frac{1}{2\pi i} \oint_{\gamma_\nu} \overline{J} D^{2j-2} e_j^t L^{-1} (\lambda) \left(e_j \beta_j f_j, 0\right) d\lambda = Q_\nu^j f_j.$$

Altogether, we obtain $Q_\nu^j f_j = e_j^t E_1 P_\nu h_j \to e_j^t E_1 h_j = f_j$ as $\nu \to \infty$. $\square$

As in subsection 3.6 we now obtain

Theorem 4.8. *Let* $1 < p < \infty$. *Let* $\lambda_0, \lambda_1, \dots,$ *be the eigenvalues of* L *defined by* (4.4), *and let* $\{y_{k,l}^{(j)} : j = 1, \dots, r(\lambda_k); l = 0, \dots, m_{k,j} - 1\}$ *and* $\{(u_{k,l}^{(j)}, d_{k,l}^{(j)}) : j = 1, \dots, r(\lambda_k); l = 0, \dots, m_{k,j} - 1\}$ *be biorthogonal* CSEAV*s of* L *and* L^* *at* λ_k. *Choose the curves* γ_ν, $\nu \in \mathbf{N}$, *according to Theorem* 4.7 *and put* $\beta_1 = f_{11}$, $\beta_2 = g_{22}$, $\beta_3 = h_{33}$. *Then for each* $f \in L_p(0,1)$ *and each* $l = 1, 2, 3$,

$$f = -\lim_{\nu \to \infty} \sum_{\substack{k \in \mathbf{N} \\ |\lambda_k| < \mu_\nu}} \left(\sum_{j=1}^{r(\lambda_k)} \sum_{h=0}^{m_{k,j}-1} D^{2l-2} e_l^t y_{k,l}^{(j)} \right.$$
$$\left. \times \int_0^1 u_{k,m_{k,j}-1-h}^{(j)} (x)^t \, e_l \beta_l (x) \, f_l(x) \, dx \right)$$

in $L_p(0,1)$.

Remark 4.9. We only obtained expansions with respect to $D^{2l-2}e_l^t y_{k,j}^{(l)}$. This can be explained by the fact that B_{ll} is a differential operator of order $2l-2$. For a detailed discussion of the general scalar case we refer to [**8**]. Note that (2.1) has the form

$$-\int_0^1 \left(u_{k,l}^{(j)}(x)\right)^t \left(\begin{pmatrix} B_{11} & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix} y_{k,m_i-1}^{(i)} \right)(x)\, dx = \delta_{ij}\delta_{0l}.$$

REFERENCES

1. R.A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

2. A.G. Aslanyan, A.V. Gulin and S.V. Kartyshov, *Calculating eigenvalues and eigenfunctions for coiled spring vibrations*, MMCE **1** (1993), 157–166.

3. F.V. Atkinson, H. Langer, R. Mennicken and A.A. Shkalikov, *The essential spectrum of some matrix operators*, Math. Nachr. **167** (1994), 5–20.

4. A.V. Goolin and S.V. Kartyshov, *Numerical study of stability and nonlinear eigenvalue problems*, Surveys Math. Ind. **3** (1993), 29–48.

5. S. Koné, *Mixed order systems of ordinary linear differential equations*, Ph.D. Thesis, University of the Witwatersrand, Johannesburg, 2001.

6. R. Mennicken and M. Möller, *Root functions of boundary eigenvalue operator functions*, Integral Equations Operator Theory **9** (1986), 237–265.

7. ———, *Non-self-adjoint boundary eigenvalue problems*, North-Holland Math. Stud., vol. 192, Elsevier, Amsterdam, 2003.

8. A.A. Shkalikov and C. Tretter, *Spectral analysis for linear pencils $N - \lambda P$ of ordinary differential operators*, Math. Nachr. **179** (1996), 275–305.

UFR de Mathématiques et Informatique, Université de Cocody–Abidjan, 22 BP 582 Abidjan 22, Ivory Coast
E-mail address: siakakone21@hotmail.com

The John Knopfmacher Centre for Applicable Analysis and Number Theory, School of Mathematics, University of the Witwatersrand, Wits 2050, South Africa
E-mail address: manfred@maths.wits.ac.za