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MONOTONICITY PROPERTIES AND INEQUALITIES OF FUNCTIONS RELATED TO MEANS

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ABSTRACT. In this paper, monotonicity properties of functions related to means are discussed and some inequalities are established.

1. Introduction. The generalized logarithmic mean (Stolarsky mean) $L_r(a,b)$ of two positive numbers a,b is defined in [**1, 2**] for $a=b$ by $L_r(a,b)=a$ and for $a\neq b$ by

$$L_r(a,b) \triangleq \left(\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r\neq -1,\ 0;$$
$$L_{-1}(a,b)=\frac{b-a}{\ln b-\ln a} \triangleq L(a,b);$$
$$L_0(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)} \triangleq I(a,b),$$

when $a\neq b$, $L_r(a,b)$ is a strictly increasing function of r. Clearly,

$$L_1(a,b)\triangleq A(a,b), \quad L_{-2}(a,b)\triangleq G(a,b),$$

where A and G are the arithmetic and geometric means, respectively.

The logarithmic mean $L(a,b)$ is generalized to the one-parameter mean in [**3**]:

$$J_r(a,b)\triangleq \frac{r(b^{r+1}-a^{r+1})}{(r+1)(b^r-a^r)}, \quad a\neq b, \quad r\neq 0,-1;$$
$$J_0(a,b)\triangleq L(a,b);$$
$$J_{-1}(a,b)\triangleq \frac{[G(a,b)]^2}{L(a,b)};$$
$$J_r(a,a)\triangleq a,$$

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when $a \neq b$, $J_r(a,b)$ is a strictly increasing function of r. Clearly,

$$J_{-2}(a,b) \triangleq H(a,b), \quad J_{-1/2}(a,b) \triangleq G(a,b), \quad J_1(a,b) \triangleq A(a,b),$$

where H is the harmonic mean.

For $a \neq b$, the following well-known inequality holds clearly:

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b).$$

2. Lemmas.

Lemma 1. *Let $a > 0$, $b > 0$. Then we have*

$$J^2_{-1/2}(a,b)\left(\frac{1}{J_{-1}(a,b)} - \frac{2}{J_0(a,b)} + \frac{1}{J_1(a,b)}\right) = J_{-2}(a,b) - 2J_{-1}(a,b) + J_0(a,b) \tag{1}$$

and

$$J^2_{-1/2}(a,b)\left(\frac{1}{J_{-2}(a,b)} - \frac{2}{J_{-1}(a,b)} + \frac{1}{J_0(a,b)}\right) = J_{-1}(a,b) - 2J_0(a,b) + J_1(a,b). \tag{2}$$

Proof. Noticing that $J_{-2}(a,b) = H(a,b)$, $J_{-1}(a,b) = G^2(a,b)/L(a,b)$, $J_{-1/2}(a,b) = G(a,b)$, $J_0(a,b) = L(a,b)$ and $J_1(a,b) = A(a,b)$, we obtain

$$\begin{aligned}
J^2_{-1/2}(a,b)&\left(\frac{1}{J_{-1}(a,b)} - \frac{2}{J_0(a,b)} + \frac{1}{J_1(a,b)}\right) \\
&= G^2(a,b)\left(\frac{L(a,b)}{G^2(a,b)} - \frac{2}{L(a,b)} + \frac{1}{A(a,b)}\right) \\
&= L(a,b) - \frac{2G^2(a,b)}{L(a,b)} + \frac{G^2(a,b)}{A(a,b)} \\
&= L(a,b) - \frac{2G^2(a,b)}{L(a,b)} + H(a,b) \\
&= J_0(a,b) - 2J_{-1}(a,b) + J_{-2}(a,b)
\end{aligned}$$

and

$$\begin{aligned}
J^2_{-1/2}(a,b)&\left(\frac{1}{J_{-2}(a,b)}-\frac{2}{J_{-1}(a,b)}+\frac{1}{J_0(a,b)}\right)\\
&=G^2(a,b)\left(\frac{1}{H(a,b)}-\frac{2L(a,b)}{G^2(a,b)}+\frac{1}{L(a,b)}\right)\\
&=\frac{G^2(a,b)}{H(a,b)}-2L(a,b)+\frac{G^2(a,b)}{L(a,b)}\\
&=A(a,b)-2L(a,b)+\frac{G^2(a,b)}{L(a,b)}\\
&=J_1(a,b)-2J_0(a,b)+J_{-1}(a,b).
\end{aligned}$$

The proof is complete. □

Corollary 1. *Let $a>0$, $b>0$. Then we have*

(3)

$$\begin{aligned}
&[J_{-2}(a,b)-2J_{-1}(a,b)+J_0(a,b)]\left(\frac{1}{J_{-2}(a,b)}-\frac{2}{J_{-1}(a,b)}+\frac{1}{J_0(a,b)}\right)\\
&\quad=[J_{-1}(a,b)-2J_0(a,b)+J_1(a,b)]\left(\frac{1}{J_{-1}(a,b)}-\frac{2}{J_0(a,b)}+\frac{1}{J_1(a,b)}\right).
\end{aligned}$$

Proof. By (1) and (2), we have

$$\begin{aligned}
&\frac{J_{-2}(a,b)-2J_{-1}(a,b)+J_0(a,b)}{J^{-1}_{-1}(a,b)-2J^{-1}_0(a,b)+J^{-1}_1(a,b)}\\
&\qquad=\frac{J_{-1}(a,b)-2J_0(a,b)+J_1(a,b)}{J^{-1}_{-2}(a,b)-2J^{-1}_{-1}(a,b)+J^{-1}_0(a,b)}\\
&\qquad=J^2_{-1/2}(a,b).
\end{aligned}$$

Hence, (3) holds. □

Lemma 2. *Let $a>0$, $b>0$ and $a\neq b$. Then we have for $r=-1$, 0,*

$$\frac{1}{J_{r-1}(a,b)}+\frac{1}{J_{r+1}(a,b)}>\frac{2}{J_r(a,b)}. \tag{4}$$

Proof. Since a and b are symmetric, without loss of generality, assume $b > a > 0$. For $r = -1$, (4) becomes

$$\frac{1}{H(a,b)} + \frac{1}{L(a,b)} > \frac{2L(a,b)}{G^2(a,b)},$$

which is equivalent to

$$\frac{2ab(\ln b - \ln a)^2 + (b^2 - a^2)(\ln b - \ln a) - 4(b-a)^2}{2ab(b-a)(\ln b - \ln a)} > 0.$$

Clearly, $2ab(b-a)(\ln b - \ln a) > 0$; thus, it is sufficient to prove that

$$\phi(x) \triangleq 2ax(\ln x - \ln a)^2 + (x^2 - a^2)(\ln x - \ln a) - 4(x-a)^2 > 0$$

for $x > a > 0$. Easy computations reveal that

$$\begin{aligned}
\phi'(x) &= 2a(\ln x - \ln a)^2 + (2x + 4a)(\ln x - \ln a) - 7x - \frac{a^2}{x} + 8a,\\
x\phi''(x) &= (2x + 4a)(\ln x - \ln a) - 5x + \frac{a^2}{x} + 4a \triangleq \psi(x),\\
\psi'(x) &= \frac{4a}{x} + 2(\ln x - \ln a) - \frac{a^2}{x^2} - 3,\\
\psi''(x) &= \frac{2(x-a)^2}{x^3} > 0.
\end{aligned}$$

Hence, we have for $x > a$,

$$\begin{aligned}
\psi'(x) > \psi'(a) = 0 &\Longrightarrow \psi(x) > \psi(a) = 0 \Longrightarrow \phi''(x) > 0\\
&\Longrightarrow \phi'(x) > \phi'(a) = 0 \Longrightarrow \phi(x) > \phi(a) = 0.
\end{aligned}$$

Thus, (4) holds for $r = -1$.

For $r = 0$, (4) becomes

$$\frac{L(a,b)}{G^2(a,b)} + \frac{1}{A(a,b)} > \frac{2}{L(a,b)},$$

which is equivalent to

$$\frac{-2ab(b+a)(\ln b - \ln a)^2 + 2ab(b-a)(\ln b - \ln a) + (b-a)^2(b+a)}{ab(b+a)((b-a))(\ln b - \ln a)} > 0.$$

Clearly, $ab(b+a)(b-a)(\ln b - \ln a) > 0$; thus it is sufficient to prove that

$$\begin{aligned} u(x) \triangleq& -2ax(x+a)(\ln x - \ln a)^2 + 2ax(x-a)(\ln x - \ln a) \\ &+ (x-a)^2(x+a) > 0 \end{aligned}$$

for $x > a > 0$. Easy computations reveal that

$$\begin{aligned} u'(x) &= -(4ax + 2a^2)(\ln x - \ln a)^2 - 6a^2(\ln x - \ln a) + 3(x^2 - a^2), \\ xu''(x) &= -4ax(\ln x - \ln a)^2 - 4a(2x+a)(\ln x - \ln a) \\ &\quad + 6(x^2 - a^2) \triangleq v(x), \\ v'(x) &= -4a(\ln x - \ln a)^2 - 16a(\ln x - \ln a) - 8a - \frac{4a^2}{x} + 12x, \\ xv''(x) &= -8a(\ln x - \ln a) - 16a + \frac{4a^2}{x} + 12x \triangleq w(x), \\ w'(x) &= \frac{4(3x+a)(x-a)}{x^2} > 0. \end{aligned}$$

Hence, we have for $x > a$,

$$\begin{aligned} w(x) > w(a) = 0 &\Longrightarrow v''(x) > 0 \Longrightarrow v'(x) > v'(a) = 0 \\ &\Longrightarrow v(x) > v(a) = 0 \\ &\Longrightarrow u''(x) > 0 \Longrightarrow u'(x) > u'(a) = 0 \\ &\Longrightarrow u(x) > u(a) = 0. \end{aligned}$$

Thus, (4) holds for $r = 0$. The proof is complete. □

By Lemma 1 and Lemma 2, the following corollary is obvious.

Corollary 2. *Let $a > 0$, $b > 0$ and $a \neq b$. Then*

$$J_{-1}(a,b) + J_1(a,b) > 2J_0(a,b), \tag{5}$$

$$J_{-2}(a,b) + J_0(a,b) > 2J_{-1}(a,b). \tag{6}$$

Lemma 3. *Let $a > 0$, $r \in (-\infty, +\infty)$. Define, for $x > 0$,*

$$R_r(x) = \begin{cases} (L_r^2(a,x))/(L_{r-1}(a,x)L_{r+1}(a,x)) & x \neq a, \\ 1 & x = a. \end{cases} \tag{7}$$

Then we have, for $x \neq a$,

$$\frac{1}{R_r(x)}\frac{dR_r(x)}{dx} = \frac{a}{x-a}\left(-\frac{2}{J_r(a,x)} + \frac{1}{J_{r-1}(a,x)} + \frac{1}{J_{r+1}(a,x)}\right). \tag{8}$$

Proof. Taking logarithm and differentiation yields

$$\begin{aligned}
&\frac{x-a}{R_r(x)}\frac{dR_r(x)}{dx}\\
&= \frac{2(rx^{r+1}-(r+1)ax^r+a^{r+1})}{r(x^{r+1}-a^{r+1})} - \frac{(r-1)x^r - rax^{r-1}+a^r}{(r-1)(x^r-a^r)}\\
&\quad - \frac{(r+1)x^{r+2}-(r+2)ax^{r+1}+a^{r+2}}{(r+1)(x^{r+2}-a^{r+2})}\\
&= 2\left(\frac{rx^{r+1}-(r+1)ax^r+a^{r+1}}{r(x^{r+1}-a^{r+1})} - 1\right)\\
&\quad - \left(\frac{(r-1)x^r - rax^{r-1}+a^r}{(r-1)(x^r-a^r)} - 1\right)\\
&\quad - \left(\frac{(r+1)x^{r+2}-(r+2)ax^{r+1}+a^{r+2}}{(r+1)(x^{r+2}-a^{r+2})} - 1\right)\\
&= -\frac{2a(r+1)(x^r-a^r)}{r(x^{r+1}-a^{r+1})} + \frac{ar(x^{r-1}-a^{r-1})}{(r-1)(x^r-a^r)} + \frac{a(r+2)(x^{r+1}-a^{r+1})}{(r+1)(x^{r+2}-a^{r+2})}\\
&= -\frac{2a}{J_r(a,x)} + \frac{a}{J_{r-1}(a,x)} + \frac{a}{J_{r+1}(a,x)}.
\end{aligned}$$

The proof is complete. □

3. Main results.

Theorem 1. *Let $a > 0$, define for $x > 0$,*

$$f(x) = \begin{cases} (G^2(a,x))/(H(a,x)L(a,x)) & x \neq a,\\ 1 & x = a.\end{cases}$$

Then f is strictly decreasing on $(0,a)$ and strictly increasing on $(a,+\infty)$.

Proof. Taking logarithm and differentiation yields

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{1}{x+a} - \frac{x(\ln x - \ln a) - (x-a)}{x(x-a)(\ln x - \ln a)} \\ &= \frac{2a\left[(x^2-a^2)/(2ax) - (\ln x - \ln a)\right]}{(x+a)(x-a)(\ln x - \ln a)} \\ &= \frac{2a}{(x+a)(x-a)}\,\frac{x-a}{\ln x - \ln a}\left(\frac{x+a}{2ax} - \frac{\ln x - \ln a}{x-a}\right) \\ &= \frac{2aL(a,x)}{(x+a)(x-a)}\left(\frac{1}{H(a,x)} - \frac{1}{L(a,x)}\right) \\ &= \frac{2a[L(a,x) - H(a,x)]}{(x+a)(x-a)H(a,x)}.\end{aligned}$$

Since $L(a,x) > H(a,x)$, it is clear that $f'(x) < 0$ for $0 < x < a$ and $f'(x) > 0$ for $x > a$. The proof is complete. □

Corollary 3. *Let $c > b > a > 0$. Then*

$$\left(\frac{G(a,b)}{G(a,c)}\right)^2 < \frac{H(a,b)L(a,b)}{H(a,c)L(a,c)}. \tag{10}$$

The inequality in (10) *is reversed for $0 < b < c < a$.*

Since f is continuous on $(0,+\infty)$ and takes its unique minimum $f(a) = 1$ at $x = a$, we get

Corollary 4. *Let $a > 0$, $b > 0$ and $a \neq b$. Then*

$$G^2(a,b) > H(a,b)L(a,b). \tag{11}$$

Theorem 2. *Let $a > 0$. Define, for $x > 0$,*

$$g(x) = \begin{cases} (L^2(a,x)/G(a,x)I(a,x)) & x \neq a, \\ 1 & x = a. \end{cases} \tag{12}$$

$$h(x) = \begin{cases} (I^2(a,x)/L(a,x)A(a,x)) & x \neq a, \\ 1 & x = a. \end{cases} \tag{13}$$

Then both g and h are strictly decreasing on $(0, a)$ and strictly increasing on $(a, +\infty)$.

Proof. By Lemma 3 (taking $r = -1, 0$, respectively), we have for $x \neq a$,

$$\frac{g'(x)}{g(x)} = \frac{a}{x-a}\left(-\frac{2}{J_{-1}(a,x)} + \frac{1}{J_{-2}(a,x)} + \frac{1}{J_0(a,x)}\right),$$
$$\frac{h'(x)}{h(x)} = \frac{a}{x-a}\left(-\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)}\right).$$

By Lemma 2, we have for $x \neq a$,

$$-\frac{2}{J_{-1}(a,x)} + \frac{1}{J_{-2}(a,x)} + \frac{1}{J_0(a,x)} > 0,$$
$$-\frac{2}{J_0(a,x)} + \frac{1}{J_{-1}(a,x)} + \frac{1}{J_1(a,x)} > 0.$$

Hence, it is clear that $g'(x) < 0$ and $h'(x) < 0$ for $0 < x < a$, and $g'(x) > 0$ and $h'(x) > 0$ for $x > a$. The proof is complete. $\square$

Corollary 5. *Let $c > b > a > 0$. Then*

$$\left(\frac{L(a,b)}{L(a,c)}\right)^2 < \frac{G(a,b)I(a,b)}{G(a,c)I(a,c)}, \tag{14}$$
$$\left(\frac{I(a,b)}{I(a,c)}\right)^2 < \frac{L(a,b)A(a,b)}{L(a,c)A(a,c)}. \tag{15}$$

The inequalities in (14) *and* (15) *are reversed for* $0 < b < c < a$.

Since both g and h are continuous on $(0, +\infty)$ and take their unique minimum $g(a) = h(a) = 1$ at $x = a$, we get

Corollary 6. *Let $a > 0$, $b > 0$ and $a \neq b$. Then*

$$L^2(a,b) > G(a,b)I(a,b), \tag{16}$$
$$I^2(a,b) > L(a,b)A(a,b). \tag{17}$$

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