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THE VALUES OF ADDITIVE FORMS AT PRIME ARGUMENTS

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ABSTRACT. New results are proved on additive forms at prime arguments of the type $\lambda_1 p_1^k + \cdots \lambda_s p_s^k$ where the λ_j are not all negative and are not all in rational ratio. The improvements come in the number of variables required and the distribution of the values. The former improvement comes from using familiar techniques in the Hardy-Littlewood method, while the latter improvement stems from recent developments in the theory of exponential sums.

1. Introduction. Given $k \geq 1$ and s nonzero real numbers $\lambda_1, \ldots, \lambda_s$ (not all in rational ratio, not all negative), we write

$$F(\mathbf{p}) = \sum_{j=1}^{s} \lambda_j p_j^k$$

where $\mathbf{p} = (p_1, \ldots, p_s)$ with each p_j a prime. Various authors have considered the distribution of values of such forms, for example, see [**14**]. Here we continue in the direction started by Brüdern, Cook and Perelli [**3**] and followed by Cook and Fox [**5**], Cook [**4**] and Harman [**9**]. We call a set of positive reals $\mathcal{V}$ a *well-spaced set* if there is a $c > 0$ such that

$$u, v \in \mathcal{V}, \quad u \neq v, \quad \Longrightarrow |u - v| > c.$$

In order to get the full-strength of the results under consideration, one should also assume that

$$|\{v \in \mathcal{V} : v \leq X\}| \gg X^{1-\varepsilon},$$

though our results are nontrivial with a weaker lower bound. Given a form F as above, let $E_k(\mathcal{V}, X, \delta)$ denote the number of $v \in \mathcal{V}$, $v \leq X$, such that the inequality

$$|F(\mathbf{p}) - v| < v^{-\delta} \tag{1.1}$$

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has no solution in primes $p_1, \dots, p_s$. As might be expected, this problem is related to the Waring-Goldbach problem, and one expects the number of variables s required to obtain a result, say $H^*(k)$, to increase with k in line with the values obtained for representing numbers as sums of prime powers. To be more precise, one might hope that $H^*(k)$ is equal to the number of prime powers needed to represent almost all integers allowed by congruence conditions, see [**11**] for example.

In [**3**] the case $k = 1$ is investigated and it is shown that one may take $s = 2$ and thereby obtain $E_1(\mathcal{V}, X, \delta) \ll X^{(2/3)+2\delta+\varepsilon}$, at least in the case λ_1/λ_2 algebraic. We shall describe an improvement of this result (at least for $\delta > 2/15$) in the final section of this paper. In [**5**] it is shown that three variables suffice for $k = 2$, and the upper bound for $E_2(\mathcal{V}, X, \delta)$ is improved in [**9**] to $\ll X^{(6/7)+2\delta+\varepsilon}$ for infinitely many X. In [**4**] it is shown that in general $2^{k-1}+1$ variables suffice ($7.2^{k-4}+1$ for $k \geq 6$), and the bound for $E_k(\mathcal{V}, X, \delta)$ will depend on the best known estimates for exponential sums over primes (or for certain double sums if sieve methods are invoked). The exponent quoted in [**4**] for X is $1 - (2/3k)4^{1-k} + 2\delta + \varepsilon$. The purpose of this note is to reduce the required number of variables for larger k and use the latest bounds for the exponential sums which arise [**12**] (these improve upon the results in [**8**] which themselves would have been strong enough to establish Theorem 1 for $k \geq 5$) to give an improved exponent for X. In this way we establish a general result (Theorem 2 below) from which other known results (or improvements upon them) follow as corollaries. We write $\sigma(3) = (14)^{-1}$, and in general $\sigma(k) = (3.2^{k-1})^{-1}$ for $k \geq 4$. We also define $H^*(k)$ by:

TABLE 1.

k	3	4	5	6	7	8	9	10
$H^*(k)$	5	8	13	19	28	38	49	62

The reader will note that these values are $[\mathcal{D}(k)/2] + 1$, where $\mathcal{D}(k)$ is given in [**14**]. For larger k the values of $H^*(k)$ can be calculated in a similar manner. Our main result is then as follows.

Theorem 1. *Let $k \geq 3$, $s = H^*(k)$, and let $\lambda_1, \lambda_2, \dots \lambda_s$ be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence. Let $\delta > 0$. Then*

$$E_k(\mathcal{V}, X, \delta) \ll X^{1-2\sigma(k)/k+2\delta+\varepsilon} \tag{1.2}$$

for any $\varepsilon > 0$.

This result is nontrivial when $\delta < \sigma(k)/k$. It will be seen from the corollaries below that this is the "expected" result (in terms of the exponent of X in [**3**]) given our current state of knowledge for other problems. One might hope that the number of variables could be reduced a little using the ideas in [**10, 11, 13**], but there are some technical problems at present. Theorem 1 follows immediately from the next general result.

Theorem 2. *Let $k \geq 3$, $s = H^*(k)$, and let $\lambda_1, \lambda_2, \dots \lambda_s$ be nonzero real numbers, not all negative. Suppose that λ_1/λ_2 is irrational. Let $\mathcal{V}$ be a well-spaced sequence. Let$\delta > 0$. Then there is a sequence $X_j \to \infty$ such that*

$$E_k(\mathcal{V}, X_j, \delta) \ll X_j^{1-2\sigma(k)/k+2\delta+\varepsilon} \tag{1.3}$$

for any $\varepsilon > 0$. Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy

$$q_{j+1}^{1-\omega} \ll q_j \quad \textit{for some} \quad \omega \in [0, 1) \tag{1.4}$$

then, for all $X \geq 1$,

$$E_k(\mathcal{V}, X, \delta) \ll X^{1-2\theta/k+2\delta+\varepsilon} \tag{1.5}$$

for any $\varepsilon > 0$ with

$$\theta = \theta(\omega, k) = \min\left(\frac{k(1-\omega)}{6-4\omega}, \sigma(k)\right). \tag{1.6}$$

Theorem 1 follows immediately from Theorem 2, since in the case of λ_1/λ_2 algebraic, we can take $\omega = \varepsilon$ and then $\theta = \sigma(k)$.

The reader should have no difficulties in deducing the following corollaries to Theorem 2.

Corollary 1. *Let* λ_j, $1 \le j \le t = 2H^*(k) - 1$, *be nonzero real numbers, not all of the same sign, with* λ_1/λ_2 *irrational,* η *real, and* $\varepsilon > 0$. *Then there are infinitely many solutions in primes* p_j *to the inequality*

$$|\eta + \textstyle\sum_{j=1}^{t}\lambda_j p_j^k| < (\max p_j)^{-\sigma(k)+\varepsilon}. \tag{1.7}$$

Corollary 2 (Baker and Harman). *Let* α *be an irrational real number,* β *real, and* $\varepsilon > 0$. *Then there are infinitely many solutions in primes* p *to the inequality*

$$||\alpha p^k + \beta|| < p^{-\sigma(k)+\varepsilon}. \tag{1.8}$$

Here $||x||$ *denotes the distance from* x *to a nearest integer.*

Corollary 1 improves Theorem 3 of [**1**], while Corollary 2 is the main theorem in [**2**], except that our exponent here is slightly weaker for $k = 3$. Corollary 2 has been improved with a sieve method [**17**], and it is possible to improve our results a little in a similar manner to Section 8 of [**9**], but the improvement obtained is not as strong as [**17**] because of our need to sieve two variables.

2. Outline of the method. We follow the modification of the Hardy-Littlewood method first stated by Davenport and Heilbronn [**6**]. Let $0 < \tau < 1$. If we write

$$A(x) = \max(0, \tau - |x|),$$

then

$$A(x) = \int_{-\infty}^{\infty} K(\alpha)e(\alpha x)\,d\alpha \tag{2.1}$$

where

$$e(\beta) = \exp(2\pi i\beta), \qquad K(\alpha) = \left(\frac{\sin \pi\tau\alpha}{\pi\alpha}\right)^2.$$

We follow Vaughan's argument [**14**], which entails treating certain variables in a rather different fashion to others, at least for $k > 3$ which we henceforth assume. The modifications for $k = 3$ will be described at the end. We first define parameters m and r which depend on k as follows:

TABLE 2.

k	4	5	6	7	8	9	10
m	4	8	13	19	28	40	51
r	3	4	5	8	9	8	10

The reader will note that $H^*(k) = m + r + 1$. Now let P be some (large) positive quantity to be chosen later, and write $X = P^k$, $P_j = P|\lambda_j|^{-1/k}$, $\mathcal{I}_j = [P_j/s, 2P_j]$ for $1 \le j \le r + 1$. We let $\mathcal{U}$ be a set of numbers representable in the form

$$\sum_{j=r+2}^{H^*(k)} \lambda_j p_j^k$$

with $p_j \le P$ as explained in [**14**]. To be more precise, $\mathcal{U}$ is a well spaced set constructed inductively with successive p_j in "diminishing ranges," and the number of elements in $\mathcal{U}$ is $\gg X^\nu$ where ν is the value given in [**14**]. For example, on page 398 there we see that for $k = 5$ we have $\nu = 0.911$.

Let $\log \mathbf{p} = \prod_j \log p_j$. If we write

$$N_v = \frac{1}{\tau} \sum_{u \in \mathcal{U}} \sum_{p_j \in \mathcal{I}_j} (\log \mathbf{p}) A(\lambda_1 p_1^k + \lambda_2 p_2^k + \cdots \lambda_{r+1} p_{r+1}^k + u - v),$$

then $0 \le N_v \le \psi(v)$ where $\psi(v)$ counts the number of solutions to

$$|\lambda_1 p_1^k + \lambda_2 p_2^k + \cdots \lambda_{r+1} p_{r+1}^k + u - v| < \tau$$

with $u \in \mathcal{U}$, weighted by a term $\log \mathbf{p}$. We shall restrict our attention to those v satisfying $X/2 \le v \le X$. In general one can consider

$X2^{-j} \le v \le X2^{1-j}$, $j = 1, 2, \dots$, and obtain a satisfactory bound for the exceptional set.

By (2.1),

$$N_v = \frac{1}{\tau} \int_{-\infty}^{\infty} S_1(\alpha) \dots S_{r+1}(\alpha) U(\alpha) K(\alpha)\, d\alpha \tag{2.2}$$

where

$$S_j(\alpha) = \sum_{p_j \in \mathcal{I}_j} (\log p_j) e(\alpha \lambda_j p_j^k), \qquad U(\alpha) = \sum_{u \in \mathcal{U}} e(\alpha u).$$

For future convenience we write

$$\Pi(\alpha) = \prod_{j=1}^{r+1} S_j(\alpha).$$

The main contribution to the integral (2.2) comes from the *major arc* $[-\phi, \phi]$, where ϕ is a small positive quantity. In such a region the exponential sums $S_j(\alpha)$ can be well approximated by integrals. The limit to the strength of a result of the present type (in terms of how small τ can be) comes from the region $1 \le |\alpha| \ll \tau^{-1}$ (the minor arc), where $K(\alpha) \gg \tau^2$. In this paper we take

$$\phi = P^{-\varepsilon - k + 5/12}, \qquad \mathfrak{M} = [-\phi, \phi], \qquad \mathbf{m} = \{\alpha : |\alpha| > \phi\}.$$

3. Preliminary lemmas. We assemble here some important results we need for the analysis of all the regions for α.

Lemma 1. *Let $k \ge 3$. Suppose that $N \ge 2$ and α satisfies*

$$|q\alpha - a| \le Q^{-1}, \quad (a, q) = 1, \quad q \in \mathbf{N}, \quad q \le Q, \quad a \in \mathbf{Z}, \tag{3.1}$$

where $Q = N^{(k^2 - 2k\sigma(k))/(2k-1)}$. Then, for any $\varepsilon > 0$,

$$\sum_{N/2 \le p \le N} (\log p)\, e(\alpha p^k) \ll N^{1+\varepsilon-\sigma(k)} + \frac{N^{1+\varepsilon}}{(q + N^k |q\alpha - a|)^{1/2}}. \tag{3.2}$$

Proof. This is given by Theorem 3 of [**12**]. □

Corollary 3. *Suppose that* $k \geq 3$, $P \geq Z \geq P^{1-\sigma(k)+\varepsilon}$ *and* $|S_j(\alpha)| > Z$. *Then there are coprime integers* a, q *satisfying*

$$1 \leq q \ll (P/Z)^2 P^{\varepsilon}, \qquad |q\alpha\lambda_j - a| \ll (P/Z)^2 P^{\varepsilon - k}. \tag{3.3}$$

Proof. By Dirichlet's theorem in Diophantine approximation, we can find a, q satisfying (3.1) by the lemma with ε replaced by $\varepsilon/2$. The conclusion of (3.3) then follows from (3.2). □

Lemma 2. *We have, for* $t = 1$ *or* 2,

$$\begin{aligned} \int_{-\infty}^{\infty} \Big| U(\alpha) \prod_{j \neq t} S_j(\alpha) \Big|^2 K(\alpha)\, d\alpha &\ll \tau |\mathcal{U}|^2 P^{2r-k+\varepsilon}, \\ \int_{-1}^{1} \Big| U(\alpha) \prod_{j \neq t} S_j(\alpha) \Big|^2 d\alpha &\ll |\mathcal{U}|^2 P^{2r-k+\varepsilon}. \end{aligned} \tag{3.4}$$

Proof. Both of these follow from [**14**]. □

4. The major arc. There are no problems in adapting the work in Section 4 of [**9**], using Lemma 2 above to prove the following result. Alternatively, one could adapt the material in Section 3 of [**4**].

Lemma 3. *We have, for all sufficiently large* P,

$$\int_{\mathfrak{M}} K(\alpha) U(\alpha) \Pi(\alpha)\, d\alpha \gg |\mathcal{U}| P^{r+1-k} \tau^2.$$

5. The minor arc. We now introduce the necessary averaging over v. The argument (which can be traced back to Watson [**16**]) is essentially the same as in [**9**] with Lemma 1 here replacing Lemma 3 there to provide information when exponential sums are large. We let a/q be a convergent to λ_1/λ_2 and choose P so that

$$q = P^{k-3\varepsilon-4\sigma(k)}.$$

By a simple argument, see [**5**, p. 142], we can restrict our attention to $\mathbf{m}^* = \{\alpha \in \mathbf{m} : |\alpha| \le \tau^{-2}\}$. Let

$$\mathbf{m'} = \{\alpha \in \mathbf{m}^* : \min(|S_1(\alpha)|, |S_2(\alpha)|) < P^{\rho+\varepsilon}\}.$$

Then, working as in Lemma 10 of [**4**], we have

(5.1)

$$\sum_{v \in \mathcal{V}} \left| \int_{\mathbf{m'}} U(\alpha)\Pi(\alpha)K(\alpha)e(-v\alpha)\,d\alpha \right|^2 \ll \tau \int_{\mathbf{m'}} |U(\alpha)\Pi(\alpha)|^2 K(\alpha)\,d\alpha$$
$$\ll \tau^2 |\mathcal{U}| P^{2\rho+2r-k+2\varepsilon}.$$

We shall take $\rho = 1 - \sigma(k)$ in this argument. We put $\hat{\mathbf{m}} = \mathbf{m}^* \setminus \mathbf{m'}$. We then divide $\hat{\mathbf{m}}$ into disjoint sets such that for $\alpha \in S(Z_1, Z_2, y)$ we have

$$Z_j \le |S_j(\alpha)| < 2Z_j, \quad y \le |\alpha| < 2y,$$

where $Z_j = P^{\rho+\varepsilon}2^t$ and $y = \phi 2^r$ for some positive integers t, r. Thus, by (3.3), there are two pairs of coprime integers $(a_1, q_1), (a_2, q_2)$ with

$$a_1 a_2 \ne 0, \quad 1 \le q_j \ll (P/Z_j)^2 P^{\varepsilon}, \quad |q_j \alpha \lambda_j - a_j| \ll (P/Z_j)^2 P^{\varepsilon - k}.$$

We further subdivide $\hat{\mathbf{m}}$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$ where $Q_j \le q_j < 2Q_j$ on each set. Then, by a familiar argument, see [**1**, p. 207] for example,

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2 \right| \ll \left(\frac{P^2}{Z_1 Z_2} \right)^2 P^{2\varepsilon - k} = o(q^{-1}).$$

Also

$$|a_2 q_1| \ll P^{2\varepsilon} y\, Q_1 Q_2.$$

Now, if $|a_2 q_1|$ took on R distinct values, we could apply the pigeon-hole principle to deduce the existence of n with

$$\left\| n \frac{\lambda_1}{\lambda_2} \right\| = o(q^{-1}), \qquad n \ll \frac{P^{2\varepsilon} y\, Q_1 Q_2}{R}.$$

This would contradict q being a convergent to λ_1/λ_2 if P is sufficiently large unless

$$R \ll \frac{P^{2\varepsilon} y\, Q_1 Q_2}{q}.$$

Of course, this actually forces $R = 0$ for many possible combinations of Q_1, Q_2, y. Since each value of $|a_2 q_1|$ corresponds to $\ll P^\varepsilon$ values of a_2, q_1 by the well-known bound on the divisor function, we conclude that $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $\ll RP^\varepsilon$ intervals of length

$$\ll P^{\varepsilon+2-k} \min\left(\frac{1}{Q_1 Z_1^2}, \frac{1}{Q_2 Z_2^2}\right) \le \frac{P^{\varepsilon+2-k}}{Z_1 Z_2 (Q_1 Q_2)^{1/2}}.$$

Thus, integrating over such a set gives

$$\begin{aligned} (5.2) \quad & \int |U(\alpha)\Pi(\alpha)|^2 \, K(\alpha)\, d\alpha \\ & \ll \min(\tau^2, y^{-2})(Z_1 Z_2)^2 \, P^{2r-2} \, |\mathcal{U}|^2 \, \frac{Q_1 Q_2 P^{3\varepsilon} y}{q} \, \frac{P^{\varepsilon+2-k}}{Z_1 Z_2 (Q_1 Q_2)^{1/2}} \\ & \ll |\mathcal{U}|^2 \, P^{2r-k+4\varepsilon} \, \frac{(Q_1 Q_2)^{1/2} Z_1 Z_2}{P^{k-3\varepsilon-4\sigma(k)}} \, \tau \\ & \ll \tau \, |\mathcal{U}|^2 \, P^{\varepsilon+2r+2-k-1/2}, \end{aligned}$$

since, by (3.3),

$$(Q_1 Q_2)^{1/2} Z_1 Z_2 \ll P^{2+\varepsilon}.$$

Summing over all possible values of y, Q_1, Q_2, Z_1, Z_2, and using (5.1) we conclude that

$$(5.3) \qquad \sum_{v \in \mathcal{V}} \left| \int_{\mathbf{m}} U(\alpha)\Pi(\alpha)K(\alpha)\, d\alpha \right|^2 \ll \tau^2 \, |\mathcal{U}|^2 \, P^{2\rho+2r-k+\varepsilon}.$$

6. Completion of the proof. As in previous work in this area, the number of $v \in [X/2, X] \cap \mathcal{V}$ for which $N_v = 0$ can be estimated by bounding the number of times

$$(6.1) \qquad \left| \int_{\mathbf{m}} U(\alpha)\Pi(\alpha)K(\alpha)\, d\alpha \right| \gg \tau^2 \, |\mathcal{U}| \, P^{r+1-k}.$$

Comparing (5.3) and (6.1) we conclude that this number is

$$\ll \tau^{-2} \, P^{2\rho+\varepsilon} \ll \tau^{-2} \, X^{1-2\sigma(k)/k+\varepsilon}.$$

Taking $\tau = X^{-\delta}$ then gives the result. Of course, there are infinitely many q we could have taken in Section 5 since λ_1/λ_2 is irrational, and this gives the sequence $X_j \to \infty$.

If the convergent denominators for λ_1/λ_2 satisfy (1.4) we can modify our work in Section 5. We now let P be any sufficiently large number and assume that

$$\min(Z_1, Z_2) > P^{1-\theta+\varepsilon}$$

with $\theta = \theta(\omega, k)$ given by (1.6). We then obtain

$$\left| a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_2 q_1 \right| \ll P^{-k+4\theta+2\varepsilon}.$$

However, we know from (1.4) that there is a convergent a/q to λ_1/λ_2 with

$$P^{(1-\omega)(k-4\theta-2\varepsilon)} \ll q \ll P^{k-4\theta-2\varepsilon}.$$

The expression corresponding to (5.2) is now

$$\int_{\mathbf{m}} |U(\alpha)\Pi(\alpha)|^2 K(\alpha)\, d\alpha \ll \tau\, |\mathcal{U}|^2\, P^{2r+2-k+A+\varepsilon}$$

where

$$A \leq -(k-4\theta)(1-\omega) < -2\theta$$

by our choice of ω. This quickly leads to the result stated.

7. The case $k = 3$. When $k = 3$ we need only use Hua's inequality (Lemma 2.5 in [**15**]) in place of Lemma 2. We now make no use of the set $\mathcal{U}$, so instead of (2.2) we have

$$N_v = \frac{1}{\tau} \int_{-\infty}^{\infty} \prod_{j=1}^{5} S_j(\alpha) K(\alpha)\, d\alpha.$$

All the arguments then go through as for $k > 3$ but with $r = 4$ and all terms $\mathcal{U}$ and $U(\alpha)$ removed.

8. The case $k = 1$. We shall briefly outline how to prove the following result which is nontrivial for $\delta < 1/5$, thus improving upon $\delta < 1/6$ required in [**3**].

Theorem 3. *Let λ_1, λ_2 be nonzero real numbers, not both negative, in irrational ratio. Let $\mathcal{V}$ be a well spaced sequence. Let $\delta > 0$. Then there is a sequence $X_j \to \infty$ such that*

$$E_1(\mathcal{V}, X_j, \delta) \ll X_j^{\varepsilon} \min\left(X_j^{(2/3)+2\delta}, X_j^{(4/5)+\delta}\right) \tag{8.1}$$

for all $\varepsilon > 0$. Moreover (8.1) *holds for all X_j if λ_1/λ_2 is an algebraic irrational.*

Proof. From the work in [**3**] it is clear that the crucial part is the estimation of the minor arc contribution

$$\int_{\mathbf{m}} |S_1(\alpha)S_2(\alpha)|^2 \, K(\alpha)\, d\alpha. \tag{8.2}$$

The expression (17) in [**3**] and the following working there give an upper bound for (8.2) which is $\ll \tau X^{(8/3)+3\varepsilon}$ for X chosen correctly in terms of a convergent denominator to λ_1/λ_2. However, the Proposition in [**7**] gives an upper bound $\ll \tau P^{13/5-c\varepsilon}$ for some c when $\tau = X^{-(1/5)+\varepsilon/2}$. If the working in that paper were valid for all τ this would lead to the result for Theorem 3 with $X^{(3/5)+2\delta}$ on the right-hand side. Unfortunately, use is made of the averaging involved in α running up to τ^{-1} (and for our application beyond) and so this result cannot be obtained. A careful examination of the analysis in [**7**, pp. 221–223] shows that it is possible to obtain

$$\tau^2 X^{14/5+\varepsilon} + \tau X^{13/5+\varepsilon}$$

as an upper bound for (8.2), and the first term of this dominates for $\delta < 1/5$ while the theorem itself is trivial for $\delta \geq 1/5$. This quickly leads to the stated result. It then comes as no surprise that we can deduce the main theorem of [**7**] as a corollary to Theorem 3! □

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