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SG-PSEUDODIFFERENTIAL OPERATORS AND GELFAND-SHILOV SPACES

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1. Introduction. Linear partial differential operators, or more generally pseudodifferential operators, of SG-type (symbol global-type) are defined in $\mathbf{R}^n$ by imposing suitable algebraic asymptotics as $x \to \infty$ for the symbols of the operators. Basic examples are $-\Delta + 1$ and for $k \geq 1$

$$H = (1 + |x|^{2k})(-\Delta + 1) + L_1(x, D) \tag{0.1}$$

where $L_1(x, D)$ is a first order operator with polynomial coefficients of degree $2k - 1$. Let us refer to Parenti [**16**], Cordes [**5**], Schrohe [**21**], Egorov and Schulze [**10**], Schulze [**22**] for a precise definition and the corresponding pseudodifferential calculus in the frame of the Schwartz spaces $\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n)$. As an application for the SG-elliptic operators P, having H in (0.1) as prototype, the above mentioned authors construct parametrices and deduce in particular the following result of global regularity: all the solutions $u \in \mathcal{S}'(\mathbf{R}^n)$ of $Pu = f \in \mathcal{S}(\mathbf{R}^n)$ are of class $\mathcal{S}(\mathbf{R}^n)$. In particular, when P is self-adjoint, as we have for H in (0.1) if $L_1(x, D)$ is suitably chosen, there exists a system of eigenfunctions in the space $\mathcal{S}(\mathbf{R}^n)$; see, for example, Maniccia and Panarese [**15**] for the corresponding eigenvalue asymptotics.

Our aim in this paper is to obtain more precise information concerning the behavior for $x \to \infty$ and the local regularity of the solutions under related assumptions on the regularity of the coefficients. The functional frame, providing the two results simultaneously, is given here by the classes $S_\theta^\theta(\mathbf{R}^n), S_\theta^{\theta'}(\mathbf{R}^n), \theta > 1$, introduced by Gelfand and Shilov [**11**]. Let us recall that $S_\theta^\theta(\mathbf{R}^n)$ is a subclass of $\mathcal{S}(\mathbf{R}^n)$, combining the exponential decay $e^{-L|x|^{1/\theta}}$, $L > 0$, with the local Gevrey estimates of order θ, i.e., $|D^\alpha u(x)| \leq C^{|\alpha|+1}(\alpha!)^\theta$. In turn, the ultradistribution space $S_\theta^{\theta'}(\mathbf{R}^n)$ contains $\mathcal{S}'(\mathbf{R}^n)$ and admits as examples functions with growth $e^{L|x|^\sigma}$, $\sigma < 1/\theta$, see Section 1 for details. Observe that the

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classes $S_\theta^\theta(\mathbf{R}^n)$ have been widely used in other contexts under different names and notations, cf. the recent papers by Biagioni and Gramchev [**2**] and by Pilipovic and Teofanov [**17, 18**].

In Section 2, we give a pseudodifferential SG-calculus in $S_\theta^\theta(\mathbf{R}^n)$, $S_\theta^{\theta'}(\mathbf{R}^n)$. Proofs are given in a shortened form, since they are variants of the standard SG versions after combining with the local Gevrey calculus in Rodino [**20**], Zanghirati [**26**], Hashimoto, Matsuzawa and Morimoto [**13**], Cappiello [**3, 4**]. As an application we obtain a result of S_θ^θ-regularity for the SG-elliptic operators P, namely all the solutions $u \in S_\theta^{\theta'}(\mathbf{R}^n)$ of $Pu = f \in S_\theta^\theta(\mathbf{R}^n)$ are of class $S_\theta^\theta(\mathbf{R}^n)$, in particular, they satisfy for $\theta > 1$ estimates of the type

$$|u(x)| \leq Ce^{-L|x|^{1/\theta}}, \quad x \in \mathbf{R}^n. \tag{0.2}$$

For H in (0.1), we intersect a number of known results, see for example the work of Agmon [**1**], the recent paper of Rabier and Stuart [**19**] and the references therein, using completely different arguments. The estimate (0.2) with $\theta = 1$ will remain unfortunately outside of our result, because of the technical difficulties coming from the analytic class.

The second part of the paper is devoted to a microlocal version of the regularity theorem above. As a preliminary step, in Section 3 we present polyhomogeneous SG-symbols according to the approach in [**10, 22**]; basic examples are operators with polynomial coefficients including (0.1) as a particular case. Finally, in Section 4, we define a particular wave front set for distributions on $S_\theta^{\theta'}(\mathbf{R}^n)$, which allows to control their behavior "at infinity" and prove the microellipticity and microregularity relations with respect to classical SG-operators defined in Section 3. The results presented here will be used in forthcoming papers to study the well posedness and the propagation of singularities for SG-hyperbolic problems in Gelfand-Shilov spaces, see Cordes [**5**], Coriasco and Rodino [**9**], Coriasco and Panarese [**8**], Coriasco and Maniccia [**7**] for the corresponding analysis in $\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n)$.

1. Gelfand-Shilov spaces. In this section we introduce the functional frame for our pseudodifferential calculus giving the basic definitions and properties of the Gelfand-Shilov spaces $S_\theta^\theta(\mathbf{R}^n)$, $\theta > 1$, and describing their relations with the Gevrey spaces. We will refer to

[**11, 12**] for proofs and details. In the following, we will denote by $\mathbf{Z}_+$ the set of all positive integers and by $\mathbf{N}$ the set $\mathbf{Z}_+ \cup \{0\}$.

Let $A, B \in \mathbf{Z}_+$ and θ be a positive real number such that $\theta > 1$.

Definition 1.1. We denote by $S^{\theta,B}_{\theta,A}(\mathbf{R}^n)$ the space of all functions u in $C^\infty(\mathbf{R}^n)$ such that

$$\sup_{\alpha,\beta\in\mathbf{N}^n} \sup_{x\in\mathbf{R}^n} A^{-|\alpha|}B^{-|\beta|}(\alpha!\beta!)^{-\theta} \left|x^\alpha \partial_x^\beta u(x)\right| < +\infty. \tag{1.1}$$

We set

$$S^\theta_\theta(\mathbf{R}^n) = \bigcup_{A,B\in\mathbf{Z}_+} S^{\theta,B}_{\theta,A}(\mathbf{R}^n).$$

For any $A, B \in \mathbf{Z}_+$, the space $S^{\theta,B}_{\theta,A}(\mathbf{R}^n)$ is a Banach space endowed with the norm given by the left-hand side of (1.1). Therefore, we can consider the space $S^\theta_\theta(\mathbf{R}^n)$ as an inductive limit of an increasing sequence of Banach spaces.

Let us give another characterization of the space $S^\theta_\theta(\mathbf{R}^n)$, providing another equivalent topology to $S^\theta_\theta(\mathbf{R}^n)$, cf. the proof of Theorem 2.2 below.

Proposition 1.2. *$S^\theta_\theta(\mathbf{R}^n)$ is the space of all functions $u \in C^\infty(\mathbf{R}^n)$ such that*

$$\sup_{\beta\in\mathbf{N}^n} \sup_{x\in\mathbf{R}^n} B^{-|\beta|}(\beta!)^{-\theta} e^{L|x|^{1/\theta}} |\partial_x^\beta u(x)| < +\infty$$

for some positive B, L.

Proposition 1.3. *The following statements hold.*

i) *$S^\theta_\theta(\mathbf{R}^n)$ is closed under the differentiation*;

ii) *If f is a function in $C^\infty(\mathbf{R}^n)$ such that, for every $\varepsilon > 0$ and for some $B > 0$*

$$|\partial_x^\alpha f(x)| \le C_\varepsilon B^{|\alpha|}(\alpha!)^\theta e^{\varepsilon|x|^{1/\theta}}$$

for all $x \in \mathbf{R}^n$, $\alpha \in \mathbf{N}^n$, then the multiplication by f is a continuous map from $S^\theta_\theta(\mathbf{R}^n)$ to $S^\theta_\theta(\mathbf{R}^n)$.

Remark 1. We have

$$G_o^\theta(\mathbf{R}^n) \subset S_\theta^\theta(\mathbf{R}^n) \subset G^\theta(\mathbf{R}^n),$$

where $G^\theta(\mathbf{R}^n)$ is the Gevrey space of all functions $u \in C^\infty(\mathbf{R}^n)$ satisfying for every compact subset K of $\mathbf{R}^n$ estimates of the form

$$\sup_{\beta\in\mathbf{N}^n} B^{-|\beta|}(\beta!)^{-\theta} \sup_{x\in K} |\partial_x^\beta u(x)| < +\infty$$

for some $B = B(K) > 0$ and $G_o^\theta(\mathbf{R}^n)$ is the space of all functions of $G^\theta(\mathbf{R}^n)$ with compact support.

We shall denote by $S_\theta^{\theta'}(\mathbf{R}^n)$ the dual space, i.e., the space of all linear continuous forms on $S_\theta^\theta(\mathbf{R}^n)$.

An equivalent characterization of the elements of $S_\theta^{\theta'}(\mathbf{R}^n)$ is given by the following proposition.

Proposition 1.4. *A linear form u on $S_\theta^\theta(\mathbf{R}^n)$ belongs to $S_\theta^{\theta'}(\mathbf{R}^n)$ if and only if for every $A, B \in \mathbf{Z}_+$, there exists $C = C(A, B) > 0$ such that*

$$|u(v)| \le C \sup_{\alpha,\beta\in\mathbf{N}^n} A^{-|\alpha|}B^{-|\beta|}(\alpha!\beta!)^{-\theta} \sup_{x\in\mathbf{R}^n} |x^\alpha\partial_x^\beta v(x)|$$

for all $v \in S_\theta^\theta(\mathbf{R}^n)$.

Remark 2. Given $u \in S_\theta^{\theta'}(\mathbf{R}^n)$, the restriction of u on $G_o^\theta(\mathbf{R}^n)$ is a Gevrey ultradistribution in $\mathcal{D}'_\theta(\mathbf{R}^n)$, topological dual of $G_o^\theta(\mathbf{R}^n)$. In this sense, we have that $S_\theta^{\theta'}(\mathbf{R}^n) \subset \mathcal{D}'_\theta(\mathbf{R}^n)$. Similarly, the space of the ultradistributions with compact support $\mathcal{E}'_\theta(\mathbf{R}^n)$ can be regarded as a subset of $S_\theta^{\theta'}(\mathbf{R}^n)$.

Theorem 1.5. *There exists an isomorphism between the space $\mathcal{L}(S_\theta^\theta(\mathbf{R}^n), S_\theta^{\theta'}(\mathbf{R}^n))$ of all linear continuous maps from $S_\theta^\theta(\mathbf{R}^n)$ to $S_\theta^{\theta'}(\mathbf{R}^n)$, and $S_\theta^{\theta'}(\mathbf{R}^{2n})$, which associates to every $T \in \mathcal{L}(S_\theta^\theta(\mathbf{R}^n), S_\theta^{\theta'}(\mathbf{R}^n))$ a distribution $K_T \in S_\theta^{\theta'}(\mathbf{R}^{2n})$ such that*

$$\langle Tu, v\rangle = \langle K_T, v \otimes u\rangle$$

for every $u, v \in S_\theta^\theta(\mathbf{R}^n)$. K_T is called the kernel of T.

Finally we give a result concerning the action of the Fourier transformation on $S_\theta^\theta(\mathbf{R}^n)$.

Proposition 1.6. *The Fourier transformation $u \to \hat{u}$ defined by*

$$\hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle x,\xi\rangle} u(x)\, dx$$

is an automorphism of $S_\theta^\theta(\mathbf{R}^n)$, and it extends to an automorphism of $S_\theta^{\theta'}(\mathbf{R}^n)$.

2. SG-calculus on Gelfand-Shilov spaces. In the following we will use the following notation:

$$\langle x\rangle = (1+|x|^2)^{1/2} \quad \text{for} \quad x \in \mathbf{R}^n$$

$$D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n} \quad \text{for all} \quad \alpha \in \mathbf{N}^n, \quad x \in \mathbf{R}^n,$$

where $D_{x_h} = -i\partial_{x_h}$, $h = 1, \dots, n$.

Let μ, ν be real numbers such that $\mu > 1$, $\nu > 1$, and let $m = (m_1, m_2) \in \mathbf{R}^2$.

Definition 2.1. For every $C > 0$, we denote by $\Gamma_{\mu\nu}^m(\mathbf{R}^{2n}; C)$ the Banach space of all functions $p(x,\xi) \in C^\infty(\mathbf{R}^{2n})$ such that

$$\sup_{\alpha,\beta \in \mathbf{N}^n} \sup_{(x,\xi)\in \mathbf{R}^{2n}} C^{-|\alpha|-|\beta|} (\alpha!)^{-\mu} (\beta!)^{-\nu} \langle \xi\rangle^{-m_1+|\alpha|} \langle x\rangle^{-m_2+|\beta|} \times \left| D_\xi^\alpha D_x^\beta p(x,\xi)\right| < +\infty$$

endowed with the norm $\|\cdot\|_C$ given by the left-hand side of (2.1).

We set

$$\Gamma_{\mu\nu}^m(\mathbf{R}^{2n}) = \varinjlim_{C\to+\infty} \Gamma_{\mu\nu}^m(\mathbf{R}^{2n}; C)$$

with the topology of inductive limit of an increasing sequence of Banach spaces.

Given a symbol $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$, we can consider the associated pseudodifferential operator defined by

$$Pu(x) = p(x,D)u(x) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\hat{u}(\xi) đ\xi, \quad u \in S_\theta^\theta(\mathbf{R}^n) \tag{2.2}$$

where $đ\xi = (2\pi)^{-n}\, d\xi$. We denote by $OPS^m_{\mu\nu}(\mathbf{R}^n)$ the space of all operators of the form (2.2) defined by a symbol $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$. We set

$$OPS_{\mu\nu}(\mathbf{R}^n) = \bigcup_{m\in\mathbf{R}^2} OPS^m_{\mu\nu}(\mathbf{R}^n).$$

This is a subclass of the SG-pseudodifferential operators in [**5, 10, 16, 21, 22**] on $\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n)$. Taking advantage of the estimates (2.1) we are able to prove continuity on $S^\theta_\theta(\mathbf{R}^n)$.

Theorem 2.2. *Let $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$, and let θ be a real number such that $\theta \geq \max\{\mu,\nu\}$. Then, the operator P defined by* (2.2) *is a linear continuous operator from $S^\theta_\theta(\mathbf{R}^n)$ to $S^\theta_\theta(\mathbf{R}^n)$ and it extends to a linear continuous map from $S^{\theta'}_\theta(\mathbf{R}^n)$ to $S^{\theta'}_\theta(\mathbf{R}^n)$.*

Proof. Let $A, B \in \mathbf{Z}_+$ and F be a bounded subset of $S^{\theta,B}_{\theta,A}(\mathbf{R}^n)$. It is sufficient to show that there exists $A_1, B_1 \in \mathbf{Z}_+$, $C > 0$, such that for every $\alpha, \beta \in \mathbf{N}^n$,

$$\sup_{x\in\mathbf{R}^n} \left|x^\alpha D^\beta_x Pu(x)\right| \leq C A_1^{|\alpha|} B_1^{|\beta|} (\alpha!\beta!)^\theta \tag{2.3}$$

for all $u \in F$, with A_1, B_1, C independent of $u \in F$. We have, for every $N \in \mathbf{Z}_+$

$$x^\alpha D^\beta_x Pu(x) = x^\alpha \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} \xi^{\beta'} D^{\beta-\beta'}_x p(x,\xi)\hat{u}(\xi) đ\xi$$
$$= x^\alpha \langle x\rangle^{-2N} \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} (1-\Delta_\xi)^N \left[\xi^{\beta'} D^{\beta-\beta'}_x p(x,\xi)\hat{u}(\xi)\right] đ\xi.$$

By Propositions 1.2 and 1.6, we easily obtain the estimate:

$$\left|x^\alpha D^\beta_x Pu(x)\right| \leq C_0 B_0^{|\beta|+2N} (2N!)^\theta \langle x\rangle^{|\alpha|+m_2-2N} \cdot \sum_{\beta'\leq\beta} \binom{\beta}{\beta'} (\beta'!)^\theta (\beta-\beta'!)^\nu \int_{\mathbf{R}^n} \langle\xi\rangle^{m_1} e^{-a|\xi|^{1/\theta}}\, đ\xi$$

for some B_0, C_0, $a > 0$ independent of $u \in F$. Hence, choosing $N = \min\{n \in \mathbf{Z}_+ : 2N \geq |\alpha| + m_2\}$, we obtain that there exist

$A_1, B_1, C > 0$ such that (2.3) holds for all $u \in F$. This concludes the first part of the proof. To prove the second part, we observe that, for $u, v \in S_\theta^\theta(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} Pu(x)v(x)dx = \int_{\mathbf{R}^n} \hat{u}(\xi)p_v(\xi)\, d\xi$$

where

$$p_v(x,\xi) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)v(x) đx.$$

By the same argument of the first part of the proof, the map $v \to p_v$ is linear and continuous from $S_\theta^\theta(\mathbf{R}^n)$ to itself. Then, we can define, for $u \in S_\theta^{\theta'}(\mathbf{R}^n)$

$$Pu(v) = \hat{u}(p_v), \quad v \in S_\theta^\theta(\mathbf{R}^n).$$

This map is linear and continuous from $S_\theta^{\theta'}(\mathbf{R}^n)$ to itself and it extends P. □

By Theorems 1.5 and 2.2, we can associate to P a kernel $K_P \in S_\theta^{\theta\prime}(\mathbf{R}^{2n})$ given as standard by

$$K_P(x,y) = \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)\, đ\xi \tag{2.4}$$

where (2.4) has the meaning of an oscillatory integral. We can prove the following result of regularity for the kernel (2.4).

Theorem 2.3. *Let $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$. For $k \in (0,1)$, define*

$$\Omega_k = \{(x,y) \in \mathbf{R}^{2n} : |x-y| > k\langle x\rangle\}.$$

Then the kernel K_P defined by (2.4) is in $C^\infty(\Omega_k)$, and there exist positive constants C, a depending on k such that

$$\left|D_x^\beta D_y^\gamma K_P(x,y)\right| \le C^{|\beta|+|\gamma|+1}(\beta!\gamma!)^\theta \exp\left[-a(|x|^{1/\theta} + |y|^{1/\theta})\right] \tag{2.5}$$

for every $(x,y) \in \overline{\Omega_k}$ and for every $\beta, \gamma \in \mathbf{N}^n$.

Lemma 2.4. *For any given* $R > 0$*, we may find a sequence* $\psi_N(\xi) \in C_0^\infty(\mathbf{R}^n)$*,* $N = 0, 1, 2, \dots$ *such that*

$$\sum_{N=0}^{\infty} \psi_N = 1 \quad \text{in} \quad \mathbf{R}^n,$$
$$\operatorname{supp} \psi_0 \subset \{\xi : \langle \xi \rangle \le 3R\}$$
$$\operatorname{supp} \psi_N \subset \{\xi : 2RN^\theta \le \langle \xi \rangle \le 3R(N+1)^\theta\}, \quad N = 1, 2, \dots,$$

and

$$\left| D_\xi^\alpha \psi_N(\xi) \right| \le C^{|\alpha|+1} (\alpha!)^\theta \left[R \sup(N^\theta, 1) \right]^{-|\alpha|}$$

for every $\alpha \in \mathbf{N}^n$ *and for every* $\xi \in \mathbf{R}^n$*.*

Proof. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ such that $\phi(\xi) = 1$ if $\langle \xi \rangle \le 2$, $\phi(\xi) = 0$ if $\langle \xi \rangle \ge 3$. Assume further $\phi \in G^\theta(\mathbf{R}^n)$, i.e.,

$$\left| D_\xi^\alpha \phi(\xi) \right| \le C^{|\alpha|+1} (\alpha!)^\theta$$

for all $\alpha \in \mathbf{N}^n$ and for all $\xi \in \mathbf{R}^n$. We may then define

$$\psi_0(\xi) = \phi\left(\frac{\xi}{R}\right)$$
$$\psi_N(\xi) = \phi\left(\frac{\xi}{R(N+1)^\theta}\right) - \phi\left(\frac{\xi}{RN^\theta}\right), \quad N \ge 1. \qquad \square$$

Proof of Theorem 2.3. We can assume without loss of generality that p is in $\Gamma^0_{\mu\nu}(\mathbf{R}^{2n})$. Let us consider a sequence $\{\psi_N\}_{N \ge 0}$ as in Lemma 2.4. We have, for $u, v \in S_\theta^\theta(\mathbf{R}^n)$,

$$\langle K_P, v \otimes u \rangle = \sum_{N=0}^{\infty} \langle K_N, v \otimes u \rangle$$

with

$$K_N(x, y) = \int_{\mathbf{R}^n} e^{i\langle x-y, \xi \rangle} p(x, \xi) \psi_N(\xi) \, đ\xi$$

so we may decompose

$$K_P = \sum_{N=0}^{\infty} K_N.$$

Let $k \in (0,1)$ and $(x,y) \in \overline{\Omega}_k$. Let $h \in \{1,\dots,n\}$ such that $|x_h - y_h| \geq k/n\langle x\rangle$. Then, for every $\alpha, \gamma \in \mathbf{N}^n$,

$$\begin{aligned} D_x^\alpha D_y^\gamma K_N(x,y) &= (-1)^{|\gamma|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} \xi^{\beta+\gamma} \psi_N(\xi) \\ &\quad \times D_x^{\alpha-\beta} p(x,\xi)\, đ\xi \\ &= (-1)^{|\gamma|+N} \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (x_h - y_h)^{-N} \\ &\quad \times \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} D_{\xi_h}^N \left[\xi^{\beta+\gamma}\psi_N(\xi) D_x^{\alpha-\beta} p(x,\xi)\right]\, d\xi. \end{aligned}$$

Now, given $\zeta > 0$, we consider the operator

$$L = \frac{1}{m_{2\theta,\zeta}(x-y)} \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} (1-\Delta_\xi)^j$$

where

$$m_{2\theta,\zeta}(x-y) = \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} \langle x-y\rangle^{2j}.$$

In view of the fact that $Le^{i\langle x-y,\xi\rangle} = e^{i\langle x-y,\xi\rangle}$, we can integrate by parts obtaining that

$$\begin{aligned} D_x^\alpha D_y^\gamma K_N(x,y) &= (-1)^{|\gamma|+N} \frac{(x_h-y_h)^{-N}}{m_{2\theta,\zeta}(x-y)} \\ &\quad \cdot \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \sum_{j=0}^{\infty} \frac{\zeta^j}{(j!)^{2\theta}} \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} \lambda_{hjN\alpha\beta\gamma}(x,\xi)\, đ\xi \end{aligned}$$

with

$$\lambda_{hjN\alpha\beta\gamma}(x,\xi) = (1-\Delta_\xi)^j D_{\xi_h}^N \left[\xi^{\beta+\gamma}\psi_N(\xi) D_x^{\alpha-\beta} p(x,\xi)\right]. \tag{2.6}$$

Let e_h be the hth vector of the canonical basis of $\mathbf{R}^n$ and $\beta_h = \langle \beta, e_h\rangle, \gamma_h = \langle \gamma, e_h\rangle$. Developing in the right-hand side of (2.6) we obtain that

$$\lambda_{hjN\alpha\beta\gamma}(x,\xi) = \sum_{\substack{N_1+N_2+N_3=N\\ N_1\le \beta_h+\gamma_h}} (-i)^{N_1}\frac{N!}{N_1!N_2!N_3!}\cdot\frac{(\beta_h+\gamma_h)!}{(\beta_h+\gamma_h-N_1)!} \cdot (1-\Delta_\xi)^j\left[\xi^{\beta+\gamma-N_1e_h}D_{\xi_h}^{N_2}\psi_N(\xi)D_{\xi_h}^{N_3}D_x^{\alpha-\beta}p(x,\xi)\right].$$

Hence

$$|\lambda_{hjN\alpha\beta\gamma}(x,\xi)| \le \sum_{\substack{N_1+N_2+N_3=N\\ N_1\le \beta_h+\gamma_h}} \frac{N!}{N_1!N_2!N_3!}\cdot\frac{(\beta_h+\gamma_h)!}{(\beta_h+\gamma_h-N_1)!}C_1^{|\alpha-\beta|+N_2+N_3+1} \cdot (N_2!)^\theta(N_3!)^\mu\left[(\alpha-\beta)!\right]^\nu C_2^j(j!)^{2\theta}\left(\frac{1}{RN^\theta}\right)^{N_2}\langle\xi\rangle^{|\beta|+|\gamma|-N_1-N_3}.$$

We observe that on the support of $\psi_N, 2RN^\theta \le \langle\xi\rangle \le 3R(N+1)^\theta$. Thus, from standard factorial inequalities, since $\theta \ge \max\{\mu,\nu\}$, it follows that

$$|\lambda_{hjN\alpha\beta\gamma}(x,\xi)| \le C_1^{|\alpha|+|\gamma|+1}(\alpha!\gamma!)^\theta C_2^j(j!)^{2\theta}\left(\frac{C_3}{R}\right)^N$$

with C_3 independent of R. Moreover, by Proposition 2.4 in [**14**], we have that

$$|m_{2\theta,\zeta}(x-y)| \ge C\exp[c'\zeta^{1/(2\theta)}|x-y|^{1/\theta}].$$

From these estimates, choosing $\zeta < C_2^{-1}$, we deduce that

$$\left|D_x^\alpha D_y^\gamma K_N(x,y)\right| \le C_1^{|\alpha|+|\gamma|+1}(\alpha!\gamma!)^\theta\left(\frac{C_4}{R}\right)^N\exp[-c'\zeta^{1/(2\theta)}|x-y|^{1/\theta}]$$

with $C_4 = C_4(k)$ independent of R. Choosing R sufficiently large and observing that $|x-y| \ge c''(\langle x\rangle+\langle y\rangle)$ on $\overline{\Omega}_k$, we obtain the estimates (2.5). □

Remark 3. From Theorem 2.3, in view of Remark 2, it is possible to deduce the pseudolocal property for the operator (2.2). Namely, we have

$$\theta - \text{sing supp}\, Pu \subset \theta - \text{sing supp}\, u$$

where $\theta - \text{sing supp}$ denotes the standard G^θ singular support, cf. [**20**]. We address the reader to Section 4 for more precise results in terms of wave front sets concerning the subclass of the polyhomogeneous operators, cf. Section 3.

Definition 2.5. A linear continuous operator from $S_\theta^\theta(\mathbf{R}^n)$ to $S_\theta^\theta(\mathbf{R}^n)$ is said to be θ-regularizing if it extends to a linear continuous map from $S_\theta^{\theta'}(\mathbf{R}^n)$ to $S_\theta^\theta(\mathbf{R}^n)$.

We now give a symbolic calculus for our symbols. We set, for $t \geq 0$,

$$Q_t = \{(x,\xi) \in \mathbf{R}^{2n} : \langle x \rangle < t,\ \langle \xi \rangle < t\}$$
$$Q_t^e = \mathbf{R}^{2n} \setminus Q_t.$$

Definition 2.6. We denote by $FS_{\mu\nu}^m(\mathbf{R}^{2n})$ the space of all formal sums $\sum_{j\geq 0} p_j(x,\xi)$ such that $p_j(x,\xi) \in C^\infty(\mathbf{R}^{2n})$ for all $j \geq 0$, and there exist $B, C > 0$ such that

(2.7)
$$\sup_{j\geq 0} \sup_{\alpha,\beta\in\mathbf{N}^n} \sup_{(x,\xi)\in Q^e_{Bj^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2j}(\alpha!)^{-\mu}(\beta!)^{-\nu}(j!)^{-\mu-\nu+1}$$
$$\cdot \langle \xi \rangle^{-m_1+|\alpha|+j} \langle x \rangle^{-m_2+|\beta|+j} \left| D_\xi^\alpha D_x^\beta p_j(x,\xi) \right| < +\infty.$$

We observe that every symbol $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$ can be identified with an element of $FS_{\mu\nu}^m(\mathbf{R}^{2n})$ by setting $p_0 = p$ and $p_j = 0$ for all $j \geq 1$.

Definition 2.7. We say that two sums $\sum_{j\geq 0} p_j$, $\sum_{j\geq 0} p'_j$ from $FS_{\mu\nu}^m(\mathbf{R}^{2n})$ are equivalent, we write

$$\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} p'_j,$$

if there exist constants $B, C > 0$ such that

$$\sup_{N\in\mathbf{Z}_+}\sup_{\alpha,\beta\in\mathbf{N}^n}\sup_{(x,\xi)\in Q^e_{BN^{\mu+\nu-1}}} C^{-|\alpha|-|\beta|-2N}(\alpha!)^{-\mu}(\beta!)^{-\nu}(N!)^{-\mu-\nu+1}$$
$$\cdot\langle\xi\rangle^{-m_1+|\alpha|+N}\langle x\rangle^{-m_2+|\beta|+N}\left|D^\alpha_\xi D^\beta_x\sum_{j<N}(p_j-p'_j)\right| < +\infty.$$

Theorem 2.8. *Given $\sum_{j\geq 0} p_j \in FS^m_{\mu\nu}(\mathbf{R}^{2n})$, there exists a symbol $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ such that*

$$p \sim \sum_{j\geq 0} p_j \quad \text{in} \quad FS^m_{\mu\nu}(\mathbf{R}^{2n}).$$

Proof. Let $\varphi \in C^\infty(\mathbf{R}^{2n})$, $0 \leq \varphi \leq 1$ such that $\varphi(x,\xi) = 0$ if $(x,\xi) \in Q_2$, $\varphi(x,\xi) = 1$ if $(x,\xi) \in Q^e_3$ and

$$\sup_{(x,\xi)\in\mathbf{R}^{2n}}\left|D^\gamma_\xi D^\delta_x\varphi(x,\xi)\right| \leq C^{|\gamma|+|\delta|+1}(\gamma!)^\mu(\delta!)^\nu. \tag{2.8}$$

We define, for $R > 0$,

$$\varphi_0(x,\xi) \equiv 1 \quad \text{on} \quad \mathbf{R}^{2n}$$
$$\varphi_j(x,\xi) = \varphi\left(\frac{x}{Rj^{\mu+\nu-1}}, \frac{\xi}{Rj^{\mu+\nu-1}}\right), \quad j \geq 1.$$

We want to prove that if R is sufficiently large, then

$$p(x,\xi) = \sum_{j\geq 0}\varphi_j(x,\xi)p_j(x,\xi)$$

is in $\Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ and $p \sim \sum_{j\geq 0} p_j$ in $FS^m_{\mu\nu}(\mathbf{R}^{2n})$.

Consider

$$D^\alpha_\xi D^\beta_x p(x,\xi) = \sum_{j\geq 0}\sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}}\binom{\alpha}{\gamma}\binom{\beta}{\delta}D^{\alpha-\gamma}_\xi D^{\beta-\delta}_x p_j(x,\xi)D^\gamma_\xi D^\delta_x\varphi_j(x,\xi).$$

If $R \geq B$, where B is the same constant of Definition 2.6, we can apply the estimates (2.7) and obtain

$$\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \leq C^{|\alpha|+|\beta|+1}\alpha!\beta!\langle\xi\rangle^{m_1-|\alpha|}\langle x\rangle^{m_2-|\beta|}\sum_{j\geq 0} H_{j\alpha\beta}(x,\xi)$$

where

$$H_{j\alpha\beta}(x,\xi) = \sum_{\substack{\gamma\leq\alpha\\ \delta\leq\beta}} \frac{(\alpha-\gamma)!^{\mu-1}(\beta-\delta)!^{\nu-1}}{\gamma!\delta!}\; C^{2j-|\gamma|-|\delta|}(j!)^{\mu+\nu-1} \cdot \langle\xi\rangle^{|\gamma|-j}\langle x\rangle^{|\delta|-j}\left|D_\xi^\gamma D_x^\delta \varphi_j(x,\xi)\right|.$$

The condition (2.8) implies that

$$H_{j\alpha\beta}(x,\xi) \leq C^{|\alpha|+|\beta|+1}(\alpha!)^{\mu-1}(\beta!)^{\nu-1}\left(\frac{C_1}{R}\right)^j,$$

with C_1 independent of R. Choosing R sufficiently large, we obtain that p is in $\Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$. It remains to prove that $p \sim \sum_{j\geq 0} p_j$ in $FS_{\mu\nu}^m(\mathbf{R}^{2n})$. Let N be a positive integer. We observe that, for $(x,\xi) \in Q^e_{3RN^{\mu+\nu-1}}$,

$$p(x,\xi) - \sum_{j<N} p_j(x,\xi) = \sum_{j\geq N} p_j(x,\xi)\varphi_j(x,\xi)$$

which we can estimate by arguing as above. □

Proposition 2.9. *Let $p \in \Gamma_{\mu\nu}^0(\mathbf{R}^{2n})$, and let $\theta \geq \mu+\nu-1$. If $p \sim 0$ in $FS_{\mu\nu}^0(\mathbf{R}^{2n})$, then the operator P is θ-regularizing.*

To prove this proposition, we need the following preliminary result.

Lemma 2.10. *Let $M, r, \varrho, \overline{B}$ be positive numbers, $\varrho \geq 1$. We define*

$$h(\lambda) = \inf_{0\leq N\leq \overline{B}\lambda^{1/\varrho}} \frac{M^{rN}(N!)^r}{\lambda^{rN/\varrho}}, \quad \lambda \in \mathbf{R}^+.$$

Then there exist positive constants C, τ such that

$$h(\lambda) \leq Ce^{-\tau\lambda^{1/\varrho}}, \quad \lambda \in \mathbf{R}^+.$$

Proof. See Lemma 3.2.4 in [**20**] for the proof. □

Proof of Proposition 2.9. It is sufficient to prove that the kernel

$$K_P(x,y) = \int_{\mathbf{R}^n} e^{i\langle x-y,\xi\rangle} p(x,\xi)\, đ\xi \tag{2.9}$$

is in $S_\theta^\theta(\mathbf{R}^{2n})$. This will easily imply that P is θ-regularizing.

If $p \sim 0$, by Definition 2.7, there exist positive constants B_1, C_1 such that, for every $(x,\xi) \in \mathbf{R}^{2n}$,

$$\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \le C_1^{|\alpha|+|\beta|+1}(\alpha!)^\mu(\beta!)^\nu \langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|} \cdot \inf_{0\le N\le B_1(\langle\xi\rangle\langle x\rangle)^{1/\mu+\nu-1}} \frac{C^{2N}(N!)^{\mu+\nu-1}}{\langle\xi\rangle^N\langle x\rangle^N}.$$

Applying Lemma 2.10, we obtain

$$\left|D_\xi^\alpha D_x^\beta p(x,\xi)\right| \le C_2^{|\alpha|+|\beta|+1}(\alpha!\beta!)^\theta \exp[-\sigma(|x|^{1/\theta}+|\xi|^{1/\theta})] \tag{2.10}$$

for some positive C_2, σ. Therefore, $p \in S_\theta^\theta(\mathbf{R}^{2n})$. Applying (2.10) in (2.9), we easily obtain that also $K_P \in S_\theta^\theta(\mathbf{R}^{2n})$. □

Proposition 2.11. *Let $P = p(x,D) \in OPS_{\mu\nu}^m(\mathbf{R}^n)$, and let ${}^tP$ be its transpose defined by*

$$\langle {}^tPu, v\rangle = \langle u, Pv\rangle, \quad u \in S_\theta^{\theta'}(\mathbf{R}^n), \quad v \in S_\theta^\theta(\mathbf{R}^n). \tag{2.11}$$

Then, ${}^tP = Q+R$, where R is a θ-regularizing operator for $\theta \ge \mu+\nu-1$ and $Q = q(x,D)$ is in $OPS_{\mu\nu}^m(\mathbf{R}^n)$ with

$$q(x,\xi) \sim \sum_{j\ge 0}\sum_{|\alpha|=j}(\alpha!)^{-1}\partial_\xi^\alpha D_x^\alpha p(x,-\xi)$$

in $FS_{\mu\nu}^m(\mathbf{R}^{2n})$.

Theorem 2.12. *Let $P = p(x,D) \in OPS_{\mu\nu}^m(\mathbf{R}^n)$, $Q = q(x,D) \in OPS_{\mu\nu}^{m'}(\mathbf{R}^n)$. Then $PQ = T+R$ where R is θ-regularizing for $\theta \ge \mu+\nu-1$ and $T = t(x,D)$ in $OPS_{\mu\nu}^{m+m'}(\mathbf{R}^n)$ with*

$$t(x,\xi) \sim \sum_{j\ge 0}\sum_{|\alpha|=j}(\alpha!)^{-1}\partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)$$

in $FS^{m+m'}_{\mu\nu}(\mathbf{R}^{2n})$.

To prove Proposition 2.11 and Theorem 2.12, it is convenient to introduce more general classes of symbols.

Let μ, ν be real numbers such that $\mu > 1$, $\nu > 1$, and let $m = (m_1, m_2, m_3) \in \mathbf{R}^3$.

Definition 2.13. For $C > 0$, we shall denote by $\Pi^m_{\mu\nu}(\mathbf{R}^{3n}; C)$ the Banach space of all functions $a(x, y, \xi) \in C^\infty(\mathbf{R}^{3n})$ such that

$$\sup_{\alpha,\beta,\gamma\in\mathbf{N}^n} \sup_{(x,y,\xi)\in\mathbf{R}^{3n}} C^{-|\alpha|-|\beta|-|\gamma|}(\alpha!)^{-\mu}(\beta!\gamma!)^{-\nu} \cdot \langle\xi\rangle^{-m_1+|\alpha|}\langle x\rangle^{-m_2+|\beta|}\langle y\rangle^{-m_3+|\gamma|}\left|D^\alpha_\xi D^\beta_x D^\gamma_y a(x, y, \xi)\right| < +\infty.$$

We set

$$\Pi^m_{\mu\nu}(\mathbf{R}^{3n}) = \varinjlim_{C\to+\infty} \Pi^m_{\mu\nu}(\mathbf{R}^{3n}; C).$$

It is immediate to verify the following relations:

i) if $a(x, y, \xi) \in \Pi^m_{\mu\nu}(\mathbf{R}^{3n})$, then the function $(x, \xi) \to a(x, x, \xi)$ belongs to $\Gamma^{\bar{m}}_{\mu\nu}(\mathbf{R}^{2n})$, where $\bar{m} = (m_1, m_2 + m_3)$.

ii) if $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ for some $m = (m_1, m_2) \in \mathbf{R}^2$, then $p(x, \xi) \in \Pi^{(m_1,m_2,0)}_{\mu\nu}(\mathbf{R}^{3n})$ and $p(y, \xi) \in \Pi^{(m_1,0,m_2)}_{\mu\nu}(\mathbf{R}^{3n})$.

Given $a \in \Pi^m_{\mu\nu}(\mathbf{R}^{3n})$, we can associate to a a pseudodifferential operator defined by

$$Au(x) = \int_{\mathbf{R}^{2n}} e^{i\langle x-y,\xi\rangle} a(x, y, \xi)u(y)\, dy\, đ\xi, \quad u \in S^\theta_\theta(\mathbf{R}^n) \tag{2.12}$$

with the standard meaning of oscillatory integral.

Theorem 2.2 and Theorem 2.3 hold also for operators (2.12).

In order to prove Proposition 2.11 and Theorem 2.12, we give the following result. The proof follows the same line of the proof of Theorem 3.9 in [**3**] and we will omit it for the sake of brevity.

Theorem 2.14. *Let A be an operator defined by an amplitude $a \in \Pi^m_{\mu\nu}(\mathbf{R}^{3n})$, $m = (m_1, m_2, m_3) \in \mathbf{R}^3$. Then we may write*

$A = P + R$, where R is a θ-regularizing operator for $\theta \geq \mu + \nu - 1$ and $P = p(x, D) \in OPS^{\bar{m}}_{\mu\nu}(\mathbf{R}^n)$, $\bar{m} = (m_1, m_2 + m_3)$ with $p \sim \sum_{j \geq 0} p_j$, where

$$p_j(x, \xi) = \sum_{|\alpha| = j} (\alpha!)^{-1} \partial^\alpha_\xi D^\alpha_y a(x, y, \xi)_{|y=x} \, . \tag{2.13}$$

Proof of Proposition 2.11. By (2.11), ${}^tP$ is defined by

$${}^tPu(x) = \int_{\mathbf{R}^{2n}} e^{i\langle x-y,\xi\rangle} p(y, -\xi) u(y) \, dy \, đ\xi, \quad u \in S^\theta_\theta(\mathbf{R}^n).$$

Thus, ${}^tP$ is an operator of the form (2.12) with amplitude $p(y, -\xi)$. By Theorem 2.14, ${}^tP = Q + R$ where R is θ-regularizing and $Q = q(x, D) \in OPS^m_{\mu\nu}(\mathbf{R}^n)$, with

$$q(x, \xi) \sim \sum_{j \geq 0} \sum_{|\alpha| = j} (\alpha!)^{-1} \partial^\alpha_\xi D^\alpha_x p(x, -\xi). \qquad \square$$

Proof of Theorem 2.12. We can write $Q = {}^t({}^tQ)$. Then, by Theorem 2.14 and Proposition 2.11, $Q = Q_1 + R_1$, where R_1 is θ-regularizing and

$$Q_1 u(x) = \int_{\mathbf{R}^{2n}} e^{i\langle x-y,\xi\rangle} q_1(y, \xi) u(y) \, dy \, đ\xi \tag{2.14}$$

with $q_1(y, \xi) \in \Gamma^{m'}_{\mu\nu}(\mathbf{R}^{2n})$, $q_1(y, \xi) \sim \sum_\alpha (\alpha!)^{-1} \partial^\alpha_\xi D^\alpha_y q(y, -\xi)$. From (2.14) it follows that

$$\widehat{Q_1 u}(\xi) = \int_{\mathbf{R}^n} e^{-i\langle y,\xi\rangle} q_1(y, \xi) u(y) \, dy, \quad u \in S^\theta_\theta(\mathbf{R}^n)$$

from which we deduce that

$$PQu(x) = \int_{\mathbf{R}^{2n}} e^{i\langle x-y,\xi\rangle} p(x, \xi) q_1(y, \xi) u(y) \, dy \, đ\xi + PR_1 u(x).$$

We observe that $p(x,\xi)q_1(y,\xi) \in \Pi_{\mu\nu}^{(m_1+m_1',m_2,m_2')}(\mathbf{R}^{3n})$, then we may apply Theorem 2.14 and obtain that

$$PQu(x) = Tu(x) + Ru(x)$$

where R is θ-regularizing and $T = t(x,D) \in OPS_{\mu\nu}^{m+m'}(\mathbf{R}^n)$ with

$$t(x,\xi) \sim \sum_{j\geq 0} \sum_{|\alpha|=j} (\alpha!)^{-1} \partial_\xi^\alpha p(x,\xi) D_x^\alpha q(x,\xi)$$

in $FS_{\mu\nu}^{m+m'}(\mathbf{R}^{2n})$. □

Remark 4. In Theorem 2.12, if $p \sim \sum_{j\geq 0} p_j$ in $FS_{\mu\nu}^m(\mathbf{R}^{2n})$ and $q \sim \sum_{j\geq 0} q_j$ in $FS_{\mu\nu}^{m'}(\mathbf{R}^{2n})$, then

$$t(x,\xi) \sim \sum_{j\geq 0} \sum_{|\alpha|+h+k=j} (\alpha!)^{-1} \partial_\xi^\alpha p_h(x,\xi) D_x^\alpha q_k(x,\xi) \quad \text{in} \quad FS_{\mu\nu}^{m+m'}(\mathbf{R}^{2n}).$$

To conclude this section, we introduce a notion of ellipticity for the elements of $OPS_{\mu\nu}(\mathbf{R}^n)$. It coincides with the definition of SG ellipticity in [**5, 10, 16, 21, 22**].

Definition 2.15. A symbol $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$ is said to be elliptic if there exist $B, C > 0$ such that

$$|p(x,\xi)| \geq C\langle\xi\rangle^{m_1}\langle x\rangle^{m_2} \quad \text{for all} \quad (x,\xi) \in Q_B^e.$$

Theorem 2.16. *If $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$ is elliptic, then there exist $E_1, E_2 \in OPS_{\mu\nu}^{-m}(\mathbf{R}^n)$ such that $E_1P = I + R_1$, $PE_2 = I + R_2$, where R_1, R_2 are θ-regularizing operators, for $\theta \geq \mu+\nu-1$.*

Proof. Let $e_0^1(x,\xi)$ be fixed such that

$$e_0^1(x,\xi) = p(x,\xi)^{-1} \quad \text{for all} \quad (x,\xi) \in Q_B^e$$

and define, by induction, for $j \geq 1$

$$e_j^1(x,\xi) = -e_0^1(x,\xi) \sum_{0<|\alpha|\leq j} (\alpha!)^{-1} \partial_\xi^\alpha e_{j-|\alpha|}^1(x,\xi) D_x^\alpha p(x,\xi).$$

It is easy to verify that $\sum_{j\geq 0} e_j^1(x,\xi) \in FS_{\mu\nu}^{-m}(\mathbf{R}^{2n})$. Applying Theorem 2.8, we can find $e^1 \in \Gamma_{\mu\nu}^{-m}(\mathbf{R}^{2n})$ such that $e^1 \sim \sum_{j\geq 0} e_j^1$. Denote by E_1 the operator with symbol e^1. By construction, Theorem 2.12 implies that $E_1 P - I$ is a θ-regularizing operator. The construction of e^2 is analogous. Proposition 2.9 gives the conclusion. □

As an immediate consequence of Theorem 2.16, we obtain the following result of global regularity.

Corollary 2.17. *Let $p \in \Gamma_{\mu\nu}^m(\mathbf{R}^{2n})$ be elliptic, and let $f \in S_\theta^\theta(\mathbf{R}^n)$ for some $\theta \geq \mu + \nu - 1$. Then, if $u \in S_\theta^{\theta'}(\mathbf{R}^n)$ is a solution of the equation*

$$Pu = f,$$

then $u \in S_\theta^\theta(\mathbf{R}^n)$.

Example. Consider the operator in (0.1):

$$H = (1 + |x|^{2k})(-\Delta + 1) + L_1(x, D)$$

where $k \geq 1$ and $L_1(x, D)$ is a first order operator with polynomial coefficients of degree $2k - 1$. H is elliptic and its symbol is in $\Gamma_{\mu\nu}^{(2,2k)}(\mathbf{R}^{2n})$ for every μ, ν such that $\mu > 1$, $\nu > 1$. By Corollary 2.17, if u is a solution of the equation $Pu = f \in S_\theta^\theta(\mathbf{R}^n)$, $\theta > 1$, and $u \in S_\theta^{\theta'}(\mathbf{R}^n)$, then $u \in S_\theta^\theta(\mathbf{R}^n)$.

Our result is not sharp for the solutions of the homogeneous equation $Hu = 0$. Namely, beside observing the well-known local analyticity of the solutions, we may test the behavior at infinity in the one-dimensional case and for $k = 1$ as follows.

Example. Consider

$$Hy = -(1 + x^2)y'' + x^2 y - 2xy', \quad x \in \mathbf{R}. \tag{2.15}$$

The operator H is L^2 self-adjoint and then there exists a sequence $\lambda_j \in \mathbf{R}$, $j = 1, 2, \ldots$, such that $Hy_j = \lambda_j y_j$ for some nontrivial $y_j \in \mathcal{S}(\mathbf{R})$, cf. [**15**]. From Corollary 2.17, we obtain $y_j \in S_\theta^\theta(\mathbf{R})$

for every $\theta > 1$. On the other hand, by the theory of asymptotic integration, see Tricomi [**24**] and Wasow [**25**], we have that

$$y_j(x) = Cx^{-1}e^{-|x|} + O(x^{-2}e^{-|x|}) \quad \text{when} \quad |x| \to +\infty,$$

giving more precise information about the behavior at infinity.

We can generalize Definition 2.15 giving the notion of ellipticity of a symbol with respect to another one, which will be applied in Section 4.

Definition 2.18. Let $m, m' \in \mathbf{R}^2$, and let $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$, $q \in \Gamma^{m'}_{\mu\nu}(\mathbf{R}^{2n})$. We say that p is elliptic with respect to q if there exist $B, C > 0$ such that

$$|p(x,\xi)| \geq C\langle\xi\rangle^{m_1}\langle x\rangle^{m_2} \quad \text{for all} \quad (x,\xi) \in Q^e_B \cap \operatorname{supp}(q).$$

We observe that a symbol p is elliptic according to Definition 2.15 if it is elliptic with respect to $q(x,\xi) \equiv 1$.

Proposition 2.19. *Let $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$, $q \in \Gamma^{m'}_{\mu\nu}(\mathbf{R}^{2n})$. If p is elliptic with respect to q, then there exist $E_1, E_2 \in OPS^{m'-m}_{\mu\nu}(\mathbf{R}^{2n})$ such that*

$$E_1 P = Q + R_1, \quad PE_2 = Q + R_2$$

where R_1, R_2 are θ-regularizing operators.

Proof. The proof follows the same lines of the one of Theorem 2.16, by defining

$$e^1_0(x,\xi) = \frac{q(x,\xi)}{p(x,\xi)} \quad \text{for all} \quad (x,\xi) \in Q^e_B,$$
$$e^1_j(x,\xi) = -\frac{1}{p(x,\xi)} \sum_{0<|\alpha|\leq j} (\alpha!)^{-1}\partial^\alpha_\xi e^1_{j-|\alpha|}(x,\xi) D^\alpha_x p(x,\xi), \quad j \geq 1. \qquad \square$$

3. Polyhomogeneous symbols. The examples at the end of Section 2 suggest the study of a subspace of $\Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$; namely, we introduce classical polyhomogeneous SG-symbols. We will refer to the approach of Egorov and Schulze [**10, 23**] in the $\mathcal{S}-\mathcal{S}'$-frame and we will define three principal subclasses of $\Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$, whose elements are respectively polyhomogeneous in x, in ξ and in both x,ξ. Before giving precise definitions for these spaces, we need to introduce in our context a notion of asymptotic expansion with respect to x and ξ separately.

Let μ,ν be real numbers such that $\mu>1$, $\nu>1$, and let $m=(m_1,m_2)$ be a vector of $\mathbf{R}^2$.

Definition 3.1. We denote by $FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ the space of all formal sums $\sum_{j\geq 0} p_j(x,\xi)$ such that $p_j\in C^\infty(\mathbf{R}^{2n})$ for all $j\geq 0$, and there exist $B,C>0$ such that

$$\sup_{j\geq 0}\ \sup_{\alpha,\beta\in\mathbf{N}^n}\ \sup_{\substack{\langle\xi\rangle\geq Bj^{\mu+\nu-1}\\ x\in\mathbf{R}^n}} C^{-|\alpha|-|\beta|-j}(\alpha!)^{-\mu}(\beta!)^{-\nu}(j!)^{-\mu-\nu+1} \cdot\langle\xi\rangle^{-m_1+|\alpha|+j}\langle x\rangle^{-m_2+|\beta|}\left|D^\alpha_\xi D^\beta_x p_j(x,\xi)\right|<+\infty. \tag{3.1}$$

As in Definition 2.7, we can define an equivalence relation among the elements of $FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$.

Definition 3.2. Two sums $\sum_{j\geq 0} p_j$, $\sum_{j\geq 0} p'_j\in FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ are said to be equivalent, we write

$$\sum_{j\geq 0} p_j \sim_\xi \sum_{j\geq 0} p'_j,$$

if there exist $B,C>0$ such that

$$\sup_{N\in\mathbf{Z}_+}\ \sup_{\alpha,\beta\in\mathbf{N}^n}\ \sup_{\substack{\langle\xi\rangle\geq BN^{\mu+\nu-1}\\ x\in\mathbf{R}^n}} C^{-|\alpha|-|\beta|-N}(\alpha!)^{-\mu}(\beta!)^{-\nu}(N!)^{-\mu-\nu+1} \cdot\langle\xi\rangle^{-m_1+|\alpha|+N}\langle x\rangle^{-m_2+|\beta|}\left|D^\alpha_\xi D^\beta_x\sum_{j<N}(p_j-p'_j)\right|<+\infty.$$

In an analogous way, we can define the space $FS^m_{\mu\nu,x}(\mathbf{R}^{2n})$ and the corresponding relation $\sim_x$.

Remark 5. We observe that

$$FS^m_{\mu\nu}(\mathbf{R}^{2n}) \subset FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n}) \cap FS^m_{\mu\nu,x}(\mathbf{R}^{2n}).$$

Furthermore, if $\sum_{j\geq 0} p_j \sim \sum_{j\geq 0} p'_j$ in $FS^m_{\mu\nu}(\mathbf{R}^{2n})$, then

$$\sum_{j\geq 0} p_j \sim_\xi \sum_{j\geq 0} p'_j \quad \text{and} \quad \sum_{j\geq 0} p_j \sim_x \sum_{j\geq 0} p'_j.$$

With the same arguments of Theorem 2.8, it is easy to prove the following result.

Proposition 3.3. *Given* $\sum_{j\geq 0} p_j \in FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$, $\sum_{j\geq 0} q_j \in FS^m_{\mu\nu,x}(\mathbf{R}^{2n})$, *then there exist* $p, q \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ *such that*

$$p \sim_\xi \sum_{j\geq 0} p_j \quad \textit{in} \quad FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n}),$$
$$q \sim_x \sum_{j\geq 0} q_j \quad \textit{in} \quad FS^m_{\mu\nu,x}(\mathbf{R}^{2n})$$

We can define the following classes of homogeneous symbols.

Definition 3.4. We denote by $\Gamma^{[m_1],m_2}_{\mu\nu}(\mathbf{R}^{2n})$ the space of all symbols $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ such that $p(x, \lambda\xi) = \lambda^{m_1} p(x, \xi)$ for all $\lambda \geq 1$, $|\xi| \geq c > 0$, $x \in \mathbf{R}^n$. Analogously, we define the space $\Gamma^{m_1,[m_2]}_{\mu\nu}(\mathbf{R}^{2n})$ by interchanging the roles of x and ξ. Finally, we set

$$\Gamma^{[m_1],[m_2]}_{\mu\nu}(\mathbf{R}^{2n}) = \Gamma^{[m_1],m_2}_{\mu\nu}(\mathbf{R}^{2n}) \cap \Gamma^{m_1,[m_2]}_{\mu\nu}(\mathbf{R}^{2n}).$$

Using Definitions 3.1, 3.2 and 3.4, we can now introduce classical polyhomogeneous symbols.

Definition 3.5. We denote by $\Gamma^{m_1,[m_2]}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n})$ the space of all $p \in \Gamma^{m_1,[m_2]}_{\mu\nu}(\mathbf{R}^{2n})$ satisfying the following condition: there exists a sum $\sum_{k\geq 0} p_k \in FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ such that $p_k \in \Gamma^{[m_1-k],[m_2]}_{\mu\nu}(\mathbf{R}^{2n})$ for all $k \geq 0$ and $p \sim_\xi \sum_{k\geq 0} p_k$ in $FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$.

Definition 3.6. We denote by $\Gamma^{m_1,m_2}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n})$ the space of all symbols $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ satisfying the following condition: there exists a sum $\sum_{k\geq 0} p_k \in FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ such that $p_k \in \Gamma^{[m_1-k],m_2}_{\mu\nu}(\mathbf{R}^{2n})$ for all $k \geq 0$ and $p \sim_\xi \sum_{k\geq 0} p_k$ in $FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$.

Analogous definitions can be given for the spaces $\Gamma^{[m_1],m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n})$ and $\Gamma^{m_1,m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n})$, by interchanging the roles of x and ξ. Finally, we define a space of symbols which are classical polyhomogeneous with respect to both the variables.

Definition 3.7. We denote by $\Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ the space of all symbols $p \in \Gamma^m_{\mu\nu}(\mathbf{R}^{2n})$ for which the following conditions hold:

i) there exists $\sum_{k\geq 0} p_k \in FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ with $p_k \in \Gamma^{[m_1-k],m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n})$ for all $k \in \mathbf{N}$, $p \sim_\xi \sum_{k\geq 0} p_k$ in $FS^m_{\mu\nu,\xi}(\mathbf{R}^{2n})$ and $p - \sum_{k<N} p_k \in \Gamma^{m_1-N,m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n})$ for all $N \in \mathbf{Z}_+$;

ii) there exists $\sum_{h\geq 0} q_h \in FS^m_{\mu\nu,x}(\mathbf{R}^{2n})$ with $q_h \in \Gamma^{m_1,[m_2-h]}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n})$ for all $h \in \mathbf{N}$, $p \sim_x \sum_{h\geq 0} q_h$ in $FS^m_{\mu\nu,x}(\mathbf{R}^{2n})$ and $p - \sum_{h<N} q_h \in \Gamma^{m_1,m_2-N}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n})$ for all $N \in \mathbf{Z}_+$.

The following inclusions hold:

$$\Gamma^{m_1,[m_2]}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n}) \subset \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n}), \quad \Gamma^{[m_1],m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n}) \subset \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n}). \tag{3.2}$$

A simple homogeneity argument shows that for every $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ and for every $k \in \mathbf{N}$, there exists a unique function $\sigma^{m_1-k}_\psi(p) \in C^\infty(\mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}))$ such that $\sigma^{m_1-k}_\psi(p)(x,\lambda\xi) = \lambda^{m_1-k}\sigma^{m_1-k}_\psi(p)(x,\xi)$ for all $\lambda > 0$, $x \in \mathbf{R}^n$, $\xi \neq 0$ and $\sigma^{m_1-k}_\psi(p)(x,\xi) = p_k(x,\xi)$ for

$|\xi| \geq c > 0$. Analogously, in view of condition ii) of Definition 3.7, we can associate to every $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ the functions $\sigma_e^{m_2-h}(p)$ for all $h \in \mathbf{N}$ such that $\sigma_e^{m_2-h}(p)$ belongs to $C^\infty((\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n)$, $\sigma_e^{m_2-h}(p)(x,\xi) = q_h(x,\xi)$ for $|x| \geq c > 0$ and $\sigma_e^{m_2-h}(p)(\lambda x,\xi) = \lambda^{m_2-h}\sigma_e^{m_2-h}(p)(x,\xi)$ for all $\lambda > 0$, $\xi \in \mathbf{R}^n$, $x \neq 0$.

We also observe that if $\omega \in G^\mu(\mathbf{R}^n)$ is an excision function, i.e., $\omega = 0$ in a neighborhood of the origin and $\omega = 1$ in a neighborhood of ∞, then $\omega(\xi)\sigma_\psi^{m_1-k}(p)(x,\xi)$ is in $\Gamma^{[m_1-k],m_2}_{\mu\nu,cl(x)}(\mathbf{R}^{2n})$. Similarly, if $\chi(x)$ is an excision function in $G^\nu(\mathbf{R}^n)$, then $\chi(x)\sigma_e^{m_2-h}(p)(x,\xi)$ is in $\Gamma^{m_1,[m_2-h]}_{\mu\nu,cl(\xi)}(\mathbf{R}^{2n})$.

By these considerations and by the inclusions (3.2), we can also consider the functions $\sigma_\psi^{m_1-k}(\sigma_e^{m_2-h}(p))$ and $\sigma_e^{m_2-h}(\sigma_\psi^{m_1-k}(p))$. It is easy to show that

$$\sigma_\psi^{m_1-k}(\sigma_e^{m_2-h}(p)) = \sigma_e^{m_2-h}(\sigma_\psi^{m_1-k}(p))$$

for all $h, k \in \mathbf{N}$.

In particular, given $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$, we can consider the triple

$$\{\sigma_\psi^{m_1}(p),\ \sigma_e^{m_2}(p),\ \sigma_{\psi e}^{m}(p)\}$$

where we denote $\sigma_{\psi e}^m(p) = \sigma_\psi^{m_1}(\sigma_e^{m_2}(p))$.

The function $\sigma_\psi^{m_1}(p)$ is called the homogeneous principal interior symbol of p and the pair $\{\sigma_e^{m_2}(p),\ \sigma_{\psi e}^m(p)\}$ is the homogeneous principal exit symbol of p.

By the previous results, it turns out that, given two excision functions $\omega(\xi)$ in $G^\mu(\mathbf{R}^n)$ and $\chi(x) \in G^\nu(\mathbf{R}^n)$, then we have

$$p(x,\xi) - \omega(\xi)\sigma_\psi^{m_1}(p)(x,\xi) \in \Gamma^{(m_1-1,m_2)}_{\mu\nu,cl}(\mathbf{R}^{2n}) \tag{3.3}$$

$$p(x,\xi) - \chi(x)\sigma_e^{m_2}(p)(x,\xi) \in \Gamma^{(m_1,m_2-1)}_{\mu\nu,cl}(\mathbf{R}^{2n}) \tag{3.4}$$

$$p(x,\xi) - \omega(\xi)\sigma_\psi^{m_1}(p)(x,\xi) - \chi(x)(\sigma_e^{m_2}(p)(x,\xi) - \omega(\xi)\sigma_{\psi e}^m(p)(x,\xi)) \in \Gamma^{m-e}_{\mu\nu,cl}(\mathbf{R}^{2n}), \tag{3.5}$$

where we denote $e = (1,1)$. We denote by $OPS^m_{\mu\nu,cl}(\mathbf{R}^n)$ the set of all operators of the form (2.2) defined by a symbol $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$, and we set, for $\theta > 1$,

$$OPS^\theta_{cl}(\mathbf{R}^n) = \bigcup_{\substack{m\in\mathbf{R}^2 \\ \mu,\nu\in(1,+\infty) \\ \mu+\nu-1\leq\theta}} OPS^m_{\mu\nu,cl}(\mathbf{R}^n).$$

Remark 6. Arguing as in the previous section and applying Remark 4, it is easy to prove that if $P \in OPS^m_{\mu\nu,cl}(\mathbf{R}^n)$, $Q \in OPS^{m'}_{\mu\nu,cl}(\mathbf{R}^n)$, then the operator PQ is in $OPS^{m+m'}_{\mu\nu,cl}(\mathbf{R}^n)$.

To conclude this section, we give an alternative definition of ellipticity for classical polyhomogeneous symbols.

Definition 3.8. A symbol $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ is said to be elliptic if the three following conditions hold:

i) $\sigma^{m_1}_\psi(p)(x,\xi) \neq 0$ for all $(x,\xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$

ii) $\sigma^{m_2}_e(p)(x,\xi) \neq 0$ for all $(x,\xi) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n$

iii) $\sigma^m_{\psi e}(p)(x,\xi) \neq 0$ for all $(x,\xi) \in (\mathbf{R}^n \setminus \{0\}) \times (\mathbf{R}^n \setminus \{0\})$.

Remark 7. A symbol $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ is elliptic if and only if it is elliptic according to Definition 2.15.

Proof. See Proposition 1.4.37 in [**23**] for the proof. □

Example. Consider a partial differential operator with polynomial coefficients

$$P = \sum_{\substack{|\alpha|\leq m_1 \\ |\beta|\leq m_2}} c_{\alpha\beta} x^\beta D^\alpha.$$

The corresponding symbol belongs to $\Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$ for every $\mu > 1$, $\nu > 1$, with $m = (m_1, m_2)$. The operator P is elliptic in the SG-sense and Corollary 2.17 applies if

$$\sigma_{\psi}^{m_1} = \sum_{\substack{|\alpha|=m_1 \\ |\beta|\le m_2}} c_{\alpha\beta} x^{\beta} \xi^{\alpha} \neq 0,$$

$$\sigma_{e}^{m_2} = \sum_{\substack{|\alpha|\le m_1 \\ |\beta|= m_2}} c_{\alpha\beta} x^{\beta} \xi^{\alpha} \neq 0,$$

$$\sigma_{\psi e}^{m} = \sum_{\substack{|\alpha|=m_1 \\ |\beta|= m_2}} c_{\alpha\beta} x^{\beta} \xi^{\alpha} \neq 0,$$

according to Definition 3.8. In particular, for H in (0.1) we have $\sigma_{\psi}^2 = (1+|x|^{2k})|\xi|^2$, $\sigma_e^{2k} = |x|^{2k}(|\xi|^2+1)$, $\sigma_{\psi e}^{2,2k} = |x|^{2k}|\xi|^2$.

4. Wave front set. In this section we introduce a notion of wave front set for the distributions of $S_{\theta}^{\theta'}(\mathbf{R}^n)$, which allows to control their behavior "at infinity," and prove the standard properties of microellipticity with respect to the polyhomogeneous operators defined in the previous section. Similar results have been proved by Coriasco and Maniccia [**7**] for Schwartz tempered distributions.

For every $\eta_o \in \mathbf{R}^n \setminus \{0\}$, we will denote by $\infty\eta_o$ the projection $\eta_o/|\eta_0|$ on the unit sphere S^{n-1}. In the following, an open set $V \subset \mathbf{R}^n$ is said to be a conic neighborhood of the direction $\infty\eta_o$ if it is the intersection of an open cone containing the direction $\infty\eta_o$ with the complementary set of a closed ball centered in the origin. The decomposition of the principal symbol into three components in the previous section suggests to define for the elements of $S_{\theta}^{\theta'}(\mathbf{R}^n)$ three sets which we will denote by WF_{ψ}^{θ}, WF_{e}^{θ}, $WF_{\psi e}^{\theta}$, $\theta > 1$.

To give precise definitions, we need to introduce two types of cut-off functions.

Definition 4.1. Let $y_o \in \mathbf{R}^n$ and fix $\nu > 1$. We denote by $\mathcal{R}_{y_o}^{\nu}$ the set of all functions $\varphi \in G_o^{\nu}(\mathbf{R}^n)$ such that $0 \le \varphi \le 1$ and $\varphi \equiv 1$ in a neighborhood of y_o.

Definition 4.2. Let $\eta_o \in \mathbf{R}^n \setminus \{0\}$ and fix $\mu > 1$. We denote by $\mathcal{Z}_{\eta_o}^{\mu}$ the set of all functions $\psi \in C^{\infty}(\mathbf{R}^n)$ such that $\psi(\lambda\xi) = \psi(\xi)$ for all $\lambda \geq 1$ and $|\xi|$ large, $0 \leq \psi \leq 1$, $\psi \equiv 1$, in a conic neighborhood V of $\infty\eta_o$, $\psi \equiv 0$ outside a conic neighborhood V' of $\infty\eta_o$, $V \subset V'$ and

$$|D^\alpha_\eta \psi(\eta)| \leq C^{|\alpha|+1}(\alpha!)^\mu \langle\eta\rangle^{-|\alpha|}, \quad \eta \in \mathbf{R}^n$$

for every $\alpha \in \mathbf{N}^n$ and for some $C > 0$.

Elements in $\mathcal{Z}^\mu_{\eta_o}$ can be constructed by considering Gevrey functions $\psi^\sharp$ of order μ on S^{n-1}, $\psi^\sharp = 1$ in a neighborhood of η_o, $\psi^\sharp = 0$ outside a larger neighborhood, extending them as homogeneous functions of order 0 on $\mathbf{R}^n$ and cutting-off in a neighborhood of the origin, cf. [**20**, pp. 153–154].

Definition 4.3. Let θ be a positive real number such that $\theta > 1$, and let $u \in S^{\theta'}_\theta(\mathbf{R}^n)$.

• We say that $(x_o, \xi_o) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ is not in $WF^\theta_\psi u$ if there exist positive numbers $\mu, \nu \in (1, +\infty)$ such that $\theta \geq \mu + \nu - 1$, and there exist cut-off functions φ_{x_o} in $\mathcal{R}^\nu_{x_o}$, $\psi_{\xi_o} \in \mathcal{Z}^\mu_{\xi_o}$ such that $\varphi_{x_o}(\psi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$.

• We say that $(x_o, \xi_o) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n$ is not in $WF^\theta_e u$ if there exist positive numbers $\mu, \nu \in (1, +\infty)$ such that $\theta \geq \mu + \nu - 1$, and there exist cut-off functions φ_{ξ_o} in $\mathcal{R}^\mu_{\xi_o}$, $\psi_{x_o} \in \mathcal{Z}^\nu_{x_o}$ such that $\psi_{x_o}(\varphi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$.

• We say that $(x_o, \xi_o) \in (\mathbf{R}^n \setminus \{0\}) \times (\mathbf{R}^n \setminus \{0\})$ is not in $WF^\theta_{\psi e} u$ if there exist positive numbers $\mu, \nu \in (1, +\infty)$ such that $\theta \geq \mu + \nu - 1$, and there exist cut-off functions $\psi_{x_o} \in \mathcal{Z}^\nu_{x_o}$, $\psi_{\xi_o} \in \mathcal{Z}^\mu_{\xi_o}$ such that $\psi_{x_o}(\psi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$.

Remark 8. We can consider $WF^\theta_\psi u$ as a subset of $\mathbf{R}^n \times S^{n-1}$, being $WF^\theta_\psi u$ invariant with respect to the multiplication of the second variable ξ by positive scalars. Analogously, we can consider $WF^\theta_e u \subset S^{n-1} \times \mathbf{R}^n$ and $WF^\theta_{\psi e} u \subset S^{n-1} \times S^{n-1}$.

Remark 9. Every $u \in S^{\theta'}_\theta(\mathbf{R}^n)$ can be regarded as an element of $\mathcal{D}'_\theta(\mathbf{R}^n)$, according to Remark 2. It is easy to show that $WF^\theta_\psi u$ coincides with the standard Gevrey wave front set of u, see for example [**20**].

Example. Consider the distribution

$$u = \sum_{\alpha \in \mathbf{N}^n} a_\alpha \delta^{(\alpha)}$$

where the coefficients a_α satisfy the following estimate. For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|a_\alpha| \leq C_\varepsilon \varepsilon^{|\alpha|} (\alpha!)^{-\theta}. \tag{4.1}$$

Under the assumption (4.1), $u \in \mathcal{E}'_\theta(\mathbf{R}^n) \subset S_\theta^{\theta'}(\mathbf{R}^n) \subset \mathcal{D}'_\theta(\mathbf{R}^n)$, and we have

$$WF_\psi^\theta u = \{0\} \times S^{n-1}, \quad WF_e^\theta u = WF_{\psi e}^\theta u = \varnothing.$$

Let us characterize the sets defined before in terms of characteristic manifolds of polyhomogeneous operators. For $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$, we define

$$\begin{aligned} \mathrm{Char}_\psi(P) &= \{(x,\xi) \in \mathbf{R}^n \times S^{n-1} : \sigma_\psi^{m_1}(p)(x,\xi) = 0\} \\ \mathrm{Char}_e(P) &= \{(x,\xi) \in S^{n-1} \times \mathbf{R}^n : \sigma_e^{m_2}(p)(x,\xi) = 0\} \\ \mathrm{Char}_{\psi e}(P) &= \{(x,\xi) \in S^{n-1} \times S^{n-1} : \sigma_{\psi e}^{m}(p)(x,\xi) = 0\}. \end{aligned}$$

Proposition 4.4. *Let $u \in S_\theta^{\theta'}(\mathbf{R}^n)$. We have the following relations*:

$$WF_\psi^\theta u = \bigcap_{\substack{P \in OPS^\theta_{cl}(\mathbf{R}^n) \\ Pu \in S^\theta_\theta(\mathbf{R}^n)}} \mathrm{Char}_\psi(P), \quad WF_e^\theta u = \bigcap_{\substack{P \in OPS^\theta_{cl}(\mathbf{R}^n) \\ Pu \in S^\theta_\theta(\mathbf{R}^n)}} \mathrm{Char}_e(P),$$

$$WF_{\psi e}^\theta u = \bigcap_{\substack{P \in OPS^\theta_{cl}(\mathbf{R}^n) \\ Pu \in S^\theta_\theta(\mathbf{R}^n)}} \mathrm{Char}_{\psi e}(P)$$

Proof. Let $(x_o, \infty\xi_o) \notin WF_\psi^\theta u$. Then, there exist μ, ν as in Definition 4.3 and φ_{x_o} in $\mathcal{R}^\nu_{x_o}$, ψ_{ξ_o} in $\mathcal{Z}^\mu_{\xi_o}$ such that $Pu = \varphi_{x_o}(\psi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$. Observe that P can be regarded as a pseudodifferential operator with symbol $\varphi_{x_o}(x)\psi_{\xi_o}(\xi) \in \Gamma^0_{\mu\nu,cl}(\mathbf{R}^{2n})$ and that $\varphi_{x_o}(x_o)\psi_{\xi_o}(\lambda\xi_o) =$

1 for $\lambda \in \mathbf{R}_+$ sufficiently large. Hence, $(x_o, \infty\xi_o)$ does not belong to $\cap_{\substack{P\in OPS^\theta_{cl}(\mathbf{R}^n)\\ Pu\in S^\theta_\theta(\mathbf{R}^n)}}\mathrm{Char}_\psi(P)$. Conversely, let us assume that there exists $P = p(x,D)$ in $OPS^\theta_{cl}(\mathbf{R}^n)$ such that $Pu \in S^\theta_\theta(\mathbf{R}^n)$ and $\sigma_\psi(p)(x_o,\infty\xi_o) \neq 0$. Then, there exists a neighborhood U of x_o and a conic neighborhood V of $\infty\xi_o$ such that $\sigma_\psi(p)(x,\infty\xi) \neq 0$ for all $(x,\xi) \in U \times V$. Furthermore, by (3.3), it turns out that if $|\xi|$ is sufficiently large, we have that

$$\frac{|p(x,\xi)|}{|\xi|^{m_1}\langle x\rangle^{m_2}} \geq \frac{|\sigma_\psi(p)(x,\xi)|}{|\xi|^{m_1}\langle x\rangle^{m_2}} - \frac{|p(x,\xi)-\sigma_\psi(p)(x,\xi)|}{|\xi|^{m_1}\langle x\rangle^{m_2}} \geq C > 0$$

for some $C > 0$. Hence, we can construct two cut-off functions φ_{x_o}, ψ_{ξ_o} respectively supported in U and in V such that p is elliptic with respect to $\varphi_{x_o}(x)\psi_{\xi_o}(\xi)$. By Theorem 2.19, there exists $E \in OPS^{-m}_{\mu\nu}(\mathbf{R}^n)$ such that $EPu = \varphi_{x_o}(\psi_{\xi_o}(D)u) + Ru$, where R is θ-regularizing. Then, $\varphi_{x_o}(\psi_{\xi_o}(D)u) = Ru - EPu \in S^\theta_\theta(\mathbf{R}^n)$. This gives the statement for $WF^\theta_\psi u$. The corresponding relation for $WF^\theta_e u$ can be obtained with the same argument by simply interchanging the roles of x and ξ. For what concerns the third relation, we obtain the inclusion $\cap_{\substack{P\in OPS^\theta_{cl}(\mathbf{R}^n)\\ Pu\in S^\theta_\theta(\mathbf{R}^n)}}\mathrm{Char}_{\psi e}(P) \subset WF^\theta_{\psi e}u$ directly again from Definition 4.3. Assume now that there exists $P \in OPS^\theta_{cl}(\mathbf{R}^n)$ such that $Pu \in S^\theta_\theta(\mathbf{R}^n)$ and $\sigma_{\psi e}(p)(\infty x_0, \infty\xi_0) \neq 0$. Then, there exist two conic neighborhoods V_{x_0}, V_{ξ_0} such that $\sigma_{\psi e}(p)(x,\xi) \neq 0$ if $(x,\xi) \in V_{x_0} \times V_{\xi_0}$. Hence, by (3.5), we have that

$$\frac{|p(x,\xi)|}{|\xi|^{m_1}|x|^{m_2}} \geq C > 0$$

if $|x|$ and $|\xi|$ are large enough. Then, we can conclude arguing as for $WF^\theta_\psi u$. □

Remark 10. By the arguments in the preceding proof, it follows easily that Definition 4.3 is independent of the choice of μ and ν. Namely, if $(x_o, \infty\xi_o) \notin WF^\theta_\psi u$, then for any given $\mu > 1$, $\nu > 1$, with $\mu + \nu - 1 \leq \theta$, we may actually find $\varphi_{x_o} \in \mathcal{R}^\nu_{x_o}$, $\psi_{\xi_o} \in \mathcal{Z}^\mu_{\psi_{\xi_o}}$ such that $\varphi_{x_o}(\psi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$, and similarly for $WF^\theta_e u, WF^\theta_{\psi e}u$.

Theorem 4.5. *Let* $u \in S_\theta^{\theta'}(\mathbf{R}^n)$ *and* $p \in \Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$, *with* $\mu+\nu-1 \leq \theta$. *Then, the following inclusions hold*:

$$WF_\psi^\theta(Pu) \subset WF_\psi^\theta u \subset WF_\psi^\theta(Pu) \cup \mathrm{Char}_\psi(P) \tag{4.2}$$

$$WF_e^\theta(Pu) \subset WF_e^\theta u \subset WF_e^\theta(Pu) \cup \mathrm{Char}_e(P) \tag{4.3}$$

$$WF_{\psi e}^\theta(Pu) \subset WF_{\psi e}^\theta u \subset WF_{\psi e}(Pu) \cup \mathrm{Char}_{\psi e}(P). \tag{4.4}$$

Proof. If $(x_o, \xi_o) \notin WF_\psi^\theta u$, then there exist cut-off functions $\varphi_{x_o} \in \mathcal{R}^\nu_{x_o}$, $\psi_{\xi_o} \in \mathcal{Z}^\mu_{\xi_o}$ such that $\varphi_{x_o}(\psi_{\xi_o}(D)u) \in S_\theta^\theta(\mathbf{R}^n)$, where, in view of Remark 10, we may take the same μ, ν of the class $\Gamma^m_{\mu\nu,cl}(\mathbf{R}^{2n})$. Shrinking the neighborhoods of $x_o, \infty\xi_o$, we can construct two cut-off functions $\widetilde{\varphi}_{x_o} \in \mathcal{R}^\nu_{x_o}$, $\widetilde{\psi}_{\xi_o} \in \mathcal{Z}^\mu_{\xi_o}$ such that $\widetilde{\varphi}_{x_o}\varphi_{x_o} = \widetilde{\varphi}_{x_o}$ and $\widetilde{\psi}_{\xi_o}\psi_{\xi_o} = \widetilde{\psi}_{\xi_o}$. Denote by Q the operator with symbol $\varphi_{x_o}\psi_{\xi_o}$ and by $\widetilde{Q}$ the operator with symbol $\widetilde{\varphi}_{x_o}\widetilde{\psi}_{\xi_o}$. By Theorem 2.12, we have that

$$\widetilde{Q}QPu = \widetilde{Q}PQu + \widetilde{Q}[Q, P]u = \widetilde{Q}PQu + Ru$$

where R is θ-regularizing. But $\widetilde{Q}Q \in OPS^\theta_{cl}(\mathbf{R}^n)$ and $\sigma_\psi(\widetilde{Q}Q)(x_o, \infty\xi_o) = \sigma_\psi(\widetilde{Q})(x_o, \infty\xi_o)\sigma_\psi(Q)(x_o, \infty\xi_o) \neq 0$. Then, by Proposition 4.4, we conclude that $(x_o, \infty\xi_o) \notin WF_\psi^\theta(Pu)$. This proves the first inclusion in (4.2). Assume now that $(x_o, \infty\xi_o) \notin WF_\psi^\theta(Pu)$. By Proposition 4.4, there exists $Q = q(x, D) \in OPS^0_{\mu\nu,cl}(\mathbf{R}^n)$ such that $QPu \in S_\theta^\theta(\mathbf{R}^n)$ and $\sigma_\psi(Q)(x_o, \infty\xi_o) \neq 0$. Furthermore, if $(x_o, \infty\xi_o) \notin \mathrm{Char}_\psi(P)$, then $\sigma_\psi(QP)(x_o, \infty\xi_o) = \sigma_\psi(Q)(x_0, \infty\xi_o)\sigma_\psi(P)(x_o, \infty\xi_o) \neq 0$. Moreover, $QP \in OPS^m_{\mu\nu,cl}(\mathbf{R}^n)$. Hence, by Proposition 4.4, we conclude that $(x_o, \infty\xi_o) \notin WF_\psi^\theta u$. The proofs of (4.3) and (4.4) are analogous. □

Let us give an example showing that the second inclusion in (4.4) can be a nontrivial identity.

Example. Consider the operator

$$P(x, D_x) = D_x - x, \quad x \in \mathbf{R}$$

for which $\mathrm{Char}_\psi(P) = \varnothing$, $\mathrm{Char}_e(P) = \varnothing$, $\mathrm{Char}\ _{\psi e}(P) = S^0 \times S^0$. A solution of the equation $Pu = 0$ is given by $u(x) = e^{i(x^2/2)}$ for which it is easy to verify that $WF^\theta_\psi u = WF^\theta_e u = \varnothing$, $WF^\theta_{\psi e} u = S^0 \times S^0$, see also [**7**]. We leave to the reader an application of Theorem 4.5 to the example at the end of Section 3, under different assumptions on $\mathrm{Char}_\psi(P)$, $\mathrm{Char}_e(P)$, $\mathrm{Char}_{\psi e}(P)$.

Proposition 4.6. *Let $u \in S^{\theta'}_\theta(\mathbf{R}^n)$. Then, $u \in S^\theta_\theta(\mathbf{R}^n)$ if and only if $WF^\theta_\psi u = WF^\theta_e u = WF^\theta_{\psi e} u = \varnothing$.*

Proof. If $WF^\theta_{\psi e} u = \varnothing$, then, for every $(\infty x_o, \infty\xi_o) \in S^{n-1} \times S^{n-1}$, there exist $\psi_{x_o} \in \mathcal{Z}^\nu_{x_o}$, $\psi_{\xi_o} \in \mathcal{Z}^\mu_{\xi_o}$ such that $\psi_{x_o}(\psi_{\xi_o}(D)u) \in S^\theta_\theta(\mathbf{R}^n)$. In view of Remark 10, we may fix μ, ν independent of $(\infty x_o, \infty \xi_o)$. Let us observe that $\sigma^{0,0}_{\psi e}(\psi_{x_o}(x)\psi_{\xi_o}(\xi)) = 1$ in a conic set in $\mathbf{R}^{2n}$, obtained as a product of conic sets of $\mathbf{R}^n_x$ and $\mathbf{R}^n_\xi$, intersecting $S^{n-1} \times S^{n-1}$ in a neighborhood V_{x_o,ξ_o} of $(\infty x_o, \infty \xi_o)$. By the compactness of $S^{n-1} \times S^{n-1}$, we can find a finite family $(\infty x_j, \infty \xi_j)$, $j = 1, \dots, N$, such that V_{x_j,ξ_j}, $j = 1, \dots, N$ cover $S^{n-1} \times S^{n-1}$. Define

$$q_o(x,\xi) = \sum_{j=1,\dots,N} \psi_{x_j}(x)\psi_{\xi_j}(\xi).$$

If $|\xi| > R$ and $|x| > R$, with R sufficiently large, then $q_o(x,\xi) \geq C > 0$. Moreover, by construction, $q_o(x,D)u \in S^\theta_\theta(\mathbf{R}^n)$. Applying similar compactness arguments to $\{x \in \mathbf{R}^n : |x| \leq R\} \times S^{n-1}$ and to $S^{n-1} \times \{\xi \in \mathbf{R}^n : |\xi| \leq R\}$ and using the assumption $WF^\theta_\psi u = WF^\theta_e u = \varnothing$, we can construct $q_1(x,\xi), q_2(x,\xi)$ such that $q_1(x,D)u \in S^\theta_\theta(\mathbf{R}^n)$, $q_2(x,D)u \in S^\theta_\theta(\mathbf{R}^n)$ and $q_1(x,\xi) \geq C_1 > 0$ if $|\xi| > R$, $|x| \leq R$, and $q_2(x,\xi) \geq C_2 > 0$ if $|x| > R$, $|\xi| \leq R$. Moreover, obviously $q_o, q_1, q_2 \in \Gamma^0_{\mu\nu,cl}(\mathbf{R}^{2n})$. Then, the function $q(x,\xi) = q_o(x,\xi) + q_1(x,\xi) + q_2(x,\xi)$ is an elliptic symbol of order 0 and $q(x,D)u \in S^\theta_\theta(\mathbf{R}^n)$. Then, $u \in S^\theta_\theta(\mathbf{R}^n)$ in view of Corollary 2.17. The inverse implication is trivial. $\square$

We conclude with a proposition which makes clear in what sense the exit components $WF^\theta_e, WF^\theta_{\psi e}$ determine the behavior of a distribution of $S^{\theta'}_\theta(\mathbf{R}^n)$ at infinity. The proof follows the same arguments of the proof of Proposition 4.6. We omit it for the sake of brevity.

Proposition 4.7. *Let* $u \in S_\theta^{\theta'}(\mathbf{R}^n)$*, and denote by* $\Pi_x : \mathbf{R}^{2n}_{x,\xi} \to \mathbf{R}^n_x$ *the standard projection on the variable* x*. If* $x_0 \notin \Pi_x(WF_e^\theta u \cup WF_{\psi e}^\theta u)$*, then there exists* $\psi_{x_0} \in \mathcal{Z}^\theta_{x_0}$ *such that* $\psi_{x_0} u \in S_\theta^\theta(\mathbf{R}^n)$*.*

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