ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 4, 2006

UNITARY GROUPS ACTING ON GRASSMANNIANS ASSOCIATED WITH A QUADRATIC EXTENSION OF FIELDS

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ABSTRACT. Let (V, H) be an anisotropic Hermitian space of finite dimension over the algebraic closure of a real closed field K. We determine the orbits of the group of isometries of (V, H) in the set of K-subspaces of V.

Throughout the paper K denotes a real closed field and $\overline{K}$ its algebraic closure. Then it is well known (see, for example, [**4**, Chapter 2], [**23**]; see also [**8**]) that $\overline{K} = K(i)$ with $i = \sqrt{-1}$. Also we let (V, H) be an anisotropic Hermitian space (with respect to the involution underlying the quadratic field extension $\overline{K}/K$) of finite dimension n over $\overline{K}$. In this context we consider the natural action of the *unitary* group $U = U(V, H)$ of isometries of (V, H) on the set X_d of all d-dimensional K-subspaces of V. The analogous problem where (V, H) is a symplectic space was treated in [**1**] (for arbitrary quadratic field extensions). It turns out that, in contrast with the symplectic case, there are infinitely many orbits for the action of the unitary group U on X_d.

In group theoretic language the stated problem turns into the determination of the double coset spaces of the form

$$G_W \setminus G \,/\, U, \tag{1}$$

where $G = \mathrm{GL}\,(V_K)$ and G_W denotes the parabolic subgroup of G stabilizing a member $W \in X_d$ (we write V_K to indicate that we are regarding V as a vector space over K). The precise structure of double coset spaces involving classical groups is of great interest in applying the classical Rankin-Selberg method for explicit construction of automorphic L-functions, as Garrett [**2**] and Piatetski-Shapiro and Rallis [**6**] worked out.

2000 AMS *Mathematics Subject Classification.* Primary 51N30, 15A21, Secondary 11E39.
Received by the editors on October 13, 2003.

Besides $G = \mathrm{GL}(V_K)$, there are further possibilities for the group G in (1), because U embeds into other classical groups over K. For instance, we have

$$H(x,y) = S(x,y) + iA(x,y) \tag{2}$$

for suitable K-bilinear forms S and A with S (anisotropic) symmetric and A alternating. Moreover, for any $\gamma \in U$ we have

$$S(\gamma(x),\gamma(y)) + iA(\gamma(x),\gamma(y)) = S(x,y) + iA(x,y),$$

which means that U embeds into the *orthogonal* group $O(V_K, S)$ of isometries of (V_K, S), as well as into the *symplectic* group $\mathrm{Sp}(V_K, A)$ of isometries of (V_K, A). Therefore in (1) we can take $G = O(V_K, S)$, or $G = \mathrm{Sp}(V_K, A)$. As $O(V_K, S)$ is transitive on X_d, double coset spaces (1) with $G = O(V_K, S)$ are essentially the same as with $G = \mathrm{GL}(V_K)$. The situation is different when $G = \mathrm{Sp}(V_K, A)$: if A restricts to $W \in X_d$ with rank r, the double coset space $G_W \setminus G / U$ corresponds to the action of U on the set $X_{d,r}$ of all d-dimensional K-subspaces on which A induces an alternating form of rank r. In this framework it has to be emphasized the fact that U has infinitely many orbits in $X_{d,r}$ for $r > 0$ and it is transitive on $X_{d,0}$, i.e., on the set of d-dimensional A-totally isotropic K-subspaces of V.

I. The set of anisotropic Hermitian forms on V maps bijectively onto a set of anisotropic bilinear forms on V_K via

$$\begin{aligned} &H \longmapsto B = S + A,\\ &B \longmapsto H = \frac{1}{2}[(B + {}^tB) + i(B - {}^tB)], \end{aligned} \tag{3}$$

where S and A are defined as in (2) and ${}^tB(x,y)$ means $B(y,x)$.

The bilinear form B *associated* to H, in the sense of (3), plays a fundamental role in this context. It turns out that the orthogonality in (V, H) is essentially the same as in (V_K, B). Indeed we have

1. Proposition. *$H(x,y) = 0$ if and only if $B(x,y) = 0$ and $B(y,x) = 0$.*

Proof. Let $H(x, y) = S(x, y) + iA(x, y) = 0$. Then

$$S(x, y) = S(y, x) = 0 = A(x, y) = A(y, x)$$

and consequently

$$B(x, y) = S(x, y) + A(x, y) = 0 = S(y, x) + A(y, x) = B(y, x).$$

Conversely, if $B(x, y) = B(y, x) = 0$, then $H(x, y) = 0$ follows from (3). $\square$

Let W be a K-subspace of V, and let $W = W_1 \oplus W_2$ be a decomposition of W into the direct sum of two nontrivial subspaces. We shall write

$$W = W_1 \perp_H W_2 \qquad (\text{resp. } W = W_1 \perp_B W_2),$$

if $H(W_1, W_2) = 0$ (respectively $B(W_1, W_2) = B(W_2, W_1) = 0$). Thanks to Proposition 1, we have then

$$W = W_1 \perp_H W_2 \Longleftrightarrow W = W_1 \perp_B W_2, \tag{4}$$

so it is superfluous to specify the form with respect to which the orthogonality occurs.

As B is anisotropic, B induces on any K-subspace W of V a nondegenerate K-bilinear form B_W:

$$B_W(x, y) = B(x, y) \quad \forall\, x, y \in W.$$

So there exists a (unique) linear mapping $\sigma_W \in \mathrm{GL}\,(W)$ (the *asymmetry* of B_W) such that

$$B_W(x, y) = B_W(y, \sigma_W(x))) \quad \forall\, x, y \in W.$$

Then $B_W(x, y) = B_W(\sigma_W(x), \sigma_W(y))$, $B_W(\sigma_W(x), y) = B_W(x, \sigma_W^{-1}(y))$ and, more generally for every polynomial $p \in K[x]$,

(5)

$$B_W(p(\sigma_W)(x), y) = B_W(x, p(\sigma_W^{-1})(y)) = B_W(x, \sigma_W^{-\deg\,(p)} p^*(\sigma_W)(y)),$$

where p^* denotes the *adjoint polynomial* of p, that is, the polynomial

$$p^*(x) := x^{\deg\,(p)} p(x^{-1}).$$

Riehm in [**7**] pointed out the importance of the asymmetry σ_W for the K-bilinear space (W, B_W). In fact, orthogonal decompositions in W correspond to decompositions into $K[\sigma_W]$-submodules, as the following proposition states.

2. Proposition. *Let $W = W_1 \oplus W_2$ be a decomposition of the K-subspace W into the direct sum of two K-subspaces with $B(W_1, W_2) = 0$. Then $W = W_1 \perp W_2$ if and only if W_1, as well as W_2, is a $K[\sigma_W]$-submodule.*

Proof. [**7**, p. 47]. $\square$

II. In view of the foregoing section, if we want to determine the U-orbit of a given K-subspace W of V, we can apply the Krull-Schmidt theorem to the $K[\sigma_W]$-module W and reduce matters to the case where such a module is indecomposable (see [**3**, p. 115]). This corresponds to say that (W, B_W) is an *indecomposable* K-bilinear space, i.e., it has no orthogonal decomposition such as (4).

We have

3. Proposition. *Let (W, B_W) be indecomposable. Then, one of the following occurs*:

a) *W is a $\overline{K}$-line*;

b) *W is a K-substructure* (*i.e., a K-subspace generated by $\overline{K}$-linearly independent vectors*).

Proof. In fact, let C be the largest $\overline{K}$-subspace of V contained in W (the $\overline{K}$-*component* of W), and let $C^{\perp}$ be the subspace of V orthogonal to the whole C. Then $V = C \perp C^{\perp}$ and we have the decomposition $W = C \perp (C^{\perp} \cap W)$. Hence, either C is trivial, i.e., W is a K-substructure, or $C = W$, and we have a line of V because a $\overline{K}$-subspace of V always possesses an orthogonal basis. $\square$

As K is really closed, to be anistropic for the Hermitian form H means that H is either *definite positive*, i.e., $H(x, x)$ is a nonzero square in K (for any $x \in V, x \neq 0$), or *definite negative*, i.e., $H(x, x)$ is the opposite of a nonzero square in K. This implies that in every one-dimensional $\overline{K}$-subspace, as well as in every one-dimensional K-subspace, there is always a vector v with $H(v,v) = 1$ (in the definite positive case), or $H(v,v) = -1$ (in the definite negative case), i.e., there is always a vector of H-norm $\varepsilon = \pm 1$. Therefore we have

4. Proposition. *The lines over $\overline{K}$ form a unique orbit for the action of U and the same occurs for the lines over K.*

Thus we have reduced matters to the determination of the U-orbit of an indecomposable K-substructure W of dimension > 1. The next proposition claims that it is the same if we determine the orbit of W for the action of the group of isometries of (V_K, B).

5. Proposition. *Let W and W' be K-substructures of V. There exists an element in U mapping W onto W' if and only if there exists an isometry of (V_K, B) mapping W onto W'.*

Proof. Assume there exists an isometry of (V_K, B) mapping the K-substructure W onto the K-substructure W'. Then there exist bases $(e_1, \dots, e_d)$ of W and $(e'_1, \dots, e'_d)$ of W' with respect to which B has the same representation in both W and W'. This means that, with respect to the above bases, the Hermitian form H $(= 1/2[(B + {}^tB) + i(B - {}^tB)])$ has the same representation in both the $\overline{K}$-vector spaces $\overline{K}W$ and $\overline{K}W'$ generated by W and W'. Hence,

$$\sum_{i=1}^{d} \lambda_i e_i \longmapsto \sum_{i=1}^{d} \lambda_i e'_i \qquad (\lambda_i \in \overline{K})$$

defines an isometry $(\overline{K}W, H) \to (\overline{K}W', H)$ which extends, by Witt's theorem, to an isometry $(V, H) \to (V, H)$ mapping W onto W'.

The converse part follows from the fact that an isometry $\varphi \in U$ satisfies the condition

$$S(\varphi(x), \varphi(y)) + iA(\varphi(x), \varphi(y)) = S(x, y) + iA(x, y),$$

giving in turn $S(\varphi(x), \varphi(y)) = S(x, y)$ and $A(\varphi(x), \varphi(y)) = A(x, y)$. Hence, φ preserves $B = S + A$, i.e., φ is an isometry of (V_K, B). $\square$

III. It turns out from Sections I and II that we have to classify the K-bilinear spaces (W, B_W) with W an indecomposable K-substructure of dimension > 1. A fundamental result in this direction is

6. Proposition. *The asymmetry σ_W of B_W has minimal polynomial $x^2 - 2bx + 1$ for a suitable element $b \in K$ such that $1 - b^2 \in K^2$, $b \neq \pm 1$.*

Proof. By [**7** Proposition 3], W decomposes orthogonally if the minimal polynomial of σ_W has two distinct prime divisors p and p' with p' and p^* relatively prime. Thus, if for each irreducible monic polynomial $p \in K[x]$ we denote by W_p the *p-primary component* of W, which is the subspace

$$W_p = \{w \in W : p^s(\sigma_W)(w) = 0 \text{ for some } s \geq 0\},$$

just two cases can occur [**7**, p. 48]:

a) $W = W_p$ for some irreducible monic $p \in K[x]$ such that $p = \pm p^*$, and in such a case the minimal polynomial of σ_W is a power p^r;

b) $W = W_p \oplus W_{p^*}$ for some irreducible monic $p \in K[x]$ such that $p \neq \pm p^*$, and in such a case the minimal polynomial of σ_W is a product $cp^r p^{*s}$ for a suitable $c \in K$, $c \neq 0$.

First we shall prove that case b) cannot occur because it requires both W_p and W_{p^*} to be totally isotropic. This can be shown as follows.

Using (5), for all $x, y \in W$ we infer

$$\begin{aligned} B(p^{*r}(\sigma_W)(x), p^{*s}(\sigma_W)(y)) &= B(x, \sigma_W^{-r\deg(p)} p^r p^{*s}(\sigma_W)(y)) \\ &= B(x, \sigma_W^{-r\deg(p)}(0)) = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} B(p^{*r}(\sigma_W)(x), p^{*s}(\sigma_W)(y)) &= B(p^{*s}(\sigma_W)(y), \sigma_W p^{*r}(\sigma_W)(x)) \\ &= B(y, \sigma_W^{1-s\deg(p)} p^s p^{*r}(\sigma_W)(x)). \end{aligned}$$

Hence, the endomorphism $p^s p^{*r}(\sigma_W)$ maps every vector to 0, which means that $p^s p^{*r}$ is the minimal polynomial of σ_W and this occurs if and only if $r = s$. Thus, $W_p = p^{*r}(\sigma_W)(W)$ and $W_{p^*} = p^r(\sigma_W)(W)$. Consequently, for all $x, y \in W$, we have

$$B(p^{*r}(\sigma_W)(x), p^{*r}(\sigma_W)(y)) = B(x, \sigma_W^{-r\deg(p)} p^r p^{*r}(\sigma_W)(y)) = 0$$

and we see that W_p is totally isotropic. Likewise, $B(W_{p^*}, W_{p^*}) = 0$.

Therefore, we are in case a). Assume now there exists a nonzero vector $w \in W$ such that $\sigma_W(w) = \lambda w$ for some $\lambda \in K$ ($\lambda \neq 0$ because $\sigma_W \in \mathrm{GL}(W)$) and let $W_1 \subset W$ with $B(W_1, w) = 0$ (hence $W = \langle w \rangle \oplus W_1$, w being anisotropic). Then we have

$$B(w, W_1) = B(W_1, \sigma_W(w)) = \lambda B(W_1, w) = 0,$$

i.e., an orthogonal decomposition of W occurs.

Thus, as K is real closed, we have $p^*(x) = p(x) = x^2 - 2bx + 1$ for a suitable element $b \in K$ such that $1 - b^2 \in K^2$, $b \neq \pm 1$ [**4**, p. 337].

Choose now a vector v such that $p^{r-1}(\sigma_W)(v) \neq 0$. Then using (5) we have

$$0 \neq B(p^{r-1}(\sigma_W)(v),\, p^{r-1}(\sigma_W)(v)) = B(v, \sigma_W^{(1-r)\deg(p)} p^{2r-2}(\sigma_W)(v)),$$

which means $2r - 2 < r$, or $r = 1$. $\square$

Now we are able to determine definitively the dimension of an indecomposable K-substructure:

7. Proposition. *An indecomposable K-substructure has dimension ≤ 2.*

Proof. In view of Proposition 2, the claim is an immediate consequence of Proposition 6. $\square$

Thanks to Propositions 6 and 7, if we are given an indecomposable K-bilinear space (W, B_W) with W a K-substructure of dimension > 1, then $\dim_K W = 2$ and the asymmetry σ_W of B_W has a representation of shape

$$\begin{pmatrix} b & \sqrt{1-b^2} \\ -\sqrt{1-b^2} & b \end{pmatrix},$$

for a suitable element $b \in K$ such that $1-b^2 \in K^2$, $b \neq \pm 1$. Let (e_1, e_2) be a basis of W giving the above representation of σ_W and put

$$a := B(e_1, e_1).$$

Then

$$B(e_1, e_1) = B(e_1, \sigma_W(e_1)) = bB(e_1, e_1) + \sqrt{1-b^2}\, B(e_1, e_2),$$

that is

$$B(e_1, e_2) = a\sqrt{\frac{1-b}{1+b}}.$$

Likewise we find

$$B(e_2, e_1) = -a\sqrt{\frac{1-b}{1+b}} \quad \text{and} \quad B(e_2, e_2) = a.$$

Now

$$\begin{aligned} b &\longmapsto k = \sqrt{\frac{1-b}{1+b}} \\ k &\longmapsto b = \frac{1-k^2}{1+k^2} \end{aligned} \tag{6}$$

is a bijective mapping from the set of elements $b \in K$ with $1-b^2$ a nonzero square onto the set of nonzero squares $k \in K^2$. Thus, with respect to the basis (e_1, e_2), B_W has the representation

$$\begin{pmatrix} a & ak \\ -ak & a \end{pmatrix},$$

for some $k \in K^2$, $k \neq 0$, and this representation can be turned in a straightforward way into

$$\begin{pmatrix} \varepsilon & k \\ -k & \varepsilon \end{pmatrix}, \tag{7}$$

where $\varepsilon = 1$ or $\varepsilon = -1$ according to whether H is positive or negative definite. By Theorem 4 in [**7**], equivalent K-bilinear forms have similar asymmetries, hence the parameter k in (7), arising via (6) from the minimal polynomial of σ_W, distinguishes the isometry class of (W, B_W).

Summing up, the restriction of the Hermitian form H to a two-dimensional indecomposable K-substructure has a representation of the shape

$$\begin{pmatrix} \varepsilon & ik \\ -ik & \varepsilon \end{pmatrix} \simeq \begin{pmatrix} \varepsilon k^{-1} & i \\ -i & \varepsilon k^{-1} \end{pmatrix} \tag{8}$$

for some $k \in K^2$, $k \neq 0$, with ε depending on the signature of H. We shall denote by $\mathbf{W}_k$ such a K-substructure of V.

IV. The above arguments say that a K-subspace $W \in X_d$ decomposes orthogonally into K-lines, $\overline{K}$-lines and two-dimensional K-substructures such as $\mathbf{W}_k$. Hence there is a decomposition $W = C \perp D \perp E$, where

• C is the largest $\overline{K}$-subspace contained in W, generated by mutually orthogonal vectors having H-norm ε,

• D is a K-substructure generated by mutually orthogonal vectors having H-norm ε,

• E is a K-substructure splitting into an orthogonal sum $E = \mathbf{W}_{k_1} \perp \cdots \perp \mathbf{W}_{k_q}$ for nonzero elements $k_1, \dots, k_q \in K$,

where $\varepsilon = 1$ or $\varepsilon = -1$ according to whether H is positive or negative definite. Let us term the set of parameters

$$\left(m = \dim_{\overline{K}} C,\ p = \dim_K D,\ q = \frac{1}{2} \dim_K E;\ k_1, \dots, k_q\right) \tag{9}$$

the *U-type* of W, where the q-tuple $(k_1, \dots, k_q)$ is ordered and $2m + p + 2q = d$. Then the Krull-Schmidt theorem allows one to state

8. Theorem. *Two K-subspaces $W', W'' \in X_d$ are in the same orbit for the action of U if and only if W' and W'' have the same U-type.*

Remarks. i) As there is no unipotent element in U, every orbit in X_d for the action of U is *negligible* in the sense of [**5**].

ii) As the K-bilinear symmetric form S is always either positive or negative definite (according to H) on any member of X_d, the group

$O(V_K, S)$ of isometries of the orthogonal space (V_K, S) acts in X_d with the same orbits as the group $\mathrm{GL}\,(V_K)$.

iii) If $W \in X_d$ has U-type (9), then the K-bilinear alternating form A restricts to W with rank $r = 2(m+q)$. Manifestly the group $\mathrm{Sp}\,(V_K, A)$ of isometries of the alternating space (V_K, A) acts in X_d with orbits $X_{d,r}$ consisting of all d-dimensional K-subspaces on which A induces an alternating form of rank r. Hence, if $r > 0$, there are infinitely many orbits for the action of U even in each of $X_{d,r}$, whereas U operates transitively on $X_{d,0}$, i.e., on the set of A-totally isotropic members of X_d.

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