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THE $\{K_i(z)\}_{i=1}^{\infty}$ FUNCTIONS

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ABSTRACT. In this paper we define and study the functions

$$K_i(z) = \frac{{}_1M_0(1;1,z+i-1) - {}_1M_0(1;1,i-1)}{{}_1M_{-1}(1;1,i)}, \quad i \in \mathbf{N},$$

where function ${}_vM_m(s;a,z)$ is defined in [**9**]. We give the recurrence relations, asymptotic and other properties. Also, we give the exponential generating function and representation of $K_i(n)$.

1. Introduction. In 1971, Professor Kurepa, see [**5, 6**], defined the *left factorial* $!n$ as the total number of nodes in a finite tree consisting of n levels with the kth level containing $k!$ nodes, $k = 0, 1, 2, \dots, n-1$. That is, Kurepa defined $!0 = 0$ and for $n \in \mathbf{N}$

$$!n = \sum_{k=0}^{n-1} k!$$

and extended it to the complex half-plane $\Re(z) > 0$ as

$$(1) \qquad !z = \int_0^{+\infty} \frac{t^z - 1}{t-1}\, e^{-t}\, dt.$$

This function can be extended analytically to the whole complex plane by

$$(2) \qquad !z = !(z+1) - \Gamma(z+1),$$

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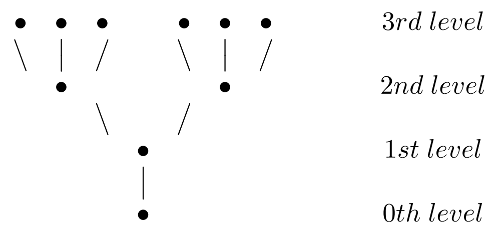

FIGURE 1. Kurepa's tree for $n = 4$.

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt, \quad \Re(z) > 0.$$

A detailed bibliography is given in Ivić and Mijajlović [**4**].

Milovanović in [**7**] defined and studied a sequence of the factorial functions $\{M_m(z)\}_{m=-1}^{+\infty}$ where $M_{-1}(z) = \Gamma(z)$ and $M_0(z) = !z$. Namely,

$$M_m(z) = \int_0^{+\infty} \frac{t^{z+m} - Q_m(t,z)}{(t-1)^{m+1}}\, e^{-t}\, dt, \quad \Re(z) > -(m+1), \tag{3}$$

where the polynomials $Q_m(t;z)$, $m = -1, 0, 1, 2, \dots$, are given by

$$Q_{-1}(t,z) = 0, \quad Q_m(t,z) = \sum_{k=0}^{m} \binom{m+z}{k} (t-1)^k.$$

For $m = -1, 0, 1, 2, \dots$, and $\Re(z) > v - m - 2$ the generalization of Milovanović's factorial function is given in [**9**]:

$$\begin{aligned} &{}_vM_m(s;a,z) \\ &\quad = \sum_{k=1}^{v} (-1)^{k-1} \binom{z+m+1-k}{m+1} \mathcal{L}[s;\, {}_2F_1(a, k-z, m+2; 1-t)], \end{aligned} \tag{4}$$

where v is a positive integer and $s,\ a,\ z$ are complex variables. The hypergeometric function ${}_2F_1(a,b,c;x)$ is defined by the series

$${}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

and has integral representation

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}\,dt, $$

in the x plane cut along the real axis from 1 to ∞, if $\Re(c) > \Re(b) > 0$. The symbols $(z)_n$ and $\mathcal{L}[s; F(t)]$ represent the Pochhammer symbol

$$ (z)_0 = 1, \quad (z)_n = z(z+1)\cdots(z+n-1) = \frac{\Gamma(z+n)}{\Gamma(z)}, $$

and Laplace transform

$$ \mathcal{L}[s; F(t)] = \int_0^{\infty} e^{-st} F(t)\,dt. $$

This function is of interest because its special cases include the following:

$$ (5) \qquad M_m(z) = {}_1M_m(1,1,z), \qquad \Gamma(z) = {}_1M_{-1}(1;1,z), $$

$$ (6) \qquad !z = {}_1M_0(1;1,z), \qquad A_n = {}_nM_{-1}(1;1,n+1), $$

$$ (7) \qquad \zeta(z) = \frac{1}{{}_1M_{-1}(1;1,z)} \sum_{n=1}^{\infty} {}_1M_{-1}(n;1,z), \quad \Re(z) > 1, $$

$$ (8) \qquad \left\{ {z \atop m} \right\} = {}_1M_{m-1}(1;0,z), $$

$$ (9) \quad \sum_{k=0}^{m} s(m,k)\cdot z^k = {}_1M_{m-1}(1;0,z-m+1)\cdot {}_1M_1(1;1,m+1), $$

$$ m \in \mathbf{N}, $$

where A_n, $\zeta(z)$, $\left\{ {z \atop m} \right\}$ and $s(n,m)$ are the alternating factorial numbers, the Riemann Zeta-function, the figured number, see [**2**], and the Stirling number of the first kind respectively, and are defined as

$$ A_n = \sum_{k=1}^{n} (-1)^{n-k} k!, $$

$$ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad \Re(z) > 1, $$

$$ \left\{ {z \atop m} \right\} = \binom{z+m-1}{m}, $$

$$ x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k) x^k. $$

However, apart from $n!$, $!n$ and A_n, 25 more well-known integer sequences in [**11**] are special cases of the function ${}_vM_m(s;a,z)$.

In this paper we define and study a sequence of functions $\{K_i(z)\}_{i=1}^{\infty}$. These functions satisfy the following equality, see Remark 3.2,

$$K_i(z) = \frac{{}_1M_0(1;1,z+i-1) - {}_1M_0(1;1,i-1)}{{}_1M_{-1}(1;1,i)}. \tag{10}$$

2. The $K_i(n)$ numbers. We now introduce a generalization of Kurepa's tree.

Definition 2.1. For $n \in \mathbf{N}$ and $i \in \mathbf{N}_0$, let $TK_i(n)$ denote a finite tree consisting of n levels with the kth level containing $(i)_k$ nodes, $k = 0, 1, 2 \ldots, n-1$. Let $K_i(n)$ denote the total number of nodes in tree $TK_i(n)$ and

$$K_0(n) \stackrel{\text{def}}{=} 1.$$

According to the previous definition we have

$$K_i(n) = \sum_{k=0}^{n-1} (i)_k, \tag{11}$$

$$K_i(n) = \frac{1}{(i-1)!} \sum_{k=i-1}^{n+i-2} k!, \tag{12}$$

$$K_i(n) = i \cdot K_{i+1}(n-1) + 1, \tag{13}$$

$$K_1(n) = !n. \tag{14}$$

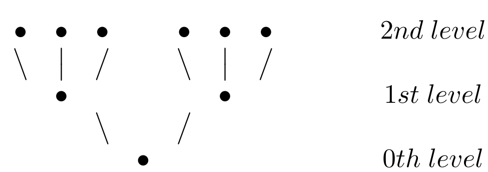

FIGURE 2. The tree $TK_2(3)$.

TABLE 1. The numbers $K_i(n)$ for $i = 0, 1, 2 \dots, 6$ and $n = 1, 2, \dots, 8$.

$i \setminus n$	1	2	3	4	5	6	7	8
0	1	1	1	1	1	1	1	1
1	1	2	4	10	34	154	874	5914
2	1	3	9	33	153	873	5913	46233
3	1	4	16	76	436	2956	23116	204556
4	1	5	25	145	985	7705	68185	672985
5	1	6	36	246	1926	17046	168246	1831446
6	1	7	49	385	3409	33649	366289	4357969

Remark 2.2. Six integer sequences in [**11**] are special cases of the function $K_i(n)$: $K_0(n) = K_i(1)$, $K_1(n)$, $K_2(n)$, $K_i(2)$, $K_i(3)$ and $K_i(4)$.

The functions $\{K_i(n)\}_{i=1}^{\infty}$ are periodical functions on (mod n), see Table 2. In this way we have the following statements:

Lemma 2.3. *For $i, j, n \in \mathbf{N}$, we have*

$$K_i(n) \equiv K_{i+jn}(n) \pmod{n}.$$

Proof. The relation

$$\prod_{k=i}^{i+t} k \equiv \prod_{k=i+n}^{i+t+n} k \pmod{n}, \quad 0 \le t \le n-1$$

yields

$$\sum_{k=0}^{n-1} (i)_k \equiv \sum_{k=0}^{n-1} (i+n)_k \pmod{n}$$
$$K_i(n) \equiv K_{i+n}(n) \pmod{n}$$

so that, after j steps we have

$$K_i(n) \equiv K_{i+jn}(n) \pmod{n}. \qquad \square$$

TABLE 2. The numbers $K_i(n) \equiv (\text{mod } n)$ for $i = 0, 1, 2 \ldots, 13$ and $n = 1, 2, \ldots, 7$.

$i \setminus n$	1	2	3	4	5	6	7
0	**0**	1	1	1	1	1	1
1	**0**	**0**	1	2	4	4	6
2	0	1	**0**	1	3	3	5
3	0	**0**	1	**0**	1	4	2
4	0	1	1	1	**0**	1	5
5	0	0	**0**	2	1	**0**	1
6	0	1	1	1	4	1	**0**
7	0	0	1	**0**	3	4	1
8	0	1	0	1	1	3	6
9	0	0	1	2	**0**	4	5
10	0	1	1	1	1	1	2
11	0	0	0	0	4	**0**	5
12	0	1	1	1	3	1	1
13	0	0	1	2	1	4	**0**

Corollary 2.4. it For $j, n \in \mathbf{N}$ we have

$$K_{jn-1}(n) \equiv 0 \pmod{n},$$
$$K_{jn-1}(n) \equiv 0 \pmod{j}.$$

Corollary 2.5. *For $i \in \mathbf{N}_0$ and $n \in \mathbf{N} \backslash \{1\}$, we have*

$$K_i(n) \equiv 0 \quad (\text{mod } i+1).$$

Using

$$K_{2n-3}(n) = \frac{1}{(2n-4)!} \sum_{k=2n-4}^{3n-5} k! \equiv \frac{1}{(2n-4)!} \sum_{k=2n-4}^{2n-1} k! \equiv -2 \pmod{n},$$

and

$$K_{2n-2}(n) = \frac{1}{(2n-3)!} \sum_{k=2n-3}^{3n-4} k! \equiv \frac{1}{(2n-3)!} \sum_{k=2n-3}^{2n-1} k! \equiv 1 \pmod{n},$$

we have

Lemma 2.6. *Let* $n \in \mathbf{N}\backslash\{1\}$. *Then*

$$K_{2n-3}(n) \equiv n-2 \pmod{n},$$
$$K_{2n-2}(n) \equiv 1 \pmod{n}.$$

2.1 Generating function for the numbers $\{K_i(n)\}_{i=0}^{\infty}$.

Definition 2.7. Let $n \in \mathbf{N}$. The exponential generating function of the sequence $\{K_i(n)\}_{i=0}^{+\infty}$ is given by

$$g_n(x) = \sum_{i=0}^{+\infty} K_i(n) \frac{x^i}{i!}. \tag{15}$$

Since

$$\frac{K_{i+1}(n)}{K_i(n)} = 1 + \frac{\sum_{k=0}^{n-1} k\Gamma(i+k)}{i\sum_{k=0}^{n-1}\Gamma(i+k)} \leq 1 + \frac{n-1}{i},$$

we have

$$1 \leq \frac{K_{i+1}(n)}{K_i(n)} \leq 1 + \frac{n-1}{i}$$

so that

$$\lim_{i\to\infty} \frac{K_{i+1}(n)}{K_i(n)} = 1,$$

which implies

$$\lim_{i\to\infty} \frac{K_{i+1}(n)i!}{K_i(n)(i+1)!} = 0,$$

and hence the expansion (15) converges for each $x \in \mathring{R}$.

Let

$$f(x) = [f(x)]^{(0)}, \quad \frac{d}{dx} f(x) = [f(x)]^{(1)}, \quad \frac{d}{dx}\left(\frac{d}{dx} f(x)\right) = [f(x)]^{(2)}, \dots .$$

Then

Theorem 2.8. *For* $n \in \mathbf{N}$ *we have*

$$g_n(x) = e^x + \sum_{k=1}^{n-1} [e^x x^k]^{(k-1)}.$$

Proof. For $n = 1$, the equality holds, since

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

For $n > 1$, from

$$g_{n+1}(x) = g_n(x) + \sum_{i=0}^{\infty} (i)_n \frac{x^i}{i!} \quad \text{and} \quad \sum_{i=0}^{\infty} (i)_n \frac{x^i}{i!} = [e^x x^n]^{(n-1)}$$

it follows

$$g_{n+1}(x) = g_n(x) + [e^x x^n]^{(n-1)}. \qquad \square$$

3. $\{K_i(z)\}_{i=1}^{\infty}$ function in the complex domain.

3.1 Basic definition and properties. Applying relation (12) these numbers can be expressed in terms of the gamma function as follows

$$\begin{aligned} K_i(n) &= \frac{1}{(i-1)!} \sum_{k=i-1}^{n+i-2} \Gamma(k+1) = \frac{1}{(i-1)!} \int_0^{\infty} e^{-x} \sum_{k=i-1}^{n+i-2} x^k \, dx \\ &= \frac{1}{(i-1)!} \int_0^{\infty} e^{-x} x^{i-1} \frac{x^n - 1}{x - 1} \, dx. \end{aligned}$$

The same relation is now used in order to define the function $K_i(z)$:

Definition 3.1. For every complex number $\Re(z) > 0$ and $i \in \mathbf{N}$, the function $K_i(z)$ is defined by

$$K_i(z) \stackrel{\text{def}}{=} \frac{1}{(i-1)!} \int_0^{\infty} e^{-x} x^{i-1} \, \frac{x^z - 1}{x - 1} \, dx.$$

Remark 3.2. Applying Definition 3.1 and relations (1), (5) and (6), we have equality (10).

The identity $(x^z - 1)/(x - 1) = (x^{z+1} - 1)/(x - 1) - x^z$ gives

$$\frac{1}{(i-1)!} \int_0^{\infty} e^{-x} x^{i-1} \, \frac{x^z - 1}{x - 1} \, dx = \frac{1}{(i-1)!} \int_0^{\infty} e^{-x} x^{i-1} \, \frac{x^{z+1} - 1}{x - 1} \, dx - \frac{1}{(i-1)!} \int_0^{\infty} e^{-x} x^{z+i-1} \, dx,$$

i.e.,

$$K_i(z) = K_i(z+1) - \frac{\Gamma(z+i)}{(i-1)!}. \tag{17}$$

On the basis of (17) and $\Gamma(z+1) = z\Gamma(z)$, functional equality

$$K_i(z+1) = (z+i)K_i(z) - (z+i-1)K_i(z-1), \quad i \in \mathbf{N}_0 \tag{18}$$

is valid.

Theorem 3.3. *For the function $K_i(z)$, the set of poles is*

$$P_{K_i} = \{-i, \, -i-2, \, -i-3, \, -i-4, \, \ldots \, , \}.$$

The infinite point is an essential singularity and every pole $z_p \in P_{K_i}$ is simple with residue

$$\operatorname{res} K_i(z_p) = \frac{1}{(i-1)!} \sum_{k=i}^{-z_p} \frac{(-1)^{k-i+1}}{(k-i)!}, \quad z_p \in P_{K_i}.$$

Proof. By means of the relation (17) and induction on $n \in \mathbf{N}$,

(19)

$$K_i(-n+\varepsilon) = K_i(\varepsilon) - \frac{1}{(i-1)!}\sum_{k=1}^{i-1}\Gamma(i-k+\varepsilon) - \frac{1}{(i-1)!}\sum_{k=i}^{n}\Gamma(i-k+\varepsilon)$$

is obtained. Hence, according to the well-known relation

$$\Gamma(z) = \frac{(-1)^m}{m!}\left[\frac{1}{z+m} + \psi(m+1) + \frac{z+m}{2} \times \left(\frac{\pi^2}{3} + \psi(m+1)^2 - \psi'(m+1)\right) + O[(z+m)^2]\right]$$

for $z = i-k+\varepsilon$ and $m = k-i$, we have

$$K_i(-n+\varepsilon) = K_i(\varepsilon) + A_i(\varepsilon) + \frac{B_i(n)}{\varepsilon} + C_i(n) + \varepsilon D_i(n) + O(\varepsilon^2), \tag{20}$$

where the functions $\psi(x)$, $A_i(\varepsilon)$, $B_i(n)$, $C_i(n)$, $D_i(n)$ are given by

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x),$$

$$A_i(\varepsilon) = -\frac{1}{(i-1)!}\sum_{k=1}^{i-1}\Gamma(i-k+\varepsilon) \tag{21}$$

$$B_i(n) = \frac{1}{(i-1)!}\sum_{k=i}^{n}\frac{(-1)^{k-i+1}}{\Gamma(k-i+1)},$$

$$C_i(n) = \frac{1}{(i-1)!}\sum_{k=i}^{n}\frac{(-1)^{k-i+1}\psi(k-i+1)}{\Gamma(k-i+1)}, \tag{22}$$

$$D_i(n) = \frac{1}{(i-1)!}\sum_{k=i}^{n}\frac{(-1)^{k-i+1}}{2\Gamma(k-i+1)} \times \left(\frac{\pi^2}{3} + \psi(k-i+1)^2 - \psi'(k-i+1)\right). \tag{23}$$

Using (17) we have

$$K_i(0) = K_i(1) - \Gamma(i)/(i-1)! = 0, \quad \text{i.e.,} \quad \lim_{\varepsilon\to 0} K_i(\varepsilon) = 0. \tag{24}$$

From (20)–(24) it follows that for $z = -i-1$, function $K_i(z)$ does not have the simple pole

$$K_i(-i-1+\varepsilon) = K_i(\varepsilon) + \frac{1}{(i-1)!} + \frac{\varepsilon}{(i-1)!}(1-\gamma) + O(\varepsilon^2), \tag{25}$$

where γ is Euler's constant. For other values $z_p \in P_{K_i}$, $B_i(-z_p) \neq 0$. An inductive argument shows that

$$K_i(-n) = -\frac{1}{(i-1)!}\sum_{k=1}^{n}\Gamma(i-k), \tag{26}$$

the residue of $K_i(z)$ at the pole $z_p \in P_{K_i}$ is given by

$$\operatorname{res} K_i(z_p) = \frac{1}{(i-1)!}\sum_{k=i}^{-z_p}\frac{(-1)^{k-i+1}}{(k-i)!}.$$

The derivation employs the fact that $\Gamma(z)$ is meromorphic with simple poles at $z=-n$ and residue $(-1)^n/n!$ there. □

Remark 3.4. Thus

$$\lim_{-z_p\to\infty}\operatorname{res} K_i(z_p) = -\frac{e^{-1}}{(i-1)!}.$$

Remark 3.5. For $i > n \in \mathbf{N}$, we have

$$K_i(-n) = \frac{1}{(i-1)!}\int_0^{\infty} e^{-x}x^{i-1}\,\frac{x^{-n}-1}{x-1}\,dx = -\frac{(i-n-1)!}{(i-1)!}\,K_{i-n}(n).$$

3.2 Asymptotic relations of $K_i(x)$.

Theorem 3.6. *For $i \in \mathbf{N}$ and $x \in \mathring{R}$, we have*

$$\lim_{x\to\infty}\frac{K_i(x)}{\Gamma(x+i-1)} = \frac{1}{(i-1)!},$$
$$\lim_{x\to\infty}\frac{K_i(x)}{\Gamma(x+i)} = 0.$$

Proof. For $i = 1$ the theorem is true, see [**6**, p. 299]. For $i > 1$ on the basis of (17) and

$$\begin{aligned} K_{i+1}(z) - \frac{K_i(z)}{i} &= \frac{1}{i!}\int_0^\infty e^{-x}x^{i-1}(x^z - 1)\,dx \\ &= \frac{1}{i!}\int_0^\infty e^{-x}x^{z+i-1}\,dx - \frac{1}{i!}\int_0^\infty e^{-x}x^{i-1}\,dx \\ &= \frac{\Gamma(z+i)}{i!} - \frac{1}{i}, \end{aligned}$$

we have

$$K_{i+1}(z) = \frac{K_i(z) - 1}{i} + \frac{\Gamma(i+z)}{i!}, \tag{27}$$

so that

$$\begin{aligned} \lim_{x\to\infty} \frac{K_i(x)}{\Gamma(x+i-1)} &= \frac{1}{(i-1)!} + \lim_{x\to\infty} \frac{K_{i-1}(x)}{(i-1)\Gamma(x+i-1)} \\ &= \frac{1}{(i-1)!} + \lim_{x\to\infty} \frac{K_{i-2}(x)}{(i-1)(i-2)\Gamma(x+i-1)} \\ &\ \vdots \\ &= \frac{1}{(i-1)!} + \lim_{x\to\infty} \frac{K_1(x)}{(i-1)!\Gamma(x+i-1)} = \frac{1}{(i-1)!}. \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{x\to\infty} \frac{K_i(x)}{\Gamma(x+i)} &= \lim_{x\to\infty} \frac{K_i(x)}{(x+i-1)\Gamma(x+i-1)} \\ &= \lim_{x\to\infty} \frac{1}{(x+i-1)(i-1)!} = 0. \qquad \square \end{aligned}$$

3.3 The representations of the function $K_i(x)$. Let the exponential integral $E_n(z)$ be defined by

$$E_n(z) = \int_1^\infty e^{-zt}t^{-n}\,dt \quad \Re(z) > 0.$$

Applying relation (24) and the well-known relation

$$E_{n+1}(z) = \frac{1}{n}\Big(e^{-z} - zE_n(z)\Big)$$

and induction on $n \in \mathbf{N}_0$, the function $K_i(n)$ can be expressed in terms of the exponential integral by

$$K_i(n) = e^{-1}\left(E_{i+n}(-1)\,\frac{(i+n-1)!}{(i-1)!} - E_i(-1)\right). \tag{28}$$

The relation $(-1)^{i-1}\Gamma(1-i,-1) = E_i(-1)$ yields

$$K_i(n) = (-1)^i e^{-1}\left[\Gamma(1-i,-1) - (-1)^n\Gamma(1-i-n,-1)\frac{(i+n-1)!}{(i-1)!}\right], \tag{29}$$

where $\Gamma(z,x)$ is the incomplete gamma function defined via

$$\Gamma(z,x) = \int_x^{+\infty} t^{z-1}e^{-t}\,dt.$$

Question 3.7. For $x \in \mathring{R}$ is it correct that

$$K_i(x) = (-1)^i e^{-1}\left[\Gamma(1-i,-1) - (-1)^x\frac{\Gamma(1-i-x,-1)\Gamma(i+x)}{(i-1)!}\right]?$$

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