ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

THE WEAK CHANG-MARSHALL INEQUALITY VIA GREEN'S FORMULA

MIROSLAV PAVLOVIĆ AND DRAGAN VUKOTIĆ

ABSTRACT. We prove the uniform Trudinger-Moser type inequality of Chang and Marshall for the Dirichlet space when $\alpha < 1$ by using only Green's formula instead of Beurling's deep inequalities.

1. Introduction. In this note we present a very short proof of the weak Chang-Marshall inequality based only on Green's formula for the disk and a standard growth estimate for the functions in the Dirichlet space $\mathcal{D}$ of the disk. By the *weak Chang-Marshall inequality* we mean the uniform estimate

$$(1)\qquad \sup\left\{\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta : \|f\|_{\mathcal{D}} \le 1,\ f(0)=0\right\} < \infty, \quad \alpha < 1.$$

This is a complex variable case of the well-known inequalities of Trudinger-Moser type. The uniform estimate (1) no longer holds when $\alpha > 1$. Its proof in the critical case $\alpha = 1$ was a deep result of Chang and Marshall [**3**] and provided an answer to a question stated on page 1079 of Moser's influential paper [**6**]. See also [**5**] for a simplified proof and [**2**] for more details and the vast literature on this topic and its relations with geometry.

The weak Chang-Marshall inequality is certainly easier to prove than the case $\alpha = 1$. However, its proofs that one encounters in the literature are based on the following deep uniform estimate from Beurling's thesis [**1**]:

$$(2)\quad \text{If } f \in \mathcal{D},\ \|f\|_{\mathcal{D}} \le 1, \quad \text{and} \quad f(0)=0, \quad \text{then} \quad |E_\lambda| \le e^{-\lambda^2+1}.$$

Here $E_\lambda = \{\theta \in [0,2\pi] : |f(e^{i\theta})| > \lambda\}$ and $|E_\lambda|$ is its normalized arc measure on the unit circle $\mathbf{T}$. Namely, a generalization of the basic

The first author is partially supported by MNTR grant No. 1863, Serbia. The second author is partially supported by MCyT grant BFM2003-07294-C02-01, Spain.
Received by the editors on December 2, 2003.

lemma on the first page of Chapter VIII of [**4**] and (2) together yield

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,\frac{d\theta}{2\pi} = 1 + 2\alpha\int_0^\infty \lambda e^{\alpha\lambda^2}|E_\lambda|\,d\lambda < \infty \tag{3}$$

for any $\alpha < 1$.

Our approach is much simpler: it relies only on the growth estimate for Dirichlet functions and Green's identity. Even though we are not able to cover the case $\alpha = 1$ as the papers [**3**] or [**5**] did, this still appears to be a novelty in the literature on the subject.

1. The proof via Green's identity. Throughout this note, $D(z,r)$ will denote the disk of radius r centered at z and $\mathbf{D} = D(0,1)$ the unit disk. We will use the notation $d\,A = (\pi)^{-1}\,dx\,dy$ for the normalized area measure so that $A(\mathbf{D}) = 1$ instead of π.

The *Dirichlet space* $\mathcal{D}$ is the Hilbert space of analytic functions with finite area integral, whose norm is given by

$$\|f\|_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2\,dA(z) = |a_0|^2 + \sum_{n=1}^\infty n|a_n|^2, \tag{4}$$

where f has the Taylor series $\sum_{n=0}^\infty a_n z^n$ in $\mathbf{D}$. Every function in $\mathcal{D}$ satisfies the (sharp) pointwise inequality:

$$|f(\zeta) - f(0)| \le \|f\|_{\mathcal{D}}\sqrt{\log\frac{1}{1-|\zeta|^2}}. \tag{5}$$

This follows by applying the Cauchy-Schwarz inequality to the Taylor series of f. One of the earliest sources that quotes this fact appears to be [**7**, pp. 218–219]. This estimate and Green's theorem in the Littlewood-Paley form will suffice to prove the Chang-Marshall inequality when $0 < \alpha < 1$.

Theorem 1. *For every positive value $\alpha < 1$, we have*

$$\sup\left\{\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,d\theta : \|f\|_{\mathcal{D}} \le 1,\ f(0) = 0\right\} < \infty.$$

Proof. Fix $\alpha < 1$. Let $f \in \mathcal{D}$, $\|f\|_{\mathcal{D}} \leq 1$ and $f(0) = 0$. Consider the function

$$W_f(z) = \exp(\alpha|f(z)|^2) - 1.$$

Its dilatations $W_{f,r}$, defined by $W_{f,r}(z) = W_f(rz)$, vanish at the origin and belong to $\mathcal{C}^\infty(\overline{\mathbf{D}})$, so we may apply the first lemma in Section D.1, Chapter X of [**4**] to get

$$\int_0^{2\pi} W_f(re^{i\theta})\, d\theta = \pi \int_{\mathbf{D}} \log \frac{1}{|z|} \cdot r^2 \cdot (\Delta W_f)(rz)\, dA(z). \tag{6}$$

A straightforward computation of the Laplacian of W_f yields:

$$\Delta W_f = 4\partial\overline{\partial} \exp(\alpha|f|^2) = 4\alpha|f'|^2(1 + \alpha|f|^2)\exp(\alpha|f|^2).$$

Since by Fatou's lemma we have

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\, d\theta \leq 2\pi + \liminf_{r\to 1^-} \int_0^{2\pi} W_f(re^{i\theta})\, d\theta,$$

the theorem will follow from (6) if we can show that the integrals over $\mathbf{D}$ of the functions

$$U_{f,r}(z) = \log \frac{1}{|z|} \cdot |f'(rz)|^2(1 + \alpha|f(rz)|^2)\exp(\alpha|f(rz)|^2)$$

are all finite and bounded by the same constant (independent of r) for each f as specified above. This can be done easily as follows.

By (5) and by our assumptions that $f(0) = 0$ and $\|f\|_{\mathcal{D}} \leq 1$, we obtain

$$\begin{aligned} U_{f,r}(z) &\leq \log \frac{1}{|z|} \cdot \frac{1 + \alpha \log 1/(1 - r^2|z|^2)}{(1 - r^2|z|^2)^\alpha} \cdot |f'(rz)|^2 \\ &\leq \log \frac{1}{|z|} \cdot \frac{1 + \alpha \log 1/1 - |z|}{(1 - |z|)^\alpha} \cdot |f'(rz)|^2. \end{aligned}$$

For R sufficiently close to one, $\log(1/|z|) \asymp 1 - |z|$ whenever $R < |z| < 1$. Since $\alpha < 1$, we get

$$U_{f,r}(z) \leq |f'(rz)|^2 \quad \text{on some annulus} \quad A_R = \{z : R < |z| < 1\}.$$

It is well known that $M_2^2(r,f') = (2\pi)^{-1}\int_0^{2\pi} |f'(re^{i\theta})|^2\,d\theta$ is an increasing function of r, hence

$$\begin{aligned}\int_{A_R} U_{f,r}\,dA &\le \int_{\mathbf{D}} |f'(rz)|^2\,dA(z) = 2\int_0^1 M_2^2(r\rho,f')\,\rho\,d\rho \\ &\le 2\int_0^1 M_2^2(\rho,f')\,\rho\,d\rho = \|f\|_{\mathcal{D}} \le 1.\end{aligned} \tag{7}$$

On the other hand, the area version of the sub-mean value property yields

$$\begin{aligned}(1-R)^2|f'(rz)|^2 \le (1-r|z|)^2|f'(rz)|^2 &\le \int_{D(rz,1-r|z|)} |f'|^2\,dA \\ &\le \|f\|_{\mathcal{D}}^2 \le 1\end{aligned}$$

whenever $|z|\le R$. Hence

(8)

$U_{f,r}(z) \le M_R \log(1/|z|)$ *on the punctured closed disk* $\overline{D(0,R)}\setminus\{0\}$,

where M_R is a constant that depends only upon R.

From (7) and (8) we finally obtain

$$\int_{\mathbf{D}} U_{f,r}(z)\,dA(z) \le 1 + M_R \int_{\overline{D(0,R)}} \log\frac{1}{|z|}\,dA$$

for all $r\in(0,1)$ and all f such that $\|f\|_{\mathcal{D}}\le 1$ and $f(0)=0$, which is what was needed. $\square$

2. Some concluding remarks. We remind the reader that we actually have

$$\int_0^{2\pi} e^{\alpha|f(e^{i\theta})|^2}\,\frac{d\theta}{2\pi} < \infty, \quad \text{whenever} \quad f\in\mathcal{D}, \quad 0<\alpha<\infty. \tag{9}$$

This observation, due to J.B. Garnett, can be found in [**3**, p. 1016]. We point out that our approach can be adapted to yield a direct proof of (9) as follows. First observe from (4) that every $f\in\mathcal{D}$ can be approximated in the norm by its Taylor polynomials. It follows from here that, for every fixed f in the Dirichlet space, we have the "little-oh" estimate

$$|f(z)| = o\left(\left(\log \frac{1}{1-|z|}\right)^{1/2}\right), \quad \text{as} \quad |z| \to 1^-.$$

This will make it possible in the estimates for the functions $U_{f,r}$ in the proof above to replace α in every appearance by $\alpha\varepsilon^2$, where $\varepsilon > 0$ is chosen so that $\alpha\varepsilon^2 < 1$. This immediately yields the proof for arbitrary positive α.

Finally, we should point out the limitations of our method, as far as proving the uniform estimate of Chang-Marshall with $\alpha = 1$ is concerned. To this end, it would be desirable in that case (and it seems crucial by inspecting the proof of Theorem 1 above) to have a uniform estimate such as

$$\frac{|f(z)|^2}{\log 1/(1-|\zeta|^2)} < \varepsilon$$

in some annulus $\{z : R < |z| < 1\}$ and for some $\varepsilon < 1$. Unfortunately, this is impossible due to the sharpness of the "big-Oh" estimate (5). Namely, for every point ζ in the unit disk we can still find a function f of norm one and vanishing at the origin so that

$$|f(\zeta)|^2 = \log \frac{1}{1-|\zeta|^2}.$$

The function $f_\zeta(z) = \log 1/(1-\overline{\zeta}z)$ does the trick. In summary, when $\alpha = 1$, a more subtle approach seems to be required, as in [**3, 5**].

Note added in proof. After the acceptance of this paper for publication, we learned that A. Aleman and A.M. Simbotin have also used Green's forumla to obtain some related results for a general class of function spaces in their paper, *Estimates in Möbius invariant spaces of analytic functions*, Complex Var. Theory Appl. **49**, no. 7–9 (2004), 487–510.

REFERENCES

1. A. Beurling, *Études sur un problème de majoration*, Thèse pour le Doctorat, Almquist & Wieksell, Upsalla, 1933.

2. S.-Y. A. Chang, *The Moser-Trudinger inequality and applications to some problems in conformal geometry*, in *Nonlinear partial differential equations in differential geometry* (R. Hardt and M. Wolf, eds.), (Park City, Utah 1992), IAS/Park City Math. Ser., vol. 2, Amer. Math. Soc., Providence, RI, 1996, pp. 65–125.

3. S.-Y. A. Chang and D.E. Marshall, *On a sharp inequality concerning the Dirichlet integral*, Amer. J. Math. **107** (1985), 1015–1033.

4. P. Koosis, *Introduction to H_p spaces*, 2nd ed., Cambridge Univ. Press, Cambridge, 1998.

5. D. Marshall, *A new proof of a sharp inequality concerning the Dirichlet integral*, Ark. Mat. **27** (1989), 131–137.

6. J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. **20** (1970/71), 1077–1092.

7. H.S. Shapiro and A.L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. **80** (1962), 217–229.

Matematički Fakultet, PP. 550, 11000 Belgrade, Serbia
E-mail address: pavlovic@poincare.matf.bg.ac.yu

Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain
E-mail address: dragan.vukotic@uam.es