ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

A SUMMATION FORMULA FOR SEQUENCES INVOLVING FLOOR AND CEILING FUNCTIONS

M.A. NYBLOM

ABSTRACT. A closed form expression for the Nth partial sum of the pth powers of $\|\sqrt{n}\|$ is obtained, where $\|\cdot\|$ denotes the nearest integer function. As a consequence, a necessary and sufficient condition for the divisibility of n by $\|\sqrt{n}\|$ is derived together with a closed form expression for the least nonnegative residue of n modulo $\|\sqrt{n}\|$. In addition an identity involving the zeta function $\xi(s)$ and the infinite series $\sum_{n=1}^{\infty} 1/\|\sqrt{n}\|^{s+1}$ for real $s>1$ is also obtained.

1. Introduction. In a recent paper, see [**3**], the author examined the problem of determining a closed form expression for those sequences $\langle b_m\rangle$ formed from an arbitrary sequence of real numbers $\langle a_n\rangle$ in the following manner. Let $d\in\mathbf{N}$ be fixed, and for each $m\in\mathbf{N}$ define b_m to be the mth term of the sequence consisting of nd occurrences in succession of the terms a_n, as follows:

$$\underbrace{a_1,\dots,a_1}_{d,\ a_1\ \text{terms}},\underbrace{a_2,\dots,a_2}_{2d,\ a_2\ \text{terms}},\underbrace{a_3,\dots,a_3}_{3d,\ a_3\ \text{terms}},\dots. \tag{1}$$

For example, if $a_n=n$ and $d=1$ then the resulting sequence $\langle b_m\rangle$ would be

$$1,2,2,3,3,3,4,4,4,4,\dots.$$

Specifically, the problem described above required the construction of a function $f:\mathbf{N}\to\mathbf{N}$ such that $b_m=a_{f(m)}$. As was shown in [**3**] the required function $f(\cdot)$ can easily be described in terms of a combination of floor and ceiling functions, that is the functions defined as $\lfloor x\rfloor=\max\{n\in\mathbf{Z}:n\le x\}$ and $\lceil x\rceil=\min\{n\in\mathbf{Z}:x\le n\}$ respectively. In particular, for the sequence in (1), we have that $b_m=a_{f(m)}$ where

$$f(m)=\left\lfloor\sqrt{\left\lceil\frac{2m}{d}\right\rceil}+\frac{1}{2}\right\rfloor. \tag{2}$$

Received by the editors on November 12, 2003.

In this note we continue our examination of those sequences defined in (1) by deriving a summation formula for the Nth partial sum $S_N = \sum_{m=1}^{N} b_m$. Our goal here will be to deduce, as a consequence of the aforementioned formula, a closed form expression for the partial sum of the pth powers of $\|\sqrt{n}\|$, where $\|x\|$ denotes the nearest integer to x. In particular, as special cases it will be shown that

$$\sum_{n=1}^{N} \frac{1}{\|\sqrt{n}\|} = \frac{N}{\|\sqrt{N}\|} + \|\sqrt{N}\| - 1 \tag{3}$$

$$\sum_{n=1}^{N} \|\sqrt{n}\| = \frac{\|\sqrt{N}\|}{3}\,(3N + 1 - \|\sqrt{N}\|^2). \tag{4}$$

As will be seen, the method used to establish (3) and (4) is quite different from that employed in establishing a closed form expression for $\sum_{n=1}^{N} \lfloor \sqrt{n} \rfloor$ as demonstrated in [**1**, p. 86]. In addition, as a consequence of (3), a necessary and sufficient condition will be derived for the divisibility of N by $\|\sqrt{N}\|$, together with a closed form expression for the least nonnegative residue of N modulo $\|\sqrt{N}\|$.

2. Main result. We begin with a technical result which will help facilitate the calculation of the Nth partial sum of the sequences defined in (1).

Lemma 2.1. *Suppose $\langle a_n \rangle$ is an arbitrary sequence of real numbers, and let $d \in \mathbf{N}$. Then, for the sequence $\langle b_m \rangle$ defined in* (1), *we have*

$$S_N = \sum_{m=1}^{N} b_m = \Big(N - \frac{d}{2}\,(f(N)-1)f(N)\Big)a_{f(N)} + d \sum_{n=1}^{f(N)-1} n a_n, \tag{5}$$

where $f(\cdot)$ is the function in (2).

Proof. For the sequence defined in (1), we have $b_m = a_n$, whenever $n(n-1)d/2 < m \leq n(n+1)d/2$, that is, $f(m) = n$ when

$$m \in I_n = \left[\frac{n(n-1)}{2}\,d + 1 \;\; \frac{n(n+1)}{2}\,d\right].$$

Now defining the mapping $S : \mathbf{N} \to \mathbf{N}$ by $S(N) = \max\{n \in \mathbf{N} : N \notin \cup_{r=1}^{n} I_r\}$ and noting that each interval I_n contains nd integers, observe the following

$$
\begin{aligned}
S_N &= \sum_{n=1}^{S(N)} \sum_{r \in I_n} a_{f(r)} + \sum_{\substack{r \in I_{S(N)+1} \\ r \leq N}} a_{f(r)} \\
&= \sum_{n=1}^{S(N)} n\, d a_n + \sum_{\substack{r \in I_{S(N)+1} \\ r \leq N}} a_{S(N)+1}.
\end{aligned} \tag{6}
$$

Our task is thus reduced to determine a closed form expression for $S(N)$ in terms of N and so evaluate the second summation in (6). Suppose $N \in I_n$ for some $n \in \mathbf{N}$, then by definition of I_n,

$$\frac{n(n-1)}{2}\, d < N \leq \frac{n(n+1)}{2}\, d. \tag{7}$$

Now if, for some $x \in \mathbf{R}^+$ we have $n_1 < x \leq n_2$ for $n_1, n_2 \in \mathbf{N}$, then $n_1 + 1 \leq \lceil x \rceil \leq n_2$. Consequently, from the inequality in (7) we have $n(n-1) + 1 \leq \lceil 2N/d \rceil \leq n(n+1)$. However, as $\sqrt{n(n+1)} < n + 1/2$ and $n - 1/2 < \sqrt{n(n-1)+1}$, one in turn deduces that

$$n < \sqrt{\left\lceil \frac{2N}{d} \right\rceil} + \frac{1}{2} < n + 1.$$

Thus, we have $n = f(N)$ and so $S(N) = f(N) - 1$. Finally as the number of integers $r \in I_{S(N)+1} = I_{f(N)}$ with $r \leq N$ is given by

$$N - f(N)(f(N)-1)\,\frac{d}{2} - 1 + 1,$$

one sees that the second summation in (6) is equal to

$$\sum_{\substack{r \in I_{f(N)} \\ r \leq N}} a_{f(N)} = \Big(N - \frac{d}{2}\,(f(N)-1)f(N)\Big) a_{f(N)}.$$

Hence (6) yields (5) as required. $\square$

Before establishing the main result it should be noted that the mapping $x \mapsto \|x\|$ is strictly, by definition, multi-valued at $x = (2n+1)/2$, where $n \in \mathbf{N}$, since $(2n+1)/2$ lies a distance of $1/2$ units from n and $n+1$. In such cases the convention, as in [**2**, p. 78], is to set $\|(2n+1)/2\| = n+1$. However this ambiguity does not arise for the mapping $N \mapsto \|\sqrt{N}\|$, where $N \in \mathbf{N}$, as $\sqrt{N} \neq (2n+1)/2$ for any $n \in \mathbf{N}$. We now prove our main result for summing the pth powers of $\|\sqrt{n}\|$, from which (3) and (4) will follow as a corollary.

Theorem 2.1. *Suppose $p \in \mathbf{R}$, then*

$$\sum_{n=1}^{N} \|\sqrt{n}\|^p = \big(N - (\|\sqrt{N}\| - 1)\,\|\sqrt{N}\|\big)\,\|\sqrt{N}\|^p + 2\sum_{n=1}^{\|\sqrt{N}\|-1} n^{p+1}. \tag{8}$$

In particular, when $p \in \mathbf{N}$, then

$$\sum_{n=1}^{N} \|\sqrt{n}\|^p = \big(N - (\|\sqrt{N}\| - 1)\,\|\sqrt{N}\|\big)\,\|\sqrt{N}\|^p + \frac{2}{p+2}\sum_{k=0}^{p+2}\binom{p+2}{k} B_k \|\sqrt{N}\|^{p+2-k},$$

where B_k denotes the kth Bernoulli number.

Proof. We first show that $\|x\| = \lfloor x + 1/2 \rfloor$ for every $x \in \mathbf{R}^+$. Indeed suppose $\|x\| = n$, taking the largest if two are equally distant. Setting $n = x + \theta$ with $-1/2 < \theta \le 1/2$, observe $\lfloor x + 1/2 \rfloor = n + \lfloor -\theta + 1/2 \rfloor = n$ since $0 \le -\theta + 1/2 < 1$. Consequently, $\|\sqrt{m}\| = \lfloor \sqrt{m} + 1/2 \rfloor$ and so from (2) we deduce that the sequence $\langle \|\sqrt{m}\|^p \rangle$ corresponds to the sequence $\langle b_m \rangle$ defined in (1), with $a_n = n^p$ and $d = 2$. Hence, in this instance, we see that (5) reduces to (8) as required. Finally if $p \in \mathbf{N}$, then the second equality follows immediately from the identity

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k n^{m+1-k}. \qquad \square$$

We now examine (8) in the case when $p = \pm 1$.

Corollary 2.1.

$$\sum_{n=1}^{N} \|\sqrt{n}\|^p = \begin{cases} N\|\sqrt{N}\|^{-1} + \|\sqrt{N}\| - 1 & \text{for } p = -1 \\ (\|\sqrt{N}\|/3)(3N + 1 - \|\sqrt{N}\|^2) & \text{for } p = 1. \end{cases}$$

Proof. Setting $p = -1$ in (8), observe that

$$\begin{aligned} \sum_{n=1}^{N} \|\sqrt{n}\|^{-1} &= \left(N - (\|\sqrt{N}\| - 1)\,\|\sqrt{N}\|\right)\|\sqrt{N}\|^{-1} + 2\sum_{n=1}^{\|\sqrt{N}\|-1} 1 \\ &= N\|\sqrt{N}\|^{-1} + \|\sqrt{N}\| - 1. \end{aligned}$$

Similarly, setting $p = 1$ in (8) and recalling $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$, one arrives, after some simplification, at the second formula. $\square$

Using the summation formula in (3) we can deduce the following divisibility property.

Corollary 2.2. *Suppose $N \in \mathbf{N}$. Then $\|\sqrt{N}\|$ divides N if and only if either $N = \|\sqrt{N}\|^2$ or $N = \|\sqrt{N}\|(\|\sqrt{N}\| + 1)$. Moreover, the least nonnegative residue of N modulo $\|\sqrt{N}\|$ is given by*

$$N - \|\sqrt{N}\|^2 + \frac{\|\sqrt{N}\|}{2}\Big((-1)^{\lfloor (N/\|\sqrt{N}\|) - \|\sqrt{N}\| + 1\rfloor} + 2(-1)^{\lfloor 1/2\left((N/\|\sqrt{N}\|) - \|\sqrt{N}\| + 1\right)\rfloor} - 1\Big).$$

Proof. From the summation formula in (3) it is immediate that $\|\sqrt{N}\|$ divides N if and only if $\sum_{n=1}^{N} 1/\|\sqrt{n}\|$ is an integer. Recalling that $\|\sqrt{m}\| = \lfloor \sqrt{m} + 1/2 \rfloor$, we deduce that the sequence $\langle \|\sqrt{m}\|^{-1} \rangle$ corresponds to the sequence $\langle b_m \rangle$ defined in (1), with $a_n = 1/n$ and $d = 2$. Consequently, from (6) we find that

$$(9) \qquad \sum_{n=1}^{N} \frac{1}{\|\sqrt{n}\|} = 2\left(\|\sqrt{N}\| - 1\right) + \sum_{\substack{r \in I_{\|\sqrt{N}\|} \\ r \le N}} \frac{1}{\|\sqrt{N}\|},$$

and so our task is reduced to determining those $N \in I_{\|\sqrt{N}\|}$ for which the summation on the righthand side of (9) is integer valued. Now since the interval $I_{\|\sqrt{N}\|}$ contains $2\|\sqrt{N}\|$ integers we see that

$$\frac{1}{\|\sqrt{N}\|} \leq \sum_{\substack{r \in I_{\|\sqrt{N}\|} \\ r \leq N}} \frac{1}{\|\sqrt{N}\|} \leq 2.$$

Furthermore, as the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ is equal to $N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1)$, we conclude that the summation in question assumes the integer values of 1 and 2 if and only if $N - \|\sqrt{N}\|(\|\sqrt{N}\| - 1) = \|\sqrt{N}\|$ and $2\|\sqrt{N}\|$, respectively. Hence, $\|\sqrt{N}\|$ divides N if and only if either $N = \|\sqrt{N}\|^2$ or $N = \|\sqrt{N}\|(\|\sqrt{N}\| + 1)$.

Denote the number of integers $r \in I_{\|\sqrt{N}\|}$ with $r \leq N$ by $R(N)$. After equating (3) with (9) and solving for $N/\|\sqrt{N}\|$, observe from the argument above that the least nonnegative residue of N modulo $\|\sqrt{N}\|$ is equal to $R(N)$, when $1 \leq R(N) < \|\sqrt{N}\|$ and $R(N) - \|\sqrt{N}\|$, when $\|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\|$, while zero, when $R(N) = 2\|\sqrt{N}\|$. Thus the desired residue can be calculated from the following formula

$$R(N) - \sigma(N)\|\sqrt{N}\| - 2\phi(N)\|\sqrt{N}\|, \tag{10}$$

where

$$\sigma(N) = \begin{cases} 0 & 1 \leq R(N) < \|\sqrt{N}\| \\ 1 & \|\sqrt{N}\| \leq R(N) < 2\|\sqrt{N}\| \\ 0 & R(N) = 2\|\sqrt{N}\| \end{cases}$$

and

$$\phi(N) = \begin{cases} 0 & 1 \leq R(N) < 2\|\sqrt{N}\| \\ 1 & R(N) = 2\|\sqrt{N}\|. \end{cases}$$

Via a simple application of the floor function, we see from inspection that the functions $\sigma(N)$ and $\phi(N)$ are given by

$$\sigma(N) = -\frac{1}{2}\left((-1)^{\lfloor R(N)/\|\sqrt{N}\| \rfloor} - 1\right)$$

and

$$\phi(N) = -\frac{1}{2}\left((-1)^{\lfloor R(N)/2\|\sqrt{N}\| \rfloor} - 1\right).$$

Finally substituting the previous expressions for $\sigma(N)$ and $\phi(N)$ into (10) produces, after some simplification, the desired residue formula. $\square$

Remark 2.1. If $N = s^2$ or $N = s(s+1)$ for some $s \in \mathbf{N}$, then in either case $s = \|\sqrt{N}\|$. Thus, the previous corollary implies that $\|\sqrt{N}\|$ divides N if and only if N is either a square or a product of two consecutive integers.

To close, we establish a curious connection between the zeta function $\zeta(s)$, for real $s > 1$, and the infinite series involving terms of the form $\|\sqrt{n}\|^{-(s+1)}$.

Corollary 2.3. *Suppose* $s > 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{1}{\|\sqrt{n}\|^{s+1}} = 2\,\zeta(s).$$

Proof. After setting $p = -(s+1)$ in (8) we need only show that

$$\left(N - (\|\sqrt{N}\| - 1)\,\|\sqrt{N}\|\right)\|\sqrt{N}\|^{-(s+1)} = o(1)$$

as $N \to \infty$. Now, by definition of the floor and ceiling functions, observe that

$$\|\sqrt{N}\| = \left\lfloor \sqrt{N} + \frac{1}{2} \right\rfloor = \left\lceil \sqrt{N} + \frac{1}{2} \right\rceil - 1 \geq \sqrt{N} + \frac{1}{2} - 1 = \sqrt{N} - \frac{1}{2}.$$

Consequently, $(\|\sqrt{N}\| - 1)\|\sqrt{N}\| \geq (\sqrt{N} - 3/2)(\sqrt{N} - 1/2) = N - 2\sqrt{N} + 3/4$, and so $N - (\|\sqrt{N}\| - 1)\|\sqrt{N}\| \leq 2\sqrt{N} - 3/4$. Thus,

$$0 < \left(N - (\|\sqrt{N}\| - 1)\,\|\sqrt{N}\|\right)\|\sqrt{N}\|^{-(s+1)} \leq \frac{2\sqrt{N} - (3/4)}{(\sqrt{N} - (1/2))^{s+1}} \longrightarrow 0,$$

as $N \to \infty$ since $s > 1$. $\square$

REFERENCES

1. R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete mathematics*, Addison-Wesley, New York, 1989.

2. I. Niven and H. Zuckerman, *An introduction to the theory of numbers*, 3rd ed., John Wiley and Sons, Inc., New York, 1972.

3. M.A. Nyblom, *Some curious sequences involving floor and ceiling functions*, Amer. Math. Monthly **109** (2002), 559–564.

RMIT University, GPO Box 2467V, Melbourne, Victoria 3001, Australia
E-mail address: michael.nyblom@rmit.edu.au