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NONLOCAL BOUNDARY VALUE PROBLEM OF HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS AT RESONANCE

ZENGJI DU, XIAOJIE LIN AND WEIGAO GE

ABSTRACT. In this paper we consider the following nth order nonlocal boundary value problem at resonance case

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad t \in (0,1),$$
$$x^{(i)}(0) = 0, \quad i = 0, 1, \dots, n-2,$$
$$x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\, dg(s),$$

where $f : [0,1] \times R^n \to R$ is a continuous function, $g : [0,1] \to [0, \infty)$ is a nondecreasing function with $g(0) = 0$. Under the resonance condition $g(1) = 1$, by applying the coincidence degree theory of Mawhin, we obtain some existence results for the boundary value problems. We also give an example to illustrate our results.

1. Introduction. In this paper, we consider the following nth order nonlocal boundary value problem at resonance case

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad t \in (0,1),$$
$$x^{(i)}(0) = 0, \quad i = 0, 1, \dots, n-2,$$
$$x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\, dg(s),$$

where $f : [0,1] \times R^n \to R$ is a continuous function, $g : [0,1] \to [0, \infty)$ is a nondecreasing function with $g(0) = 0$. In boundary condition (3), the integral is meant in the Rieman-Stieljes sense.

Similar to [**4, 15**], if the linear equation $x^{(n)}(t) = 0$, with boundary conditions (2), (3) has only zero solution, and the differential operator

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defined in a suitable Banach space, with boundary conditions taken into account, is invertible, the so-called nonresonance case; otherwise, is noninvertible, than the so-called resonance case.

Nonlocal boundary value problems were first considered by Bitsadze and Samarskii [**1**] and later by Il'in and Moiseev [**8, 9**]. We refer the reader to [**2, 3, 10**–**12**] for recent results of nonlocal boundary value problems. Multi-point boundary value problems, as a special case of this class of problems, were considered extensively by some authors. For example, Feng [**4, 5**], Liu [**13**] and Gupta [**6, 7**] studied the existence results for some second order multi-point boundary value problems at resonance case. However to our best knowledge, there is no paper to consider the higher order nonlocal boundary value problems (1)–(3) at resonance case.

The purpose of this paper is to study the existence of solutions for nonlocal BVP (1)–(3) at resonance case and establish some existence results under nonlinear growth restriction of f. Our method is based upon the coincidence degree theory of Mawhin [**14**].

2. Preliminary. In this section, we recall some notation and an abstract existence result.

Let Y, Z be real Banach spaces, and let $L : \operatorname{dom} L \subset Y \to Z$ be a linear operator which is a Fredholm map of index zero, that is, $\operatorname{Im} L$, the image of L, $\operatorname{Ker} L$, the kernel of L is finite-dimensional with the same dimension as $Z/(\operatorname{Im} L)$, and let $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $Y = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible, we denote the inverse of that map by K_P. Let Ω be an open bounded subset of Y such that $\operatorname{dom} L \cap \Omega \neq \varnothing$, the map $N : Y \to Z$ is said to be L-compact on $\overline{\Omega}$ if the map $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact. Let $J : \operatorname{Im} Q \to \operatorname{Ker} L$ be a linear isomorphism.

The theorem we use in the following is Theorem IV.13 of [**14**].

Theorem 1. *Let L be a Fredholm operator of index zero, and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, ; 1)$.*

(ii) $Nx \notin \operatorname{Im} L$ *for every* $x \in \operatorname{Ker} L \cap \partial\Omega$.

(iii) $\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$;

where $Q : Z \to Z$ *is a projection with* $\operatorname{Im} L = \operatorname{Ker} Q$. *Then the equation* $Lx = Nx$ *has at least one solution in* $\operatorname{dom} L \cap \overline{\Omega}$.

Throughout this paper, we use the classical spaces $C[0,1], C^1[0,1], \dots, C^{n-1}[0,1]$ and $L^1[0,1]$. For $x \in C^{n-1}[0,1]$, we use the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$ and $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n-1)}\|_\infty\}$, and denote the norm in $L^1[0,1]$ by $\|\cdot\|_1$. We will use the Sobolev space $W^{n,1}(0,1)$ which may be defined by

$$W^{n,1}(0,1) = \{x : [0,1] \longrightarrow R | x, x', \dots, x^{(n-1)} \text{ are absolutely continuous on } [0,1] \text{ with } x^{(n)} \in L^1[0,1]\}.$$

Let $Y = C^{n-1}[0,1]$, $Z = L^1[0,1]$, L is the linear operator from $\operatorname{dom} L \subset Y$ to Z with

$$\operatorname{dom} L = \left\{ x \in W^{n,1}(0,1) : x^{(i)}(0) = 0,\ i = 0,1,\dots,n-2,\ x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\, dg(s) \right\}$$

and $Lx = x^{(n)}$, $x \in \operatorname{dom} L$. We define $N : Y \to Z$ by setting

$$Nx = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad t \in (0,1).$$

Then BVP (1)–(3) can be written as $Lx = Nx$.

It is clear that

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x = ct^{n-1},\ c \in R,\ t \in [0,1]\}.$$

3. Main results.

Theorem 2. *Let* $f : [0,1] \times R^n \to R$ *be a continuous function and assume that*

(H1) *There exist functions* $a_1(t), a_2(t), \dots, a_n(t)$, $b(t), r(t) \in L^1[0,1]$, *and constant* $\theta \in [0,1)$ *such that for all* $(x_1, x_2, \dots, x_n) \in R^n$, $t \in [0,1]$, *satisfying*

$$|f(t, x_1, x_2, \dots, x_n)| \le \left(\sum_{i=1}^{n} a_i(t)|x_i|\right) + b(t)\left(\sum_{i=1}^{n} |x_i|^{\theta}\right) + r(t). \tag{4}$$

(H2) *There exists a constant* $M > 0$ *such that, for* $x \int \operatorname{dom} L$, *if* $|x^{(n-1)}(t)| > M$, *for all* $t \in [0,1]$, *then*

$$\int_0^1 f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds - \int_0^1 \int_0^s f(v, x(v), x'(v), \dots, x^{(n-1)}(v))\, dv\, dg(s) \neq 0. \tag{5}$$

(H3) *There exists a constant* $M^* > 0$ *such that, for any* $|c| > M^*$, *either*

$$c \cdot \left[\int_0^1 f(s, cs^{n-1}, c(n-1)s^{n-2}, \dots, c\cdot(n-1)!)\, ds - \int_0^1 \int_0^s f(v, cv^{n-1}, c(n-1)v^{n-2}, \dots, c\cdot(n-1)!)\, dv\, dg(s)\right] < 0, \tag{6}$$

or else

$$c \cdot \left[\int_0^1 f(s, cs^{n-1}, c(n-1)s^{n-2}, \dots, c\cdot(n-1)!)\, ds - \int_0^1 \int_0^s f(v, cv^{n-1}, c(n-1)v^{n-2}, \dots, c\cdot(n-1)!)\, dv\, dg(s)\right] > 0. \tag{7}$$

Then BVP (1)–(3) *with* $g(1) = 1$, $\int_0^1 s\, dg(s) \neq 1$ *has at least one solution in* $C^{n-1}[0,1]$ *provided that*

$$\sum_{i=1}^{n} \|a_i\|_1 < \frac{1}{2}. \tag{8}$$

We will show Theorem 2 via the following lemmas.

Lemma 1. *If* $g(1) = 1$, $\int_0^1 s\,dg(s) \neq 1$, *then* $L : \operatorname{dom} L \subset Y \to Z$ *is a Fredholm operator of index zero. Furthermore, the linear continuous projector operator* $Q : Z \to Z$ *can be defined by*

$$Qy = \frac{1}{1 - \int_0^1 s\,dg(s)} \left[\int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) \right] \tag{9}$$

and the linear operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ *can be written by*

$$K_P y = \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} y(s_1)\,ds_1 \cdots ds_n. \tag{10}$$

Furthermore

$$\|K_P y\| \leq \|y\|_1, \quad \textit{for every} \quad y \in \operatorname{Im} L.$$

Proof. Obviously, the problem

$$x^{(n)} = y \tag{11}$$

has a solution $x(t)$ satisfying $x^{(i)}(0)=0$ $(i=0,1,\dots,n-2)$, $x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\,dg(s)$, if and only if

$$\int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) = 0, \tag{12}$$

which implies

$$\operatorname{Im} L = \left\{ y \in Z : \int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) \right\} = 0. \tag{13}$$

In fact, if (11) has solution $x(t)$ satisfied $x^{(i)}(0) = 0$ $(i = 0, 1, \dots, n-2)$, $x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\,dg(s)$, then from (11), we have

$$x(t) = \frac{1}{(n-1)!}\, x^{(n-1)}(0) t^{n-1} + \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} y(s_1)\,ds_1 \cdots ds_n.$$

According to $x^{(n-1)}(1) = \int_0^1 x^{(n-1)}(s)\,dg(s),\ g(0) = 0,\ g(1) = 1$, we obtain

$$\begin{aligned} x^{(n-1)}(1) &= x^{(n-1)}(0) + \int_0^1 y(v)\,dv \\ &= \int_0^1 x^{(n-1)}(s)\,dg(s) \\ &= \int_0^1 \left[x^{(n-1)}(0) + \int_0^s y(v)\,dv\right] dg(s) \\ &= x^{(n-1)}(0)g(1) + \int_0^1 \int_0^s y(v)\,dv\,dg(s), \end{aligned}$$

then

$$\int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) = 0.$$

On the other hand, if (12) holds, setting

$$x(t) = c\,t^{n-1} + \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} y(s_1)\,ds_1 \cdots ds_n,$$

where c is an arbitrary constant, then $x(t)$ is a solution of (11), and $x^{(i)}(0) = 0$ $(i = 0, 1, \dots, n-2)$, and from (12) and $g(1) = 1$, we have

$$\begin{aligned} &x^{(n-1)}(1) - \int_0^1 x^{(n-1)}(s)\,dg(s) \\ &\quad = (n-1)!\,c + \int_0^1 y(v)\,dv - \int_0^1 \left[(n-1)!\,c + \int_0^s y(v)\,dv\right] dg(s) \\ &\quad = (n-1)!\,c[1 - g(1)] + \int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) \\ &\quad = 0 \end{aligned}$$

Hence (13) is valid.

For $y \in Z$, define

$$Qy = \frac{1}{1 - \int_0^1 s\,dg(s)} \left[\int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s)\right], \quad 0 \le t \le 1.$$

Letting $y_1 = y - Qy$, we obtain

$$\begin{aligned}
&\left[1 - \int_0^1 s\,dg(s)\right] Qy_1 \\
&\qquad = \int_0^1 (y - Qy)(s)\,ds - \int_0^1 \int_0^s (y - Qy)(v)\,dv\,dg(s) \\
&\qquad = \int_0^1 y(s)\,ds - Qy - \int_0^1 \int_0^s y(v)\,dv\,dg(s) + Qy \int_0^1 s\,dg(s) \\
&\qquad = \int_0^1 y(s)\,ds - \int_0^1 \int_0^s y(v)\,dv\,dg(s) - Qy\left[1 - \int_0^1 s\,dg(s)\right] \\
&\qquad = 0;
\end{aligned}$$

thus, $y_1 \in \operatorname{Im} L$. Hence $Z = \operatorname{Im} L + Z_1$, where $Z_1 = \{x(t) \equiv c : t \in [0,1],\ c \in R\}$, also $\operatorname{Im} L \cap Z_1 = \{0\}$. So we have $Z = \operatorname{Im} L \oplus Z_1$, and

$$\dim \operatorname{Ker} L = \dim Z_1 = \operatorname{co}\dim \operatorname{Im} L = 1.$$

Thus L is a Fredholm operator of index zero.

We define a projector $P : Y \to \operatorname{Ker} L$ by

$$(Px)(t) = x^{(n-1)}(0)t^{n-1}. \tag{14}$$

Then we show that K_P defined in (10) is a generalized inverse of $L : \operatorname{dom} L \cap Y \to Z$. In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_P)y(t) = [(K_P y)(t)]^{(n)} = y(t),$$

and for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we know

$$\begin{aligned}
(K_P L)x(t) &= \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} x^{(n)}(s_1)\,ds_1 \cdots ds_n \\
&= x(t) - x(0) - \frac{1}{1!}\,x'(0)t - \cdots - \frac{1}{(n-1)!}\,x^{(n-1)}(0)t^{n-1},
\end{aligned}$$

in view of $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $x^{(i)}(0) = 0$ $(i = 0, 1, \dots, n-2)$ and $Px = 0$, thus

$$(K_P L)x(t) = x(t).$$

This shows that $K_P = (L|_{\operatorname{dom} L \operatorname{Ker} P})^{-1}$. Also we have

$$\|K_P y\|_\infty \le \int_0^1 \cdots \int_0^1 |y(s_1)|\, ds_1 \cdots ds_n = \|y\|_1,$$

and from

$$(K_P y)^{(i)}(t) = \int_0^t \int_0^{s_{n-i}} \cdots \int_0^{s_2} y(s_1)\, ds_1 \cdots ds_{n-i}, \quad i = 1, 2, \dots, n-1.$$

Then we obtain

$$\|(K_P y)'\|_\infty \le \|y\|_1, \dots, \|(K_P y)^{(n-1)}\|_\infty \le \|y\|_1,$$

then $\|K_P y\| \le \|y\|_1$. This completes the proof of Lemma 1. □

Lemma 2. *Under condition* (4)*, there are nonnegative functions* $\bar{a}_i$*,* $i = 1, 2, \dots, n$*,* $\bar{r} \in L^1[0,1]$ *satisfying*

$$|f(t, x_1, x_2, \dots, x_n)| \le \left(\sum_{i=1}^n \bar{a}_i(t)|x_i| \right) + r(t).$$

Proof. Without loss of generality, we suppose that $\|b\|_1 = \int_0^1 |b(t)|\, dt = \beta > 0$. Take $\gamma \in \big(0, (\beta/n)\big((1/2) - \sum_{i=1}^n \|a_i\|_1\big)\big)$; then there exists $\overline{M} > 0$ such that

$$|x_i|^\theta \le \gamma |x_i| + \overline{M}, \quad i = 1, 2, \dots, n. \tag{15}$$

Let

$$\bar{a}_i(t) = a_i(t) + \gamma b(t), \quad i = 1, 2, \dots, n,$$
$$\bar{r}(t) = r(t) + n\overline{M} b(t).$$

Obviously, $\bar{a}_i$, $i = 1, 2, \dots, n$, $\bar{r} \in L^1[0,1]$ and

$$\|\bar{a}_i\|_1 \le \|a_i\|_1 + \gamma \|b\|_1, \quad i = 1, 2, \dots, n;$$

then

$$\sum_{i=1}^{n} \|\bar{a}_i\|1 \le \sum_{i=1}^{n} \|a_i\|_1 + n\beta\gamma < \frac{1}{2}.$$

From (4) and (15), we have

$$\begin{aligned} |f(t, x_1, x_2, \dots, x_n)| &\le \sum_{i=1}^{n} [a_i(t) + \gamma b(t)]|x_i| + n\overline{M} b(t) + r(t) \\ &= \sum_{i=1}^{n} \bar{a}_i(t)|x_i| + \bar{r}(t). \end{aligned}$$

Hence we can take $\bar{a}_i$, $(i = 1, 2, \dots, n), 0, \bar{r}$ to replace a_i, $(i = 1, 2, \dots, n)$, b, r, respectively in (4) and for convenience omit the bar above a_i, $(i = 1, 2, \dots, n)$ and r, i.e.,

$$|f(t, x_1, x_2, \dots, x_n)| \le \sum_{i=1}^{n} a_i(t)|x_i| + r(t). \tag{16}$$

Lemma 3. *Let* $\Omega_1 = \{x \in \operatorname{dom} L \setminus \operatorname{Ker} L : Lx = \lambda N x$ *for some* $\lambda \in [0, 1]\}$. *Then* Ω_1 *is a bounded subset of* Y.

Proof. Suppose that $x \in \Omega_1$ and $Lx = \lambda N x$. Thus, $\lambda \ne 0$ and $QNx = 0$, so that

$$\int_0^1 y(s)\, ds - \int_0^1 \int_0^s y(v)\, dv\, dg(s) = 0;$$

thus, from (H2), there exists $t_0 \in [0, 1]$ such that $|x^{(n-1)}(t_0)| \le M$. In view of

$$x^{(n-1)}(0) = x^{(n-1)}(t_0) - \int_0^{t_0} x^{(n)}(t)\, dt;$$

then

$$\|Px\| = |x^{(n-1)}(0) \le M + \|x^{(n)}\|_1 = M + \|Lx\|_1 \le M + \|Nx\|_1. \tag{17}$$

Again for $x \in \Omega_1$, $x \in \operatorname{dom} L \setminus \operatorname{Ker} L$, then $(I - P)x \in \operatorname{dom} L \cap \operatorname{Ker} P$, $LPx = 0$; thus, from Lemma 1, we know

$$\|(I-P)x\| = \|K_P L(I-P)x\| \le \|L(I-P)x\|_1 = \|Lx\|_1 \le \|Nx\|_1. \tag{18}$$

From (17) and (18), we have

$$\|x\| \leq \|Px\| + \|(I-P)x\| \leq 2\|Nx\|_1 + M. \tag{19}$$

From (4), (16) and (19), we obtain

$$\|x\| \leq 2\bigg[\bigg(\sum_{i=1}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty\bigg) + \|r\|_1 + \frac{M}{2}\bigg]. \tag{20}$$

Thus, from $\|x\|_\infty \leq \|x\|$ and (20), we have

$$\|x\|_\infty \leq \frac{2}{1-2\|a_1\|1}\bigg[\bigg(\sum_{i=2}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty\bigg) + \|r\|_1 + \frac{M}{2}\bigg]. \tag{21}$$

From $\|x'\|_\infty \leq \|x\|$, (20) and (21), one has

$$\begin{aligned}\|x'\|_\infty \leq \|x\| &\leq 2\bigg[1 + \frac{2\|a_1\|_1}{1-2\|a_1\|_1}\bigg] \\ &\quad \times \bigg[\bigg(\sum_{i=2}^{n} \|a_i\|_1 x^{(i-1)}\|_\infty\bigg) + \|r\|_1 + \frac{M}{2}\bigg] \\ &= \frac{2}{1-2\|a_1\|_1}\bigg[\bigg(\sum_{i=2}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty + \|r\|_1 + \frac{M}{2}\bigg],\end{aligned}$$

then

(22)

$$\|x'\|_\infty \leq \frac{2}{1-2(\|a_1\|_1 + \|a_2\|_1)}\bigg[\bigg(\sum_{i=3}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty\bigg) + \|r\|_1 + \frac{M}{2}\bigg].$$

Similarly, we get

(23)

$$\|x^{(j)}\|_\infty \leq \frac{2}{1-2\sum_{i=1}^{j+1} \|a_i\|_1}\bigg[\bigg(\sum_{i=j+2}^{n} \|a_i\|_1 \|x^{(i-1)}\|_\infty\bigg) + \|r\|_1 + \frac{M}{2}\bigg],$$
$$j = 2, \dots, n-2,$$

and

$$\|x^{(n-1)}\|_\infty \leq \frac{2(\|r\|_1 + (M/2))}{1-2\sum_{i=1}^{n} \|a_i\|_1} := M_1. \tag{24}$$

Similarly, from (23), (22) and (21), there exist $M_i > 0$ $(i = 2, \dots, n)$, such that

$$\|x^{(n-i)}\|_\infty \le M_i, \quad i = 2, \dots, n; \tag{25}$$

hence,

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n-1)}\|_\infty\} \le \max\{M_1, M_2, \dots, M_n\}.$$

Again from (4), (24) and (25), we have

$$\|x^{(n)}\|_1 \le \|a_1\|_1 M_n + \cdots + \|a_{n-1}\|_1 \|M_2 + \|a_n\|_1 M_1 + \|r\|_1.$$

Then we show that Ω_1 is bounded.

Lemma 4. *The set* $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$ *is bounded.*

Proof. Let $x \in \Omega_2$; then $x \in \operatorname{Ker} L = \{x \in \operatorname{dom} L : x = c\, t^{n-1},\ c \in R,\ t \in [0,1]\}$, and $QNx = 0$. Therefore

$$\int_0^1 f(s, cs^{n-1}, c(n-1)s^{n-2}, \dots, c \cdot (n-1)!)\, ds - \int_0^1 \int_0^s f(v, cv^{n-1}, c(n-1)v^{n-2}, \dots, c \cdot (n-1)!)\, dv\, dg(s) = 0.$$

From (H2), there exists $t_0 \in [0,1]$ such that $|x^{(n-1)}(t_0)| \le M$. Then we have $\|x^{(n-1)}\|_\infty = |x^{(n-1)}| = |(n-1)!\, c| < M$, then

$$\|x\| = \max\{\|x\|_\infty, \dots, \|x^{(n-1)}\|_\infty\} = \|x^{(n-1)}\|_\infty \le M.$$

Thus Ω_2 is bounded. □

Lemma 5. *If the condition* (H3) *holds, i.e., there exists* $M^* > 0$ *such that*

$$\frac{1}{1 - \int_0^1 s\, dg(s)} \Big[\int_0^1 f(s, x(s), x'(s), \dots, x^{(n-1)}(s))\, ds - \int_0^1 \int_0^s f(v, x(v), x'(v), \dots, x^{(n-1)}(v))\, dv\, dg(s) \Big] < 0 \tag{26}$$

or

$$\frac{1}{1-\int_0^1 s\,dg(s)}\bigg[\int_0^1 f(s,x(s),x'(s),\dots,x^{(n-1)}(s))\,ds - \int_0^1\int_0^s f(v,x(v),x'(v),\dots,x^{(n-1)}(v))\,dv\,dg(s)\bigg]>0 \tag{27}$$

for all $|c|>M^*$. *If* (26) *holds, let*

$$\Omega_3=\{x\in \operatorname{Ker} L: -\lambda x+(1-\lambda)JQN_x=0,\ \lambda\in[0,1]\};$$

here $J:\operatorname{Im} Q\to \operatorname{Ker} L$ *is the linear isomorphism given by* $J(c)=c\,t^{n-1}$, *for all* $c\in R$, $t\in[0,1]$.

Proof. Suppose that $x=c_0t^{n-1}\in\Omega_3$; then we obtain

$$\begin{aligned}\lambda c_0t^{n-1} &= \frac{(1-\lambda)t^{n-1}}{1-\int_0^1 s\,dg(s)}\bigg[\int_0^1 f(s,c_0s^{n-1},c_0(n1)s^{n-2},\dots,c_0(n-1)!)\,ds \\ &\quad -\int_0^1\int_0^s f(v,c_0v^{n-1},c_0(n-1)v^{n-2},\dots,c_0(n-1)!)\,dv\,dg(s)\bigg], \\ &\qquad t\in[0,1]\end{aligned}$$

or equivalently

$$\begin{aligned}\lambda_{c_0} &= \frac{1-\lambda}{1-\int_0^1 s\,dg(s)}\bigg[\int_0^1 f(s,c_0s^{n-1},c_0(n-1)s^{n-2},\dots,c_0(n-1)!)\,ds \\ &\quad \int_0^1\int_0^s f(v,c_0v^{n-1},c_0(n-1)v^{n-2},\dots,c_0(n-1)!)\,dv\,dg(s)\bigg].\end{aligned}$$

If $\lambda=1$, then $c_0=0$. Otherwise, if $|c_0|>M^*$, in view of (26), one has

$$\begin{aligned}\lambda c_0^2 &= \frac{(1-\lambda)c_0}{1-\int_0^1 s\,dg(s)}\bigg[\int_0^1 f(s,c_0s^{n-1},c_0(n-1)s^{n-2},\dots,c_0(n-1)!)\,ds \\ &\quad -\int_0^1\int_0^s f(v,c_0v^{n-1},c_0(n-1)v^{n-2},\dots,c_0(n-1)!)\,dv\,dg(s)\bigg]<0,\end{aligned}$$

which contradicts $\lambda c_0^2 \geq 0$. Then we obtain $\Omega_3 \subset \{x \in \operatorname{Ker} L : \|x\| \leq (n-1)!M^*\}$ is bounded.

Then the proof of Theorem 2 is now an easy consequence of the above lemmas and Theorem 1.

Proof of Theorem 2. Let $\Omega = \{x \in Y : \|x\| < d\}$ such that $\cup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. By the Ascoli-Arzela theorem, it can be shown that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact, thus N is L-compact on $\overline{\Omega}$. Then by the above lemmas, we have

(i) $Lx \neq \lambda Nx$ for every $(x, y) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$.

(ii) $Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$.

(iii) Let $H(x, \lambda) = -\lambda x + (1-\lambda)JQNx$, with J as in Lemma 5. We know $H(x, \lambda) \neq 0$, for $x \in \operatorname{Ker} L \cap \partial\Omega$. Thus, by the homotopy property of degree, we get

$$\begin{aligned} \deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(-I, \Omega \cap \operatorname{Ker} L, 0). \end{aligned}$$

According to the definition of degree on a space which is isomorphic to R^m, $m < \infty$, and

$$\Omega \cap \operatorname{Ker} L = \{c\, t^{n-1} : |c| < d\},$$

we have

$$\begin{aligned} \deg(-I, \Omega \cap \operatorname{Ker} L, 0) &= \deg(-J^{-1}IJ, J^{-1}(\Omega \cap \operatorname{Ker} L), J^{-1}\{0\}) \\ &= \deg(-I, (-d, d), 0) = -1 \neq 0, \end{aligned}$$

and then

$$\deg(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0.$$

Then, by Theorem 1, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, so that the BVP (1)–(3) has at least one solution in $C^{n-1}[0, 1]$. The proof is completed. □

Remark 3. If the inequality (27) holds in Lemma 5, then we take

$$\Omega_3 = \{x \in \operatorname{Ker} L : \lambda x + (1-\lambda)JQN_x = 0,\ \lambda \in [0, 1]\}.$$

Let $H(x,\lambda) = \lambda x + (1-\lambda)JQNx$, with J as in Lemma 5, and, exactly as there, we can prove that Ω_3 is bounded. Then, in the proof of Theorem 2, we have

$$\deg\,(JQN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) = \deg\,(I, \Omega \cap \operatorname{Ker} L, 0) = 1,$$

since $0 \in \Omega \cap \operatorname{Ker} L$. The remainder of the proof is the same.

4. An example.

Consider the following fourth-order nonlocal boundary value problem:

(28)

$$x^{(iv)} = 2t^2 + 5 + \sin(x)^2 + \arctan x' + \cos(x'')^{1/3} + \frac{1}{5}\,(t^2+1)x''', \quad t \in (0,1),$$

(29)
$$x(0) = x'(0) = x''(0) = 0,$$

(30)
$$x'''(1) = \int_0^1 x'''(s)\,dg(s),$$

where

$$f(t, x_1, x_2, x_3, x_4) = 2t^2 + 5 + \sin(x_1)^2 + \arctan x_2 + \cos(x_3)^{1/3} + \frac{1}{5}\,(t^2+1)x_4, \quad t \in (0,1),$$

and $g(s) = s^2$ satisfying $g(0) = 0$, $g(1) = 1$ and $\int_0^1 s\,dg(s) = 2/3 \neq 1$. Then we can choose $a_1(t) = a_2(t) = a_3(t) = 0$, $a_4(t) = 2/5$, $r(t) = 11$, for $t \in [0,1]$; thus,

$$|f(t, x_1, x_2, x_3, x_4)| \leq \frac{2}{5}\,|x_4| + 11,$$
$$\|a\|_1 + \|a\|_2 + \|a\|_3 + \|a\|_4 = \frac{2}{5} < \frac{1}{2}.$$

Since

$$\begin{aligned}
&\int_0^1 f(s, x(s), x'(s))\,ds - \int_0^1 \int_0^s f(v, x(v), x'(v))\,dv\,dg(s)\\
&= \int_0^1 \int_0^1 f(v, x(v), x'(v))\,dv\,dg(s) - \int_0^1 \int_0^s f(v, x(v), x'(v))\,dv\,dg(s)\\
&= \int_0^1 \int_s^1 f(v, x(v), x'(v))\,dv\,dg(s),
\end{aligned}$$

and f has the same sign as $x'''(t)$ when $|x'''(t)| > 55$, we may choose $M = M^* = 55$, and then the conditions (H1)–(H3) of Theorem 2 are satisfied. Theorem 2 implies that the BVP (28)–(30) has at least one solution $x \in C^3[0, 1]$.

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Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu, 221116, P.R. China
E-mail address: duzengji@163.com

Department of Mathematics, Xuzhou Normal University, Xuzhou, Jiangsu, 221116, P.R. China

Department of Mathematics, Beijing Institute of Technology Beijing 100081, P.R. China