ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

CONTINUOUS HOMOMORPHISMS BETWEEN TOPOLOGICAL ALGEBRAS OF HOLOMORPHIC GERMS

LUCIANO O. CONDORI AND M. LILIAN LOURENÇO

ABSTRACT. We study τ_w-continuous homomorphisms on algebras of holomorphic germs. In this setting we give conditions for these homomorphisms to be composition operators. We also present equivalent conditions for the above homomorphisms to be Montel or reflexive.

1. Introduction. Let E be a Banach space, U an open subset of E and $\mathcal{H}(U)$ the space of all holomorphic functions on U. A holomorphic germ on a compact subset K of E is an equivalence class determined in the set of all holomorphic functions on open neighborhoods of K by the relation, $f \cong g$ if f and g coincide on an open neighborhood of K. We will denote by $\mathcal{H}(K)$ the algebra of all holomorphic germs on K. The natural topology on spaces of holomorphic germs is the Nachbin ported topology τ_w. It is defined on $H(U)$ by the family of all semi-norms ported by the compact subsets of U. A semi-norm p on $H(U)$ is ported by the compact subset L of U if for every open subset V, $L \subset V \subset U$, there is a $c > 0$ such that $p(f) \leq c\|f\|_V = c\sup_{x\in V}|f(x)|$ for every $f \in H(U)$.

Now, the topology τ_w on $\mathcal{H}(K)$ is the locally convex topology defined by the inductive limit of the spaces $(H(U), \tau_w)$, where U varies over all open neighborhoods of K. We remark that the space $(\mathcal{H}(K), \tau_w)$ can be represented as an inductive limit of Banach spaces, namely the inductive limit of the spaces $H^\infty(U)$, where $H^\infty(U)$ denotes the Banach space of all bounded holomorphic functions on U. We are interested in the study of continuous homomorphisms between locally m-convex algebras of holomorphic germs. The continuous homomorphisms between topological algebras of holomophic functions have been extensively studied lately. For example see [**4**–**6**]. In [**16**] Nicodemi has

2000 AMS *Mathematics Subject Classification.* Primary 46G20, 32A38, 47B33.
Key words and phrases. Holomorphic germs, continuous homomorphisms.
Research of the first author supported by a doctoral fellowship from CNPq, Brazil.
Received by the editors on October 3, 2003, and in revised form on Feb. 18, 2004.

studied the continuous homomorphisms between algebras of holomorphic germs with the compact open topology. She proved that, under some conditions, the continuous homomorphisms $A : \mathcal{H}(K) \to \mathcal{H}(L)$, where L is a compact subset of a Banach space F, are exactly those induced by a holomorphic mapping $\Phi : V \to E$ such that $\Phi(L) \subset K$ and V is an open set in F containing L, in the sense that $A(f) = f \circ \Phi$, for every $f \in \mathcal{H}(K)$. It is natural to study the continuous homomorphisms between algebras of holomorphic germs endowed with the τ_w-topology.

We show that every τ_w-continuous homomorphism between locally m-convex algebras of holomorphic germs factors through a continuous homomorphism between Fréchet algebras of holomorphic functions of bounded type. Using this factorization we establish when the continuous homomorphisms between locally m-convex algebras of holomorphic germs are composition operators. As a consequence we get that if E is a Tsirelson space each continuous homomorphism between locally m-convex algebras of holomorphic germs is a composition operator. Furthermore, this factorization also allows us to characterize when continuous homomorphisms between locally m-convex algebras of holomorphic germs are Montel or reflexive.

We refer to the books of Dineen [**2**] or Mujica [**13**] for background information from infinite dimensional complex analysis.

2. The space of holomorphic germs. Let $(H, \tau) = \varinjlim_{i \in I} (H_i, \tau_i)$ denote the locally convex inductive limit of locally convex spaces (H_i, τ_i). The inductive limit (H, τ) is called *regular* if, for each bounded set B in (H, τ), there exists $i = i_{(B)} \in I$ such that $B \subset H_i$ and B is τ_i-bounded. The inductive limit (H, τ) is called *strongly boundedly retractive* if (H, τ) is regular and, for each $i \in I$, there exists $j > i$ such that (H, τ) and (H_j, τ_j) induce the same topology on each bounded subset B of (H_i, τ_i).

Let τ_w denote the compact-ported topology introduced by Nachbin [**15**] on the space $H(U)$ of all holomorphic functions on an open subset U of a complex Banach space E. Let $\mathcal{H}(K)$ denote the space of all germs of holomorphic functions on a compact subset K of E and let us also denote by τ_w the locally convex inductive limit topology on $\mathcal{H}(K)$ which is defined by $(\mathcal{H}(K), \tau_w) = \varinjlim_{U \supset K} (H(U), \tau_w)$.

Let $B_{1/n}(0)=\{x\in E : ||x||<1/n\}$. If we fix a decreasing fundamental sequence of open neighborhoods $(U_n)_{n\in\mathbf{N}}$ in E with $U_n=K+B_{1/n}(0)\subset E$, we remark that the space $(\mathcal{H}(K),\tau_w)$ can be represented as an inductive limit of Banach spaces, namely, $(\mathcal{H}(K),\tau_w)=\lim_{\substack{\longrightarrow \\ U_n\supset K}} H^\infty(U_n)$ where $H^\infty(U_n)$ denotes the Banach space of all bounded holomorphic functions on U_n, with the supremum norm. Mujica showed in [**10**] that the inductive limit $(\mathcal{H}(K),\tau_w) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}} H^\infty(U_n)$ is strongly boundedly retractive, since all the spaces $H^\infty(U_n)$, $n \in \mathbf{N}$, are normed spaces.

Before stating our results, let us fix some additional notation and terminology. For $m \in \mathbf{N}$ we let $\mathcal{J}_m^K$ denote the inclusion mapping from $H^\infty(U_m)$ into $\mathcal{H}(K)$ given by $\mathcal{J}_m^K(f) = [f]$.

If U is an open subset of E, then a set $A \subset U$ is said to be U-bounded if A is bounded and is bounded away from the boundary of U. As usual the space of all holomorphic mappings from an open set U of E into $\mathbf{C}$ that are bounded on U-bounded subsets of U, endowed with the topology τ_b of uniform convergence on U-bounded sets is denoted by $H_b(U)$.

Since $H^\infty(U_n) \hookrightarrow H_b(U_n) \hookrightarrow (\mathcal{H}(K),\tau_w)$ for each $n \in \mathbf{N}$, we have $(\mathcal{H}(K),\tau_w) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}} H_b(U_n)$ and we define the natural inclusion from $H_b(U_n)$ into $(\mathcal{H}(K),\tau_w)$ by $\mathcal{I}_n^K(f) = [f]$, where $[f]$ is the equivalence class determined by f.

Now, we are going to give some results about inductive limits, which will be useful in Section 4.

Proposition 2.1. *Let $(H_n,\tau_n)_{n\in\mathbf{N}}$ denote a sequence of locally convex spaces and $(H,\tau) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}}(H_n,\tau_i)$ denote a strongly boundedly retractive inductive limit. Then for each $m \in \mathbf{N}$ there exists $n > m$ such that $\sigma(H,H')$ and $\sigma(H_n,H_n')$ induce the same topology on each bounded subset of (H_m,τ_m).*

Proof. Given $B \subset (H_m,\tau_m)$ a bounded subset, the closed absolutely convex hull of B will be denoted by $\overline{\Gamma(B)}$. Since (H,τ) is a strongly boundedly retractive limit, then there exists $n > m$ such that (H,τ) and (H_n,τ_n) induce the same topology on $\overline{\Gamma(B)}$. Now, by Grothendieck's lemma [**8**, Proposition 3.11.1] we have that for each $f \in H_n'$ and $\varepsilon > 0$

there exists $g \in H'$ with $|f(x) - g(x)| < \varepsilon$, for all $x \in \overline{\Gamma(B)}$. Hence given $f_1, \dots, f_p \in H'_n$, there exist $g_1, \dots, g_p \in H'$ such that

$$\{x \in \overline{\Gamma(B)} : |f_j(x)| < 2\varepsilon \text{ for } j = 1, \dots, p\} \supset \{x \in \overline{\Gamma(B)} : |g_j(x)| < \varepsilon \text{ for } j = 1, \dots, p\}$$

and therefore $\sigma(H_n, H'_n)|_B \leq \sigma(H, H')|_B$.

Since the opposite inequality is always true, we conclude that $\sigma(H_n, H'_n)|_B = \sigma(H, H')|_B$. ◻

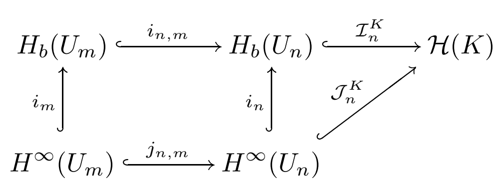
Proposition 2.2. *Let E be a Banach space, and let $K \subset E$ be a compact subset. Then the inductive limit $(\mathcal{H}(K), \tau_w) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}} H_b(U_n)$ is strongly boundedly retractive.*

Proof. $(\mathcal{H}(K), \tau_w)$ is the strongly boundedly retractive inductive limit of an increasing sequence of Banach spaces

$$(1) \qquad (\mathcal{H}(K), \tau_w) = \varinjlim_{n\in\mathbf{N}} H^\infty(U_n).$$

Therefore, given $\mathcal{X} \subset \mathcal{H}(K)$ bounded set there exists $m_0 \in \mathbf{N}$ such that $\mathcal{X}$ is a bounded subset of $H^\infty(U_{m_0})$. Since the inclusion mapping $H^\infty(U_{m_0}) \overset{i_{m_0}}{\hookrightarrow} H_b(U_{m_0})$ is continuous, we have that $\mathcal{X}$ is a bounded subset of $H_b(U_{m_0})$ and consequently $(\mathcal{H}(K), \tau_w) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}} H_b(U_n)$ is a regular inductive limit. Now we can prove that $(\mathcal{H}(K), \tau_w) = \lim_{\substack{\longrightarrow \\ n\in\mathbf{N}}} H_b(U_n)$ is strongly boundedly retractive. Let $m \in \mathbf{N}$ and let $\mathcal{X} \subset H_b(U_m)$ be bounded. So $\mathcal{I}_m^K(\mathcal{X})$ is a bounded subset of $\mathcal{H}(K)$. By (1), we can find $n > m$ such that $(\mathcal{H}(K), \tau_w)$ and $H^\infty(U_n)$ induce the same topology on the bounded set $\mathcal{X}$ of $H^\infty(U_m)$.

Now, it follows at once from the commutative diagram:



that $(\mathcal{H}(K), \tau_w)$ and $(H_b(U_n), \tau_b)$ induce the same topology on the bounded subset $\mathcal{X}$ of $H_b(U_m)$. □

An immediate consequence of Proposition 2.1 and Proposition 2.2 is the following:

Corollary 2.3. *Let E be a Banach space and let $K \subset E$ be a compact subset. Let $(\mathcal{H}(K), \tau_w) = \lim_{\substack{\longrightarrow \\ n \in \mathbf{N}}} H_b(U_n)$ denote a strongly boundedly retractive inductive limit. Then for each $n \in \mathbf{N}$ there exists $m > n$ such that $\sigma(\mathcal{H}(K), \mathcal{H}(K)')$ and $\sigma(H_b(U_m), H_b(U_m)')$ induce the same topology on each bounded set $\mathcal{X}$ of $(H_b(U_n), \tau_b)$.*

3. Homomorphisms between algebras of holomorphic germs. A topological algebra is locally m-convex if its topology is defined by the family of the convex 0-neighborhoods W with $W^2 \subset W$.

For any compact subset K of a Banach space E, Mujica in [**12**] showed that the inductive limit $(\mathcal{H}(K), \tau_w) = \lim_{\substack{\longrightarrow \\ n \in \mathbf{N}}} H^\infty(U_n)$ is a locally m-convex algebra. From now on $(\mathcal{H}(K), \tau_w)$ stands for a locally m-convex algebra and $(H_b(U_n), \tau_b)$ stands for a Fréchet algebra. By a homomorphism between topological algebras we mean an algebra homomorphism, which is not identically zero.

Next we show that each continuous homomorphism between locally m-convex algebras of holomorphic germs factors through some continuous homomorphism between Fréchet algebras of holomorphic functions of bounded type.

Theorem 3.1. *Let E and F be Banach spaces. Let $K \subset E$ and $L \subset F$ be compact sets, and let $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ be a continuous homomorphism. Then for each open subset $U_n \supset K$ there exists an open subset $V_{m_n} \supset L$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$.*

Proof. Given $U_n \supset K$ an open subset of E, we consider the continuous homomorphism $A \circ \mathcal{I}^K_n : H_b(U_n) \to \mathcal{H}(L)$. By Grothendieck's theorem, see [**7**, p. 16], there exist an open subset $V_{m_n} \supset L$ and a continuous linear mapping $\overline{A \circ \mathcal{I}_n^K} : H_b(U_n) \to H_b(V_{m_n})$ such that the diagram

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{A} & \mathcal{H}(L) \\ {\scriptstyle \mathcal{I}_n^K}\uparrow & & \uparrow{\scriptstyle \mathcal{I}_{m_n}^L} \\ H_b(U_n) & \xrightarrow{\overline{A \circ \mathcal{I}_n^K}} & H_b(V_{m_n}) \end{array}$$

is commutative. Since A, $\mathcal{I}_n^K$ and $\mathcal{I}_{m_n}^L$ are continuous homomorphisms, we have that

$$A \circ \mathcal{I}_n^K(f \cdot g) = A \circ (\mathcal{I}_n^K(f) \cdot \mathcal{I}_n^K(g))$$

and therefore $\overline{A \circ \mathcal{I}_n^K}(f \cdot g) \sim \overline{A \circ \mathcal{I}_n^K}(f) \cdot \overline{A \circ \mathcal{I}_n^K}(g)$, for all $f, g \in H_b(U_n)$. So, there exists an open set $V_{m_{l_n}} \subset V_{m_n}$ such that $\overline{A \circ \mathcal{I}_n^K}(f \cdot g)_{|V_{m_{l_n}}} = (\overline{A \circ \mathcal{I}_n^K}(f) \cdot \overline{A \circ \mathcal{I}_n^K}(g))_{|V_{m_{l_n}}}$. Since each connected component of V_{m_n} contained an open set of $V_{m_{l_n}}$, by the Identity Principle for holomorphic functions we have $\overline{A \circ \mathcal{I}_n^K}(f \cdot g) = (\overline{A \circ \mathcal{I}_n^K}(f) \cdot \overline{A \circ \mathcal{I}_n^K}(g))$ on V_{m_n} for all $f, g \in H_b(U_n)$.

Now, the mapping $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ defined by $\widetilde{A}_n(f) = \overline{A \circ \mathcal{I}_n^K}(f)$ for each $f \in H_b(U_n)$ is clearly a continuous homomorphism such that $A \circ \mathcal{I}_n^K = \mathcal{I}_{m_n}^L \circ \widetilde{A}_n$. Thus, for every open set $U_n \supset K$, there exist an open set $V_{m_n} \supset L$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ such that the diagram

$$\begin{array}{ccc} \mathcal{H}(K) & \xrightarrow{A} & \mathcal{H}(L) \\ {\scriptstyle \mathcal{I}_n^K}\uparrow & & \uparrow{\scriptstyle \mathcal{I}_{m_n}^L} \\ H_b(U_n) & \xrightarrow{\widetilde{A}_n} & H_b(V_{m_n}) \end{array}$$

is commutative. $\square$

Before stating and proving Theorem 3.2 we need some preparation. Let U and V denote open subsets of complex Banach spaces E and F, respectively. Given a holomorphic mapping of bounded type $\Phi : V \to E$ such that $\Phi(V) \subset U$ where Φ takes a V-bounded set in a U-bounded set, the mapping $A_\Phi : H_b(U) \to H_b(V)$ defined by $A_\Phi(f) = f \circ \Phi$ is called a composition operator.

Let K and L be compact subsets of E and F respectively. A homomorphism $A : \mathcal{H}(K) \to \mathcal{H}(L)$ is a composition operator if there exist an open subset $V \supset L$ and a holomorphic mapping $\Phi : V \to E$ such that $\Phi(L) \subset K$ and $A([f]) = [f \circ \Phi]$ for each holomorphic function f defined on a neighborhood of K. We denote a composition operator A by A_Φ.

Now, we give some conditions for τ_w-continuous homomorphisms between algebras of holomorphic germs to be composition operators.

Theorem 3.2. *Let E and F be Banach spaces. Let $K \subset E$ and $L \subset F$ be compact subsets, and let $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ be a continuous homomorphism. If, for each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and a composition operator $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$, then A is a composition operator.*

Proof. We suppose that for each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ a composition operator such that $\mathcal{I}^L_{m_n} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$. We shall prove that A is a composition operator.

Without loss of generality we may assume that $V_{m_1} \supset V_{m_2} \supset \cdots$.

Since $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ is a composition operator, then there exists a mapping $\Phi_n \in H_b(V_{m_n}, E)$ such that $\widetilde{A}_n(f) = f \circ \Phi_n$, for every $f \in H_b(U_n)$ and for each $n \in \mathbf{N}$. So,

$$(2) \quad A([f]) = (A \circ \mathcal{I}^K_n)(f) = (\mathcal{I}^L_{m_n} \circ \widetilde{A}_n)(f) = [f \circ \Phi_n], \quad \forall\, f \in H_b(U_n),$$

for each $n \in \mathbf{N}$.

Now, we consider the open subset V_{m_1} as above and define a mapping $\Phi : V_{m_1} \to E$ by $\Phi(y) = \Phi_1(y)$, for all $y \in V_{m_1}$. We claim that $\Phi(y) = \Phi_n(y)$ for all $y \in V_{m_n}$ and $A = A_\Phi$. Since $(E', \|\ \|) \hookrightarrow H_b(U_n)$, for all $n \in \mathbf{N}$, we have that by equality (2)

$$(3) \quad \begin{cases} A([x']) = [x' \circ \Phi_1], & \text{for every} \quad x' \in E' \quad \text{and} \\ A([x']) = [x' \circ \Phi_n], & \text{for every} \quad x' \in E', \ \text{for every} \quad n \in \mathbf{N}; \end{cases}$$

consequently, $[x' \circ \Phi_1] = [x' \circ \Phi_n]$, for every $x' \in E'$ and for every $n \in \mathbf{N}$. Then there exists an open subset W such that $L \subset W \subset V_{m_n}$

and $(x' \circ \Phi_1)_{|W} = (x' \circ \Phi_n)_{|W}$ for all $x' \in E'$. Now, for each $y \in V_{m_n}$ there exists $z \in L$ such that $y \in B_{1/m_n}(z)$. Since $L \subset W$ we have that $W \cap B_{1/m_n}(z) \neq \varnothing$ and consequently $(x' \circ \Phi_1)_{|W \cap B_{1/m_n}(z)} = (x' \circ \Phi_n)_{|W \cap B_{1/m_n}(z)}$ for all $x' \in E'$ and $(x' \circ \Phi_1)(y) = (x' \circ \Phi_n)(y)$ for all $y \in V_{m_n}$ and for all $x' \in E'$. By the Hahn-Banach theorem we have that

$$\Phi(y) = \Phi_1(y) = \Phi_n(y), \quad \text{for every} \quad y \in V_{m_n}. \tag{4}$$

Thus to show our claim it suffices to prove that $\Phi(L) \subset K$. By (4) we have that $\Phi(L) \subset U_n$ for all $n \in \mathbf{N}$. So, for each $y \in L$, there is a $\xi_n \in K$ such that $\|\Phi(y) - \xi_n\| < 1/n$, for each $n \in \mathbf{N}$. Then $d(\Phi(y), K) = 0$ for each $y \in L$. Now the proof of the theorem is complete. $\square$

Our next corollary shows that in the case of the Banach space E to be the Tsirelson space, defined by Tsirelson in [**17**], every continuous homomorphism between algebras of holomorphic germs is a composition operator.

Corollary 3.3. *Let E be a Tsirelson space and F a Banach space. Let $K \subset E$ be an absolutely convex and compact subset and $L \subset F$ a compact subset. Then every continuous homomorphism $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ is a composition operator.*

Proof. Let $K \subset E$ be an absolutely convex and compact subset. Since $U_n = K + B_{1/n}(0)$ for each $n \in \mathbf{N}$, U_n is also absolutely convex. Now, by Theorem 3.1 there exist an open subset $V_{m_n} \supset L$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$. By [**4**, Proposition 3] we have that $\widetilde{A}_n$ is a composition operator. Now it suffices to apply Theorem 3.2. $\square$

Before giving our next corollary we need additional notation. Let E be a complex Banach space. We denote $P_f(^nE)$ the space generated by all polynomials of the form $P(x) = \psi(x)^m$, with $\psi \in E'$.

Corollary 3.4. *Let E be a reflexive Banach space such that $\mathcal{P}_f(^nE)$ is dense in $\mathcal{P}(^nE)$ for each $n \in \mathbf{N}$. Let $K \subset E$ be an absolutely* *convex and compact subset of E. Let F be a Banach space and $L \subset F$ be a compact subset. Then every continuous homomorphism $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ is a composition operator.*

Proof. The result follows arguing as in Corollary 3.3 and using a result of Mujica [**14**, Theorem 1.6]. □

4. Montel and reflexive homomorphisms. Our purpose in this section is to study the Montel and reflexive homomorphisms between the algebras of holomorphic germs and their relationship.

Let U be an open subset of a complex Banach space E. We say that a mapping $g : U \to F$ is *compact*, respectively *weakly* compact, if it takes U-bounded sets into relatively compact, respectively *weakly compact*, sets. A linear operator between locally convex spaces is *Montel*, respectively *reflexive*, if it takes bounded subsets into relatively compact, respectively *weakly compact*, subset. All these definition can be found in the paper [**1**].

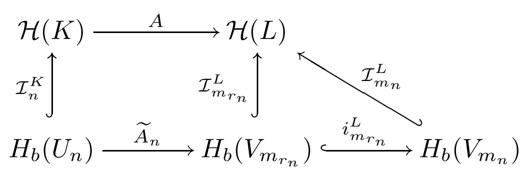
Let us recall that a linear mappings $T : X \to Y$, where X and Y are locally convex spaces, is called *weakly compact*, respectively *compact*, if it maps some 0-neighborhood into relatively weakly compact, respectively relatively compact sets. If X and Y are normed space T is weakly compact, respectively compact, if and only if T is reflexive, respectively Montel.

Theorem 4.1. *Let E and F be Banach spaces. Let $K \subset E$ and $L \subset F$ be compact subsets, and let $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ be a continuous homomorphism. Then A is a Montel homomorphism if and only if for each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and a Montel continuous homomorphism $\Psi_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \Psi_n = A \circ \mathcal{I}^K_n$.*

Proof. Suppose A is a Montel homomorphism. Given $U_n \subset E$ an open subset, by Theorem 3.1 there exist an open subset $V_{m_{r_n}} \supset L$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_{r_n}})$ such that $\mathcal{I}^L_{m_{r_n}} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$.

By Proposition 2.2 we can find $m_n > m_{r_n}$ such that the spaces $(\mathcal{H}(L), \tau_w)$ and $H_b(V_{m_n})$ induce the same topology on each bounded subset B of $H_b(V_{m_{r_n}})$.

Since $H_b(V_{m_{r_n}}) \overset{i^L_{m_{r_n}}}{\hookrightarrow} H_b(V_{m_n})$ and $\mathcal{I}^L_{m_n} \circ i^L_{m_{r_n}} = \mathcal{I}^L_{m_{r_n}}$, we can define a continuous homomorphism $\Psi_n : H_b(U_n) \to H_b(V_{m_n})$ by $\Psi_n(f) = i^L_{m_{r_n}} \circ \widetilde{A}_n(f)$ for each $f \in H_b(U_n)$. So $A \circ \mathcal{I}^K_n = \mathcal{I}^L_{m_n} \circ \Psi_n$ and this means the following diagram



commutes. We claim that the homomorphism Ψ_n is Montel. Indeed, let $\mathcal{X} \subset H_b(U_n)$ be a bounded subset. Then $\widetilde{A}_n(\mathcal{X}) \subset H_b(V_{m_{r_n}})$ is a bounded subset. Since A is a Montel homomorphism, $\Psi_n(\mathcal{X}) = i^L_{m_{r_n}}(\widetilde{A}_n(\mathcal{X})) \subset H_b(V_{m_n})$ and the topology induced by $(\mathcal{H}(L), \tau_w)$ and $H_b(V_{m_n})$ on $\Psi_n(\mathcal{X})$ is the same. Then $\Psi_n(\mathcal{X})$ is a relatively compact subset of $H_b(V_{m_n})$ and therefore Ψ_n is Montel.

Conversely, suppose that for each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and a Montel homomorphism $\Psi_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \Psi_n = A \circ \mathcal{I}^K_n$. We will prove that A is a Montel homomorphism.

Let $\mathcal{X} \subset \mathcal{H}(K)$ be a bounded subset. By Proposition 2.2 there exists an $r \in \mathbf{N}$ such that $\mathcal{X}$ is a bounded subset of $H_b(U_r)$. Then using the hypothesis, there exist an open subset $V_{m_r} \supset L$ and a Montel homomorphism $\Psi_r : H_b(U_r) \to H_b(V_{m_r})$ such that $\mathcal{I}^L_{m_r} \circ \Psi_r = A \circ \mathcal{I}^K_r$. As $\Psi_r(\mathcal{X})$ is relatively compact subset and $A(\mathcal{X}) = \mathcal{I}^L_{m_r}(\Psi_r(\mathcal{X}))$, we have that A is a Montel operator and the proof is complete. $\square$

We now discuss under which conditions a homomorphism between algebras of holomorphic germs is reflexive.

Theorem 4.2. *Let E and F be Banach spaces. Let $K \subset E$ and $L \subset F$ be compact subsets, and let $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ be a continuous homomorphism. Then A is reflexive if and only if for* *each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and a reflexive continuous homomorphism $\Psi_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \Psi_n = A \circ \mathcal{I}^K_n$.*

Proof. Suppose A is reflexive. Given $U_n \subset E$ an open subset, by Theorem 3.1, there exist an open subset $V_{m_{r_n}} \supset L$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_{r_n}})$ such that $\mathcal{I}^L_{m_{r_n}} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$. By Corollary 2.3 there exists $m_n > m_{r_n}$ such that $\sigma(\mathcal{H}(L), \mathcal{H}(L)')_{|B} = \sigma(H_b(V_{m_n}), H_b(V_{m_n})')_{|B}$, for each bounded subset B of $H_b(V_{m_{r_n}})$. Now, a slight modification of arguments from Theorem 4.1 gives the proof. □

Before proving Proposition 4.3 we need some preparation. Let $\mathcal{P}(E)$ denote the algebra of all continuous polynomials on E. We denote by $(E, \sigma(E, \mathcal{P}(E)))$ the space E endowed with the coarsest topology making all $P \in \mathcal{P}(E)$ continuous. The topology $\sigma(E, \mathcal{P}(E))$ is a regular Hausdorff topology such that $(E, \|\cdot\|) \geq (E, \sigma(E, \mathcal{P}(E))) \geq (E, \sigma(E, E'))$. Thus it follows that $\sigma(E, \mathcal{P}(E))$ is angelic, see [**3**, Corollary 1], and consequently the concepts (relatively) countably compact, (relatively) sequentially compact and (relatively) compact all agree with respect to this toplogy. A Banach space E is called a Λ-space, if all null sequences in $\sigma(E, \mathcal{P}(E))$ are norm convergent, and hence, convergent sequences in $\sigma(E, \mathcal{P}(E))$ are also norm convergent. All superreflexive spaces and ℓ_1 are Λ-spaces [**9**].

Proposition 4.3. *Let E be a Λ-space with the approximation property and F be a Banach space. Let $K \subset E$ be an absolutely convex compact set, $L \subset F$ a compact set and $A : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(L), \tau_w)$ a continuous homomorphism. If for each open subset $U_n \supset K$ there exist an open subset $V_{m_n} \supset L$ and a reflexive operator $\widetilde{A}_n : H_b(U_n) \to H_b(V_{m_n})$ such that $\mathcal{I}^L_{m_n} \circ \widetilde{A}_n = A \circ \mathcal{I}^K_n$, then A is a Montel operator.*

Proof. Since $U_n = K + B_{1/n}(0)$ is an absolutely convex subset and E has the approximation property, by [**4**, Proposition 9] we have that Φ_n maps V_{m_n}-bounded into $\sigma(E, \mathcal{P}(E))$-relatively compact subsets in $U_n \subset E$. Since the topology $\sigma(E, \mathcal{P}(E))$ is angelic, see [**3**, Corollary 1], we have that Φ_n maps V_{m_n}-bounded subset into relatively compact subset in $U_n \subset E$. Consequently Φ_n is Montel and $\widetilde{A}_{n_{\Phi_n}} : H_b(U_n) \to H_b(V_{m_n})$ is a Montel homomorphism by [**4**, Proposition 13]. Now, by Theorem 4.1 the proof is complete. □

The next example gives a non Montel composition operator which is a reflexive composition operator.

Example 4.1. Let E be a Tsirelson space, and let $\Phi \in H(E,E)$ be defined by $\Phi(z) = z/2$ for all $z \in E$. If we take U and V as open sets such that $\Phi(V) \subset U$, then it is not difficult to see that Φ takes V-bounded sets in U-bounded sets. If $K \subset E$ is a balanced compact set, then the composition operator $A_\Phi : (\mathcal{H}(K), \tau_w) \to (\mathcal{H}(K), \tau_w)$ given by $A_\Phi(f) = f \circ \Phi$ is reflexive and however A_Φ is not Montel.

Indeed, for each open subset $U_n \supset K$ by Theorem 3.1 there exist an open subset $U_{m_n} \supset K$ with $m_n \geq n$ and a continuous homomorphism $\widetilde{A}_n : H_b(U_n) \to H_b(U_{m_n})$ such that $\mathcal{I}^K_{m_n} \circ \widetilde{A}_n = A_\Phi \circ \mathcal{I}^K_n$. For each $n \in \mathbf{N}$, let $\Phi_n = \Phi_{|U_{m_n}}$. Then it is possible to prove $\widetilde{A}_n(f) = f \circ \Phi_n$, for each $f \in H_b(U_n)$. As E is a Tsirelson space, we have that $H_b(U_n)$ is reflexive and consequently $\widetilde{A}_n$ is reflexive. Therefore, according to Theorem 4.2, A_Φ is a reflexive operator.

Now suppose that A_Φ is a Montel operator. By Theorem 4.1 we have that $\widetilde{A}_n$ is a Montel operator. From Proposition 13 of [**4**] we have that Φ_n is a compact operator. As $m_n > n$, then $\Phi(U_{m_n})$ is U_n-bounded and so $(1/4)B_E \subset m_n \cdot \overline{\Phi_n(U_{m_n})}$ is a compact subset. Thus E has a finite dimension. This is a contradiction; consequently, A_Φ is not a Montel operator.

Acknowledgment. We would like to thank Prof. Jorge Mujica for helpful comments on the subject of this paper.

REFERENCES

1. J. Bonet, P. Domanski, M. Lindström and M. S. Ramanujan, *Operator spaces containing c_0 or ℓ^∞*, Results Math. **28** (1995), 250–269.

2. S. Dineen, *Complex analysis on infinite dimensional spaces*, Springer-Verlag, London, 1999.

3. K. Floret, *Weakly compact sets*, Lecture Notes in Math., vol. 801, Springer, New York, 1978.

4. P. Galindo, M.L. Lourenço and L.A. Moraes, *Compact and weakly compact homomorphisms on Fréchet algebras of holomorphic functions*, Math. Nachr. **236** (2002), 109–118.

5. M. González and J. Gutiérrez, *Schauder type theorems for differentiable and holomorphic mappings*, Math. Nachr. **122** (1996), 325–343.

6. ———, *Gantmacher type theorems for holomorphic mappings*, Math. Nachr. **186** (1997), 131–145.

7. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

8. J. Horvath, *Topological vector spaces and distributions*, Vol. 1., Addison-Wesley, Massachusetts, 1966.

9. J. Jaramillo and A. Prieto, *The weak-polynomial convergence in a Banach space*, Proc. Amer. Math. Soc. **118** (1993), 463–468.

10. J. Mujica, *Gérmenes holomorfos y funciones holomorfas en espacios de Fréchet*, Publicaciones del Depto. de Teoria de Funciones, Universidad de Santiago de Compostela, $N^{\underline{\text{ro}}}$1, 1978.

11. ———, *Spaces of germs of holomorphic functions*, Adv. Math. Suppl. Stud., vol. 4, Academic Press, New York, 1979, pp. 1–41.

12. ———, *Ideals of holomorphic functions on Fréchet spaces*, in *Advances in holomorphy* (J.A. Barroso, ed.), North-Holland Publ. Co., Amsterdam, 1979.

13. ———, *Complex analysis in Banach spaces*, North-Holland Math. Stud. **120**, North-Holland, Amsterdam, 1986.

14. ———, *Ideals of holomorphic functions on Tsirelson's space*, Arch. Math. **76** (2001), 292–298.

15. L. Nachbin, *On the topology of the space of all holomorphic function on a given open subset*, Indag. Math. **29** (1967), 366–368.

16. O. Nicodemi, *Homomorphisms of algebras of germs of holomorphic functions*, in *Functional analysis, holomorphy and approximation theory* (S. Machado, ed.), Lecture Notes in Math., vol. 843, Springer-Verlag, New York, 1981, pp. 534–546.

17. B. Tsirelson, *Not every Banach space contains an imbedding of ℓ_p or c_0*, Funct. Anal. Appl. **8** (1974), 138–144.

ICET, Universidade Paulista–CEP: 06500-000 São Paulo, Brazil
E-mail address: `lch@ime.usp.br`

Departamento de Matemática, Universidade de São Paulo, CP 66281 - CEP: 05315-970 São Paulo, Brazil
E-mail address: `mllouren@ime.usp.br`