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THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY FOR AN ARBITRARY LENGTH TWO MODULE

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ABSTRACT. Let V be an arbitrary R-module of length 2 with $n \geq 3$ submodules of length 1. Then every permutation of the length 1 submodules is induced by an isomorphism $V \xrightarrow{\sim} V$ if and only if $n = 3$ or 4.

1. Introduction. In this note all rings R have an identity and all R-modules V are unital. We write $\mathcal{L}(V)$ for the lattice of all submodules of V. Every module isomorphism $f : V \xrightarrow{\sim} V$ clearly induces a lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ where $F(W) := f(W)$. Call V *linearly induced* if conversely for each lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ there is a module isomorphism $f : V \xrightarrow{\sim} V$ such that $F(W) = f(W)$ for all $W \in \mathcal{L}(V)$. A variant of the *fundamental theorem of projective geometry* can be phrased as follows:

Theorem 1 [**1**, p. 62]. *Let K be a division ring such that every automorphism is inner. Then each K-vector space of finite dimension ≥ 3 is linearly induced.*

(In the classic fundamental theorem of projective geometry [**1**, p. 44] there is *no restriction* on the division ring but then the lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ perhaps is only induced by a *semilinear* bijection $f : V \to V$. We do not wish to bother about semilinearity in this article.)

In particular, in Theorem 1 division rings without proper automorphisms, such as $K = \mathbf{R}$, comply. The lattice $\mathcal{L}(V)$ of subspaces of the K-vector space V is often called the *projective geometry* associated with K. The dimension $1, 2, 3$ subspaces are the *points, lines, planes* of the projective geometry. Lattice isomorphisms $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ are called

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projectivities in [**1**] (but in modern texts the meaning of collineation and projectivity may be switched).

Theorem 1 fails for two-dimensional vector spaces. In this case *von Staudt* type theorems take over. They essentially assert that it works for permutations of the points that *preserve cross ratios* as defined in [**1**, p. 71]. This condition is necessary in the sense that each K-linear isomorphism $K^2 \xrightarrow{\sim} K^2$ induces a cross ratio preserving permutation $\mathcal{L}(K^2) \to \mathcal{L}(K^2)$. Here is an example of a von Staudt type theorem:

Theorem 2 [**1**, p. 87]. *The division ring K is commutative if and only if the identity is the only permutation of the K-projective line which preserves cross ratios and possesses three fixed points.*

As a consequence, note the following: Suppose K is commutative and the K-projective line $\mathcal{L}(V)$ has $n \geq 5$ points (this amounts to $|K| \geq 4$). Then obviously there is a non-identity permutation F of $\mathcal{L}(V)$ that fixes three points. By Theorem 2 each linear isomorphism $f : V \xrightarrow{\sim} V$ fixing three proper subspaces must be the identity. Therefore V cannot be linearly induced.

Theorem 1 has been generalized in many ways in order to accommodate rings R other than K. For instance, the m-dimensional "projective geometry" associated with a *ring* R is often defined as the set of all direct summands of the module R^{m+1}. Theorem 2 has been generalized to a lesser extent; usually the concept of cross ratio is somehow adapted to the relevant ring.

Rather than looking at R-modules R^2, which have length bigger than two unless R is a field, in this paper we let V be any R-module of length two. Also, as opposed to the usual generalizations of Theorem 2, we do not focus on special types of permutations $F : \mathcal{L}(V) \to \mathcal{L}(V)$, but focus on those numbers $n := |\mathcal{L}(V)| - 2$ for which Theorem 1 holds unconditionally.

So let V be an arbitrary length two module. Then the lattice $\mathcal{L}(V)$ is isomorphic to the length two modular lattice M_n completely characterized by the number $n = n(V)$ of atoms. When n is infinite we write $n = \infty$ rather than distinguishing between infinite cardinals. For $n(V) = 1$ the only lattice isomorphism $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ is the identity, which is induced by the identity $V \xrightarrow{\sim} V$. Let $n(V) = 2$, so $\mathcal{L}(V) = \{\langle 0\rangle, U_1, U_2, V\}$. Such a V is necessarily cyclic. The only nontrivial lattice isomorphism F switches U_1 and U_2. Clearly F is induced by a module isomorphism $f : V \xrightarrow{\sim} V$ if and only if $U_1 \simeq U_2$. Things get more interesting for $n(V) \geq 3$; here is our (not so fundamental) *theorem of projective lines*.

Theorem 3. *Let R be an arbitrary ring and let V be an R-module of length two with $n(V) \geq 3$. Then V is linearly induced if and only if $n(V) \leq 4$.*

Proof. The following fact will be crucial.

(1) Let $\mathcal{L}(V) = \{\langle 0\rangle, U_1, U_2, U_3, \ldots, U_n, V\}$. Then the map T which sends ϕ to $\{u + \phi(u) \mid u \in U_1\}$ is a bijection between the set of isomorphisms $\phi : U_1 \xrightarrow{\sim} U_2$ and the set $\{U_3, \ldots, U_n\}$.

We omit the straightforward verification; notice that for given U_i ($3 \leq i \leq n$) the ϕ with $T(\phi) = U_i$ is the map which sends $u \in U_1$ to the *unique* $u' \in U_2$ with $u + u' \in U_i$. It follows from (1) that

$$(2) \qquad |\mathrm{Aut}\,(U_1)| = |\{\phi : U_1 \xrightarrow{\sim} U_2\}| = n - 2 \quad (\text{note } \infty - 2 = \infty).$$

By Schur's lemma $\mathrm{End}\,(U_1) = \mathrm{Aut}\,(U_1) \cup \{0\}$ is a division ring. Thus, if $|\mathrm{End}\,(U_1)| < \infty$, then Wedderburn's theorem yields $\mathrm{End}\,(U_1) \simeq GF(q)$ where the latter is the Galois field of cardinality q (= power of a prime). Summarizing, either $n = \infty$ or $q - 1 = n - 2$. So $n = q + 1$.

$$(3) \qquad |\mathrm{Aut}\,(V)| = n(n-1)(n-2)^2$$

Indeed, the module automorphisms $f : V \xrightarrow{\sim} V$ are exactly the maps $f_1 \oplus f_2 : U_1 \oplus U_2 \to U_i \oplus U_j$, $i \neq j$, where $f_1 : U_1 \xrightarrow{\sim} U_i$ and $f_2 : U_2 \xrightarrow{\sim} U_j$ are module isomorphisms. The number of pairs (i, j) is $n(n-1)$ and by (2) the number of f_1's, respectively f_2's, is $n - 2$.

One checks that $n(n-1)(n-2)^2 < n!$ for all $n \geq 6$. This includes infinite cardinals n since then $n(n-1)(n-2)^2 = n < 2^n \leq n!$. Thus, for $n \geq 6$, the mere cardinality argument (3) guarantees lattice automorphisms $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ which are not induced by any $f \in \mathrm{Aut}\,(V)$. Can one explicitly pinpoint such an F? Provided the division ring $\mathrm{End}\,(U_1)$ has a nontrivial center and $n \geq 5$ we shall manage to do so. In particular this will settle the case $n = 5$ since then End $(U_1) \simeq GF(4)$. So suppose $n \geq 5$ and let $F : \mathcal{L}(V) \overset{\sim}{\to} \mathcal{L}(V)$ be any fixed lattice isomorphism such that $F(U_i) = U_i$, $1 \leq i \leq 3$, $F(U_i) \neq U_i$, $4 \leq i \leq n$. Such an F exists because $n \geq 5$. (Recall that for the very special case of a two-dimensional *vector space* V over a *commutative* division ring this F does the job due to Theorem 2.) Suppose $f : V \overset{\sim}{\to} V$ is a module isomorphism that induces F. We want to derive a contradiction. According to (1) we have

$$U_3 = \{u + \phi(u) \mid u \in U_1\} \tag{4}$$

for some unique isomorphism $\phi : U_1 \overset{\sim}{\to} U_2$. Because by assumption $f(U_1) = U_1$, we have

$$U_3 = \{f(u) + \phi(f(u)) \mid u \in U_1\}. \tag{5}$$

Using (4), one derives

$$U_3 = f(U_3) = \{f(u) + f(\phi(u)) \mid u \in U_1\}. \tag{6}$$

Because of $f \circ \phi : U_1 \overset{\sim}{\to} U_2 \overset{\sim}{\to} U_2$, both (5) and (6) are representations of U_3 of type (1). Hence

$$\phi \circ f(u) = f \circ \phi(u) \quad (u \in U_1) \tag{7}$$

by the uniqueness of this representation. By assumption End (U_1) contains a central element $\psi \neq 0, 1$. By (1) we have

$$\{u + \phi \circ \psi(u) \mid u \in U_1\} = U_j \tag{8}$$

for some $j \in \{4, \dots, n\}$. Now

$$\begin{aligned} f(U_j) &\overset{(8)}{=} \{f(u) + f \circ \phi \circ \psi(u) \mid u \in U_1\} \\ &\overset{(7)}{=} \{f(u) + \phi \circ f \circ \psi(u) \mid u \in U_1\} \\ &= \{f(u) + \phi \circ \psi \circ f(u) \mid u \in U_1\} \overset{(8)}{=} U_j \end{aligned}$$

which contradicts $F(U_j) \neq U_j$.

Now we show that V is linearly induced when $n = 3$ or 4. The case $n = 3$ being analogous, we only do $n = 4$, so $\mathcal{L}(V) =$

$\{0, U_1, U_2, U_3, U_4, V\}$. Then $\mathrm{End}\,(U_1) \simeq GF(3)$, so $\mathrm{Aut}\,(U_1) = \{\mathrm{id}, \psi\}$. Analogous to (4) and (8) above we have

$$U_3 = \{u + \phi(u) |\ u \in U_1\}$$
$$U_4 = \{u + \phi \circ \psi(u) |\ u \in U_1\}$$

where $\phi : U_1 \xrightarrow{\sim} U_2$ is a unique isomorphism. Since the symmetric group of degree 4 is generated by 2-cycles, it suffices to show that the lattice isomorphisms $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ determined by the permutation

$$U_1 \longmapsto U_1, \quad U_2 \longmapsto U_2, \quad U_3 \longmapsto U_4 \longmapsto U_3$$

is linearly induced. Put $f := \psi \oplus id :\ U_1 \oplus U_2 \xrightarrow{\sim} U_1 \oplus U_2$. Using $\psi \circ \psi = id$ we get

$$f(U_3) = \{\psi(u) + \phi(u) \mid u \in U_1\} = \{\psi(u) + \phi \circ \psi(\psi(u)) \mid u \in U_1\} = U_4.$$

Because f induces a bijection $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$, this forces $f(U_4) = U_3$. □

What else can be said about an arbitrary length two module V? As to its endomorphism ring, if $n(V) \geq 3$ then $V \simeq U_1 \oplus U_1$, and so $\mathrm{End}\,(V)$ is isomorphic to the ring $M_2(\mathrm{End}\,(U_1))$ of 2×2 matrices with entries from the division ring $\mathrm{End}\,(U_1)$. In particular, when $n = n(V) < \infty$, then $n = q + 1$ and $\mathrm{End}\,(V) \simeq M_2(GF(q))$. The reader may check that the number of invertible 2×2 matrices over $GF(q)$ is indeed $n(n-1)(n-2)^2$ in accordance with (3). What can be said about the Abelian group $(V, +)$? Not much, but if $n(V) < \infty$ and $V = {}_RV$ is *noncyclic*, then $(V, +)$ turns out to be $(GF(q)^2, +)$. This does not imply that $R \simeq GF(q)$. Whether V is cyclic or not, $n(V) < \infty$ always implies that $n = q+1$ for some prime power q. Now 7 is the first integer ≥ 3 not of type $q + 1$, and so there cannot be a length two module V with $n(V) = 7$.

This relates to a major unsolved problem of universal algebra: Which finite lattices occur as congruence lattices of a finite algebra? A breakthrough was made in [**3**] where the problem is reduced to intervals in subgroup lattices of finite groups. In particular, which lattices M_n occur as such an interval? It has, e.g., been shown in [**2**] that the answer is affirmative for $n = q + 2$. Thus, $n = 7$ works, but not with modules.

When is a module V of length at least three linearly induced? Theorem 3 suggests that this is unlikely, unless either $\mathcal{L}(V)$ has no interval sublattice M_5 or the identity is the only lattice isomorphism of $\mathcal{L}(V)$. As mentioned in the introduction, a module isomorphism $V \overset{\sim}{\to} V$ trivially induces a lattice isomorphism $\mathcal{L}(V) \overset{\sim}{\to} \mathcal{L}(V)$. But what if $f : V \to V$ is merely a *homogeneous* bijection, i.e., satisfying $f(\lambda x) = \lambda f(x)$ but not necessarily $f(x + y) = f(x) + f(y)$? Call V hom-proj if such a f nevertheless always induces a lattice isomorphism $\mathcal{L}(V) \overset{\sim}{\to} \mathcal{L}(V)$. It is easy to see that every length two module is hom-proj, but many others are as well [**4**].

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