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GREEN'S THEOREM WITHOUT DERIVATIVES

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ABSTRACT. The result is established for a Jordan measurable region with rectifiable boundary. The entity F to be integrated by the new plane integral is a function of axis-parallel rectangles, finitely additive on non-overlapping ones, hence unambiguously defined and additive on "figures," i.e., finite unions of axis-parallel rectangles. Define its integral over Jordan measurable S as the limit of its value on the figures, which contain a subfigure of S and are contained in a figure containing S, as the former/complements of the latter expand directedly to fill out S/the complement of S. The integral over every Jordan measurable region exists when additive F is "absolutely continuous" in the sense of converging to zero as the area enclosed by its argument does, or with F the circumferential line integral $\oint P\,dx + Q\,dy$ for P, Q continuous at the rectifiable boundary of S and integrable along axis-parallel line segments. Thus, the equality of this area integral with the line integral around the boundary, to be proved, follows for the various integrals of divergence presented in: *The Riemann approach to integration*, W.F. Pfeffer, Cambridge University Press, New York, 1993.

1. Introduction. In advanced calculus texts, Green's theorem is presented for continuous vector fields with continuous first partial derivatives, defined in a region containing a simple piecewise smooth curve enclosing an area of not too complicated shape. More careful treatments, e.g., [**1**, Sections 10–14], dispense with the continuity of the derivatives in favor of their (bounded existence and) integrability over the interior; recently this requirement has been successively weakened further to integrability of the partials in the "generalized Riemann" sense [**6**, subsection 7.12] and beyond to "gage integrability" [**7**] which even follows from the mere existence of the derivative. By modifying this last integral further, it proves possible to obtain the theorem for a continuous vector field with no differentiability assumption whatsoever.

The "integral" of an additive rectangle function over a Jordan measurable set. For plane Jordan content, see [**1**, subsection

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10-4], [**3**] or [**4**]: Inner Jordan content of a bounded set S is the sup of areas of finite unions of axis-parallel rectangles in the interior of S; outer Jordan content, the inf of areas of such which cover closure of S. Their equality, "Jordan measurability," comes to the boundary of S having Jordan content (equivalently, by compactness, Lebesgue measure) zero.

Let F be a function of axis-parallel rectangles, finitely additive on non-overlapping ones, hence unambiguously defined and additive on "figures," i.e., finite unions of such. Following this pattern, we propose to define its integral over S as the common limit, on covers of S by partitions into axis-parallel rectangles, as the partition is refined, of its values on the partition's smallest subfigure which contains the closure of S and largest subfigure contained in its interior. (This is "Riemann- style"; one comes back to the usual Riemann formalism when the rectangle function is the area of the rectangle times one of the values that a point function takes in it.) The integral over every Jordan measurable set S exists when additive F is "absolutely continuous" in the sense of converging to zero as the area enclosed by its argument does—indeed, the difference between a containing and a contained figure of S eventually has small area and so the value of F on the difference would be small, hence the values of F on these figures eventually close. This obtains for the more usual kinds of area integral of a bounded integrand. However, independent of this absolute continuity, the circumferential line integral $\oint P\,dx + Q\,dy$, for P, Q integrable along axis-parallel line segments and continuous at the boundary (if rectifiable), construed as a rectangle function, will now be shown integrable in this sense to the usual line integral around the boundary as value.

Green's theorem for Jordan regions with rectifiable boundaries. Cover the boundary Γ with non-overlapping axis-parallel rectangles of sufficiently small side length (finitely many by compactness) so that each encloses only a connected piece of Γ (which is locally connected, since it is the continuous image of a compact such) and enters and leaves the rectangle through the shorter opposite sides (if part of Γ is a horizontal or vertical interval, count this as a degenerate rectangle). Thus, the sum of their perimeters is at most four times the arc length of Γ. Construe the boundary rectangle sides in the bounded component of S as the boundary of a figure in the interior of S, those in the unbounded component as the boundary of a cover.

It is classical that the line integral of a continuous vector-valued function along a rectifiable curve Γ exists [**1**, subsection 10-10]. Since a constant vector line integrates around a closed curve to zero, every circumferential line integral is bounded by the oscillation of the vector integrand times the arc length of the circumference. Adding up the line integrals (of a vector function continuous at Γ and integrable over axis-parallel lines) around the circumferences (of pieces of Γ and adjacent rectangle sides) shows the line integral along Γ close to that along each of the boundaries (if sufficiently close to Γ)—and of course along boundaries derived from any smaller rectangle cover of Γ, which thus bound larger interior figures and smaller covering figures of S—hence equal to the above defined surface integral for the "flux": i.e., the circumferential line integral of the vector field construed as a rectangle function.

Supplement. The same argument works in higher dimensions; I carry it through for dimension three (thus replacing "Green" with "Gauss"). Let Γ be a homeomorph of the 2-sphere, which bounds a Jordan measurable volume S in 3-space and is "rectifiable," i.e., of finite total surface area and a.e. (for surface area) C': then every continuous vector-valued function's normal component is integrable over Γ. Cover Γ with non-overlapping axis-parallel paralellepipeds of sufficiently small side area (finitely many by compactness) so that each meets Γ in a connected piece and whose boundary meets Γ in opposite sides of area no greater than that of the non-meeting pair (if part of Γ is a horizontal or vertical rectangle, count this as a degenerate paralellepiped). Thus, the sum of their perimeters is at most six times the area of Γ. Construe the boundary rectangles in the bounded component of S as the boundary of a figure in the interior of S, those in the unbounded component as the boundary of a cover. Since a constant vector's normal component integrates around a closed surface to zero, every circumferential surface integral is bounded by the oscillation of the vector integrand times the area of the perimeter. Adding up the surface integrals (of a vector field's normal component, continuous at Γ and integrable over axis-parallel rectangles) around the circumferences (of pieces of Γ and adjacent rectangles) shows the surface integral of the normal component along Γ close to that along each of the boundaries (if sufficiently close to Γ)—and of course along boundaries derived from any smaller paralellepiped cover of Γ, which thus bounds larger interior figures and smaller covering figures of S—hence, equal to the volume integral for the "flux": i.e., the circumferential surface integral of the vector field construed as a function of axis-parallel paralellepipeds.

This has applications to the integrals discussed in [**7**], which all feature additive rectangle functions under various supplementary hypotheses of differentiability. His "solids" are Jordan measurable and so his "divergence theorems" would receive a better specified formulation from the above.

Addendum. The integration domains are closed bounded Jordan measurable sets. One can define an "inner" ("lower") integral as the refinement limit over figures in the interior; an "outer" ("upper") over covering figures. By additivity only the region near the boundary affects convergence. Integrability means that both these limits exist and are equal. The integral is linear in its function argument and additive on disjoint domains (on non-overlapping ones if it integrates to zero over the common part of the boundary).

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