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EXACT STRUCTURE OF POSITIVE SOLUTIONS FOR A P-LAPLACIAN PROBLEM INVOLVING SINGULAR AND SUPERLINEAR NONLINEARITIES

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ABSTRACT. We study the structure of positive solutions for a p-Laplacian boundary value problem involving singular and superlinear nonlinearities. We prove that there exists $\lambda^* > 0$ such that the problem has exactly two positive solutions for $0 < \lambda < \lambda^*$, exactly one positive solution for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. More precisely, we give a complete description of the structure of the solution set. Our result partially generalizes some results of Wei [**12**].

1. Introduction. In this paper we study the structure of positive solutions $u \in C^1[-1,1] \cap C^2(-1,1)$ of the nonlinear two point boundary value problem

(1.1)
$$\begin{cases} (\varphi_p(u'(x)))' + \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j} = 0, \quad -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $p > 1$, $\varphi_p(y) = |y|^{p-2}y$, $(\varphi_p(u'))'$ is the one-dimensional p-Laplacian, $\lambda > 0$ is a bifurcation parameter, and $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfies

$$(1.2) \quad \begin{cases} -1/(p+1) \leq q_1 < q_2 < \cdots < q_m < p-1 \\ \leq r_1 < r_2 < \cdots < r_n, \quad m, n \geq 1, \\ q_1 < 0, \quad r_n > p-1, \quad a_i > 0 \quad \text{for } i = 1, 2, \dots, m \\ \text{and } b_j > 0 \quad \text{for } j = 1, 2, \dots, n, \\ \text{and (either } r_1 > p-1 \text{ or } b_1 < (p-1)((\pi/p)\csc(\pi/p))^p). \end{cases}$$

Note that, in (1.2),

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(a) If $r_1 = p - 1$ and $b_1 \geq (p-1)((\pi/p)\csc(\pi/p))^p$, then it can be easily proved that (1.1) has no positive solution for any $\lambda > 0$. (Note that $(p-1)((\pi/p)\csc(\pi/p))^p$ is the first eigenvalue of the one-dimensional operator $-(\varphi_p(u'))'$ on $(-1,1)$ with zero Dirichlet boundary conditions.)

(b) Assume that $p = 2$. If $r_1 = p - 1 = 1$, then nonlinearity $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ contains a linear term. In addition, for fixed $\lambda > 0$, $f_\lambda(u)$ is either a *convex-concave-convex* or a *convex* function on $(0,\infty)$ if $0 < q_m < 1$. If $-1/3 \leq q_m \leq 0$, then $f_\lambda(u)$ is a *convex* function on $(0,\infty)$.

(c) We allow q_m to be positive, zero or negative.

Sun, Wu, and Long [**5**] studied combined effects of singular and superlinear nonlinearities in a singular problem

$$\begin{cases} \Delta u + \lambda u^q + \sigma u^r = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset R^N$, $N \geq 3$, is a bounded domain. They [**5**, Theorem 2] mainly proved

Theorem 1.1. *Let $-1 < q < 0$, $1 < r < (N+2)/(N-2)$ and $N \geq 3$. Then, for every $\sigma > 0$, there exists $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*]$, problem* (1.3) *possesses at least one weak positive solution $u \in H_0^1(\Omega)$.*

Recently, Wang and Yeh [**9**, Theorem 2.2] studied the exact structure of positive solutions of (1.1) by applying modified time-map techniques for $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfying

$$\begin{cases} 0 < q_1 < q_2 < \cdots < q_m < p-1 \\ \leq r_1 < r_2 < \cdots < r_n, \quad m,n \geq 1, \quad r_n > p-1, \\ a_i > 0 \quad \text{for } i = 1,2,\ldots,m \quad \text{and} \quad b_j > 0 \quad \text{for } j = 1,2,\ldots,n, \\ \text{and (either } r_1 > p-1 \text{ or } b_1 < (p-1)((\pi/p)\csc(\pi/p))^p). \end{cases} \tag{1.4}$$

To (1.1), we generalize [**9**, Theorem 2.2] for nonlinearities $f_\lambda \in C[0,\infty) \cap C^2(0,\infty)$ satisfying (1.4) to nonlinearities $f_\lambda \in C^2(0,\infty)$ satisfying (1.2) as in Theorem 2.1 stated behind. Note that, to study

(1.1), we first study the structure of positive solutions of

(1.5)
$$\begin{cases} (\varphi_p(u'(x)))' + \lambda \left(\sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}\right) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $f = \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfies

(1.6)
$$\begin{cases} -1/(p+1) \le q_1 < q_2 < \cdots < q_m < p-1 \\ \le r_1 < r_2 < \cdots < r_n, \quad m, n \ge 1, \quad r_n > p-1, \\ a_i > 0 \quad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad b_j > 0 \quad \text{for } j = 1, 2, \ldots, n. \end{cases}$$

Very recently, motivated by a result of Agarwal and O'Regan [**1**], Wei [**12**] studied the exact multiplicity and properties of positive solutions of the singular problem

$$(1.7) \qquad \begin{cases} u''(x) + \lambda(u^q + ku + u^r) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ is a bifurcation parameter, $k \ge 0$, and q, r satisfy either

(A1) $-1/3 \le q \le 0$, $1 < r < \infty$, or

(A2) $-1 < q < -1/3$, and

$$1 < r < 1 + \left[\frac{q-1}{2(1+3q)}\right] \left[(3+5q) + \sqrt{(3+5q)^2 - 8(1+3q)(1+q)}\right].$$

Wei [**12**, Theorem 1] mainly proved

Theorem 1.2. *Consider* (1.7) *and assume that* (A1) *and* (A2) *are satisfied. Then there exists* $\lambda^* > 0$ *such that* (1.7) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $0 < \lambda < \lambda^*$, *exactly one positive solution* u_{λ^*} *for* $\lambda = \lambda^*$ *and no positive solution for* $\lambda > \lambda^*$. *Moreover, if we denote* $u_{\lambda^*} = v_{\lambda^*}$ *when* $\lambda = \lambda^*$, *then for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$, *the positive solutions of* (1.7) *satisfy*

(a) $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$,

(b) $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ *for* $-1 < x < 1$,

(c) $v_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/2}\, v_{\lambda_2}(x)$ *for* $-1 < x < 1$,

(d) $\lim_{\lambda\to 0^+} u_\lambda(x) = 0$ *and* $\lim_{\lambda\to 0^+} v_\lambda(x) = \infty$ *for* $-1 < x < 1$.

Remark 1. Consider (1.7). If $-1 < q < 0$, then (classical) positive solutions $u \in C^1[-1,1]$. However, if $q \le -1$, then positive solutions $u \notin C^1[-1,1]$. See [**3, 6, 7**].

The paper is organized as follows. Section 2 contains the statement of Theorems 2.1 and 2.2 which are the main results in this paper. Section 3 contains the lemmas needed to prove Theorems 2.1 and 2.2. Section 4 contains the proofs of Theorems 2.1 and 2.2. Finally, in Section 5, to Theorem 2.1 and (1.2), we give an example to demonstrate that the hypotheses of positive coefficients a_i and b_j in nonlinearities $f_\lambda = \lambda\sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ can be weakened.

2. Main results. The main results in this paper are following Theorems 2.1 and 2.2. In Theorem 2.2, we first study the exact structure of positive solutions of (1.5) for $f = \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfying (1.6). It extends Wang and Yeh [**9**, Theorem 2.1] for p-Laplacian problem (1.5) from $q_1 > 0$ to $q_1 \ge -1/(p+1)$, and it partially generalizes some results of Theorem 1.2 for Laplacian problem (1.7) to p-Laplacian problem (1.5). Then in Theorem 2.1, we apply Theorem 2.2 to study the exact multiplicity and structure of positive solutions of (1.1) for $f_\lambda = \lambda\sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfying (1.2). It partially generalizes some results of Theorem 1.1 in the one-dimensional case, and it extends Wang and Yeh [**9**, Theorem 2.2] for the p-Laplacian problem (1.1).

Recall the Beta function as follows, see e.g., [**4**, p. 18]:

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \quad x, y > 0.$$

Theorem 2.1. (*See Figure* 1). *Consider* (1.1) *where* $f_\lambda = \lambda\sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ *satisfies* (1.2). *Then*

(i) *There exists* $\lambda^* > 0$ *such that* (1.1) *has exactly two positive solutions* u_λ, v_λ, *with* $u_\lambda < v_\lambda$ *for* $0 < \lambda < \lambda^*$, *exactly one positive*

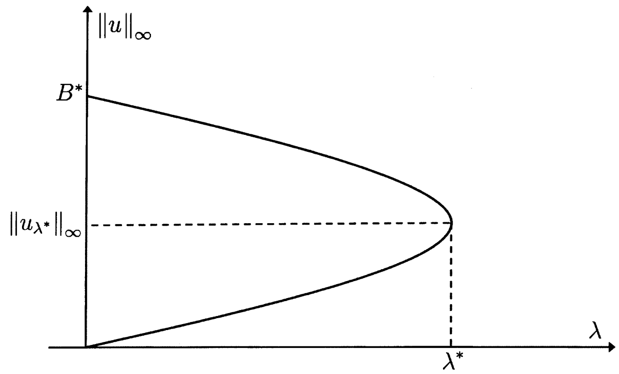

FIGURE 1. Bifurcation diagram of (1.1).

solution u_{λ^*} *for* $\lambda = \lambda^*$*, and no positive solution for* $\lambda > \lambda^*$*. Moreover, if we denote* $u_{\lambda^*} = v_{\lambda^*}$ *when* $\lambda = \lambda^*$*, then for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$*, the positive solutions of* (1.1) *satisfy*

(a) $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ *and* $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$*,*

(b) $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ *for* $-1 < x < 1$*,*

(c) $v_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p}\, v_{\lambda_2}(x)$ *for* $-1 < x < 1$*.*

(ii) *Let* u *be a positive solution of* (1.1)*. Then there exists a unique positive number* B^* *defined by* (4.6) *below such that* $\|u\|_\infty < B^*$*. In addition, if* $n = 1$*,*

$$B^* = \left[\left(\frac{p-1}{pb_1\,(r_1+1)^{p-1}}\right)^{1/p} B\left(\frac{p-1}{p}, \frac{1}{r_1+1}\right)\right]^{p/(r_1-p+1)}.$$

(iii) *For* $0 < \lambda < \lambda^*$*, let* u_λ*,* v_λ *be the two positive solutions of* (1.1) *with* $u_\lambda < v_\lambda$*. Then* $\|u_\lambda\|_\infty < \|u_{\lambda^*}\|_\infty < \|v_\lambda\|_\infty$*,* $\lim_{\lambda\to 0^+}\|u_\lambda\|_\infty = 0$*, and* $\lim_{\lambda\to 0^+}\|v_\lambda\|_\infty = B^*$*.*

(iv) *If* $r_1 = p-1$*, then for fixed* a_i*,* b_j*,* q_i*,* r_j*,* $1 \le i \le m$ *and* $2 \le j \le n$*, positive numbers* $\lambda^* = \lambda^*(b_1)$ *and* $B^* = B^*(b_1)$ *are both strictly decreasing in* $b_1 \in (0, (p-1)((\pi/p)\csc(\pi/p))^p)$*. In addition,*

(2.1)

$$\lambda^*(b_1) \longrightarrow 0 \quad \text{and} \quad B^*(b_1) \longrightarrow 0 \text{ as } b_1 \longrightarrow \left(\left(p-1\right)\left(\frac{\pi}{p}\csc\frac{\pi}{p}\right)^p\right)^-.$$

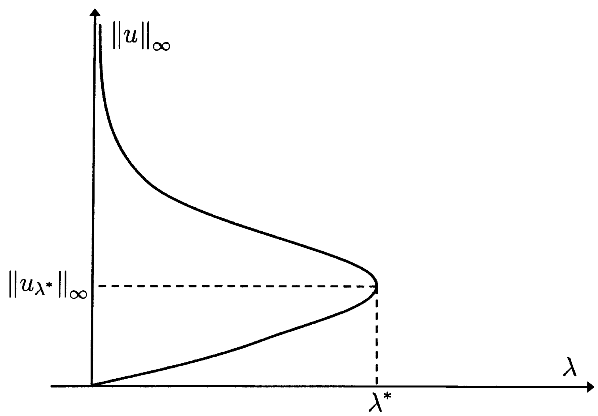

FIGURE 2. Bifurcation diagram of (1.5).

Theorem 2.2. (*See Figure* 2.) *Consider* (1.5) *where* $f = \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{r_j}$ *satisfies* (1.6). *Then*

(i) *There exists* $\lambda^* > 0$ *such that* (1.5) *has exactly two positive solutions* u_λ, v_λ *with* $u_\lambda < v_\lambda$ *for* $0 < \lambda < \lambda^*$, *exactly one positive solution* u_{λ^*} *for* $\lambda = \lambda^*$, *and no positive solution for* $\lambda > \lambda^*$. *Moreover, if we denote* $u_{\lambda^*} = v_{\lambda^*}$ *when* $\lambda = \lambda^*$, *then for* $0 < \lambda_1 < \lambda_2 \le \lambda^*$, *the positive solutions of* (1.5) *satisfy*

(a) $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ *and* $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$,

(b) $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ *for* $-1 < x < 1$,

(c) $v_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p} v_{\lambda_2}(x)$ *for* $-1 < x < 1$.

(ii) *For* $0 < \lambda < \lambda^*$, *let* u_λ *and* v_λ *be the two positive solutions of* (1.5) *with* $u_\lambda < v_\lambda$. *Then* $\|u_\lambda\|_\infty < \|u_{\lambda^*}\|_\infty < \|v_\lambda\|_\infty$, $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$, *and* $\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = \infty$. *More precisely,*

(2.2)

$$\|u_\lambda\|_\infty \sim \left[\frac{pa_1(q_1+1)^{p-1}}{(p-1)\left(B\left(1/(q_1+1),(p-1)/p\right)\right)^p}\right]^{1/(p-1-q_1)} \lambda^{1/(p-1-q_1)} \quad \text{as} \quad \lambda \to 0^+,$$

(2.3)

$$\|v_\lambda\|_\infty \sim \left[\frac{pb_n(r_n+1)^{p-1}}{(p-1)\left(B\left(1/(r_n+1),(p-1)/p\right)\right)^p}\right]^{1/(p-1-r_n)} \lambda^{1/(p-1-r_n)} \quad \textit{as} \quad \lambda \to 0^+.$$

3. Lemmas. To prove Theorem 2.1, we modify the time-map techniques applied to prove [**10**, Theorems 2.1 and 2.2]. We need the following six lemmas. Consider

$$\begin{cases} (\varphi_p(u'(x)))' + \lambda f(u) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{3.1}$$

where $\lambda > 0$ is a bifurcation parameter. Assume that $f \in C^2(0,\infty)$ satisfies $f(u) > 0$ for $u > 0$ and $\lim_{u\to 0^+} u^\beta f(u) = 0$ for some constant $0 < \beta < 1$. Let $F(u) := \int_0^u f(t)\,dt$. Then $F(0) := \lim_{u\to 0^+} F(u) = 0$.

The time-map formula for (3.1) takes the form as follows:

(3.2)

$$\lambda^{1/p} = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [F(\alpha) - F(u)]^{-1/p}\,du := T(\alpha) \quad \text{for } 0 < \alpha < \infty;$$

see [**2**, equation (2.4)]. Positive solutions u of (3.1) correspond to $\|u\|_\infty = \alpha$ and $T(\alpha) = \lambda^{1/p}$. Thus, to study the number of positive solutions of (3.1) is equivalent to study the shape of the time map $T(\alpha)$ on $(0,\infty)$.

The following lemma is a generalization of [**9**, Lemma 4.1]; we omit the proof.

Lemma 3.1. *Suppose that* $f \in C(0,\infty)$ *satisfies* $f(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to 0^+} u^\beta f(u) = 0$ *for some constant* $0 < \beta < 1$.

(i) *If* $\lim_{u\to 0^+} f(u)/u^{p-1} := m_0 \in (0,\infty]$ *and* $\lim_{u\to\infty} f(u)/u^{p-1} := m_\infty \in (0,\infty]$, *then*

$$\lim_{\alpha\to 0^+} T(\alpha) = \left(\frac{p-1}{m_0}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \ge 0,$$

$$\lim_{\alpha\to \infty} T(\alpha) = \left(\frac{p-1}{m_\infty}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \ge 0.$$

(ii) *If* $f(u) \sim \widetilde{m}_0 u^{s_1}$ *as* $u \to 0^+$ *and* $f(u) \sim \widetilde{m}_\infty u^{s_2}$ *as* $u \to \infty$ *for some constants* $1-p < s_1,\ s_2 < \infty$, $0 < \widetilde{m}_0,\ \widetilde{m}_\infty < \infty$, *then*

$$\begin{aligned} T(\alpha) \sim &\left(\frac{p-1}{p\tilde{m}_0}\right)^{1/p} (s_1+1)^{(1-p)/p} \\ &\times B\left(\frac{1}{s_1+1}, \frac{p-1}{p}\right)\alpha^{(p-1-s_1)/p} \quad \text{as } \alpha \longrightarrow 0^+ \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} T(\alpha) \sim &\left(\frac{p-1}{p\tilde{m}_\infty}\right)^{1/p} (s_2+1)^{(1-p)/p} \\ &\times B\left(\frac{1}{s_2+1}, \frac{p-1}{p}\right)\alpha^{(p-1-s_2)/p} \quad \text{as } \alpha \longrightarrow \infty. \end{aligned} \tag{3.4}$$

The following key lemma is a generalization of [**9**, Theorem 1.1]; we omit the proof. Let $\theta_f(u) := pF(u) - uf(u)$.

Lemma 3.2. *Suppose that* $f \in C^2(0,\infty)$ *satisfies*

(H1) $f(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to 0^+} u^\beta f(u) = 0$ *for some constant* $0 < \beta < 1$,

(H2) $\lim_{u\to 0^+} f(u)/u^{p-1} = m_0 \in (0,\infty]$ *and* $\lim_{u\to\infty} f(u)/u^{p-1} = m_\infty \in (0,\infty]$,

(H3) *there exist positive numbers* $A < B$ *such that*

$$\begin{cases} \theta_f'(u) = (p-1)f(u) - uf'(u) > 0 & \text{on } (0,A), \\ \theta_f'(A) = (p-1)f(A) - Af'(A) = 0, \\ \theta_f'(u) = (p-1)f(u) - uf'(u) < 0 & \text{on } (A,\infty), \end{cases} \tag{3.5}$$

and

$$\begin{cases} \theta_f(u) = pF(u) - uf(u) > 0 & \text{on } (0,B), \\ \theta_f(B) = pF(B) - Bf(B) = 0, \\ \theta_f(u) = pF(u) - uf(u) < 0 & \text{on } (B,\infty), \end{cases} \tag{3.6}$$

(H4) $uf'(u)/f(u) \geq -1/(p+1)$ *on* $(0,A)$ *and* $uf'(u)/f(u)$ *is increasing on* (A,B).

Then

$$\lim_{\alpha\to 0^+} T(\alpha) = \left(\frac{p-1}{m_0}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \geq 0,$$
$$\lim_{\alpha\to \infty} T(\alpha) = \left(\frac{p-1}{m_\infty}\right)^{1/p} \frac{\pi}{p} \csc \frac{\pi}{p} \geq 0,$$

and $T(\alpha)$ has exactly one critical point, a maximum, on $(0,\infty)$. Let α^* be the critical point for $T(\alpha)$. Then $A < \alpha^* < B$.

Lemma 3.3. *Consider* (3.1) *where $f \in C(0,\infty)$ satisfies $f(u) > 0$ for $u > 0$ and $\lim_{u\to 0^+} u^\beta f(u) = 0$ for some constant $0 < \beta < 1$. Suppose that, for two fixed positive numbers $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$, is a positive solution of* (3.1) *for $\lambda = \lambda_1$, $u_{\lambda_2}(x)$ is a positive solution of* (3.1) *for $\lambda = \lambda_2$. Then*

(i) *If $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$, then $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ for $-1 < x < 1$.*

(ii) *If $\|u_{\lambda_1}\|_\infty > \|u_{\lambda_2}\|_\infty$, then $u_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p} u_{\lambda_2}(x)$ for $-1 < x < 1$.*

The proof of Lemma 3.3 is exactly the same as that of [**9**, Theorem 1.2]. We omit it.

Consider

$$\begin{cases} (\varphi_p\,(u'(x)))' + f_\lambda(u(x)) = 0, & -1 < x < 1, \\ u(-1) = u(1) = 0, \end{cases} \tag{3.7}$$

where $f_\lambda(u) = \lambda g(u) + h(u)$, $g \in C(0,\infty)$, $h \in C[0,\infty)$ and g, h satisfy $g(u), h(u) > 0$ for $u > 0$ and $\lim_{u\to 0^+} u^\beta g(u) = 0 = h(0)$ for some constant $0 < \beta < 1$. Define

$$F_\lambda(u) = \int_0^u f_\lambda(t)\,dt, \tag{3.8}$$

$$T_\lambda(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha [F_\lambda(\alpha) - F_\lambda(u)]^{-1/p}\,du \quad \text{for } 0 < \alpha < \infty. \tag{3.9}$$

The following lemma is a generalization of [**11**, Lemma 3.2]; we omit the proof.

Lemma 3.4. *Consider* (3.7) *where* $f_\lambda(u) = \lambda g(u) + h(u)$, $g \in C(0,\infty)$, $h \in C[0,\infty)$ *and* g, h *satisfy* $g(u), h(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to 0^+} u^\beta g(u) = 0 = h(0)$ *for some constant* $0 < \beta < 1$. *Then, for each fixed* $\alpha > 0$, $T_\lambda(\alpha)$ *is a continuous function of* $\lambda \geq 0$ *and* $\lim_{\lambda\to\infty} T_\lambda(\alpha) = 0$.

Consider (3.7) where $f_\lambda(u) = \lambda g(u) + h(u)$, $g \in C(0,\infty)$, $h \in C[0,\infty)$ and g, h satisfy $g(u), h(u) > 0$ for $u > 0$ and $\lim_{u\to 0^+} u^\beta g(u) = 0 = h(0)$ for some constant $0 < \beta < 1$. Let λ_1, λ_2 be two positive constants. Suppose that, for $\lambda_1 \leq \lambda \leq \lambda_2$, $T_\lambda(\alpha)$ has exactly one critical point, a maximum at some α_λ^*, on $(0,\infty)$. Then for $\lambda_1 \leq \lambda \leq \lambda_2$, let

$$M(\lambda) := T_\lambda(\alpha_\lambda^*) = \max_{\alpha\in(0,\infty)} T_\lambda(\alpha).$$

For $u > 0$, $f_\lambda(u) = \lambda g(u) + h(u)$ is strictly increasing in $\lambda > 0$ since $g(u) > 0$ for $u > 0$. This and (3.9) imply that, for any fixed $\alpha > 0$, $T_\lambda(\alpha)$ is strictly decreasing in $\lambda > 0$. Thus $M(\lambda)$ is strictly decreasing in $\lambda \in [\lambda_1, \lambda_2]$. The following lemma is a generalization of [**11**, Lemma 3.3]; we omit the proof.

Lemma 3.5. *Consider* (3.7) *where* $f_\lambda(u) = \lambda g(u) + h(u)$, $g \in C(0,\infty)$, $h \in C[0,\infty)$ *and* g, h *satisfy* $g(u), h(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to 0^+} u^\beta g(u) = 0 = h(0)$ *for some constant* $0 < \beta < 1$. *Assume that there exist two positive numbers* $\lambda_1 < \lambda_2$ *such that*

(i) *for* $\lambda_1 \leq \lambda \leq \lambda_2$, $T_\lambda(\alpha)$ *has exactly one critical point, a maximum at some* α_λ^*, *on* $(0,\infty)$,

(ii) $M(\lambda_2) < 1 < M(\lambda_1)$,

(iii) $0 < \inf\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} \leq \sup\{\alpha_\lambda^* \mid \lambda \in [\lambda_1, \lambda_2]\} < \infty$.

Then there exists a unique number $\lambda^* \in (\lambda_1, \lambda_2)$ *such that* $M(\lambda^*) = 1$.

Lemma 3.6. *Consider* (3.7) *where* $f_\lambda(u) = \lambda g(u) + h(u)$, $g \in C(0,\infty)$, $h \in C[0,\infty)$ *and* g, h *satisfy* $g(u), h(u) > 0$ *for* $u > 0$ *and* $\lim_{u\to 0^+} u^\beta g(u) = 0 = h(0)$ *for some constant* $0 < \beta < 1$. *Suppose that, for two fixed positive numbers* $\lambda_1 < \lambda_2$, $u_{\lambda_1}(x)$ *is a positive solution of* (3.7) *for* $\lambda = \lambda_1$, $u_{\lambda_2}(x)$ *is a positive solution of* (3.7) *for* $\lambda = \lambda_2$. *Then*

(i) *If* $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$, *then* $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ *for* $-1 < x < 1$.

(ii) *If* $\|u_{\lambda_1}\|_\infty > \|u_{\lambda_2}\|_\infty$, *then* $u_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p} u_{\lambda_2}(x)$ *for* $-1 < x < 1$.

The proof of Lemma 3.6 (ii) is similar to that of [**10**, Lemma 3.5]. A similar argument as in the proof of [**10**, Lemma 3.5] can apply to prove Lemma 3.6 (i). We omit the proofs.

4. Proofs of Theorems 2.1 and 2.2. To prove Theorem 2.1 we first prove Theorem 2.2 by mainly applying Lemma 3.2.

Proof of Theorem 2.2. (i) Suppose that

$$f = f_{m,n}(u) := \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{r_j}, \quad m, n \geq 1,$$

satisfies (1.6).

It is easy to check that, for $m, n \geq 1$, $f_{m,n} \in C^2(0,\infty)$ satisfies (H1)–(H3) for some positive numbers $A < B$ and $m_0 = \infty = m_\infty$; we omit the proofs.

For (H4), we compute that

$$\begin{aligned} u f'_{m,n}(u) + \frac{1}{p+1} f_{m,n}(u) &= \sum_{i=1}^{m} a_i \left(q_i + \frac{1}{p+1} \right) u^{q_i} \\ &\quad + \sum_{j=1}^{n} b_j \left(r_j + \frac{1}{p+1} \right) u^{r_j} > 0 \quad \text{on } (0,\infty) \end{aligned}$$

by (1.6). So, to complete the proof of (H4), it suffices to prove that $u f'_{m,n}(u)/f_{m,n}(u)$ is increasing on (A, B). Actually, we prove that

$$\frac{uf'_{m,n}(u)}{f_{m,n}(u)} \quad \text{is increasing on } (0,\infty) \tag{4.1}$$

by the principle of double induction on positive integers m, n as follows.

Note that, since $f \in C^2(0,\infty)$, it is easy to see that $uf'(u)/f(u)$ is increasing on $(0,\infty)$ if and only if $(uf'(u))'f(u) - u(f'(u))^2 \geq 0$ on $(0,\infty)$.

First we prove (4.1) for $f = f_{1,n}(u) := a_1u^{q_1} + \sum_{j=1}^{n} b_j u^{p_j}$ by induction on n. For $n = 1$, $f = f_{1,1}(u) = a_1u^{q_1} + b_1u^{p_1}$ with $0 < q_1 < 1 < p_1$ and $a_1, b_1 > 0$, we compute that

$$\begin{aligned}(uf'_{1,1}(u))'f_{1,1}(u) - u(f'_{1,1}(u))^2 &= (a_1q_1^2u^{q_1-1} + b_1p_1^2u^{p_1-1})(a_1u^{q_1} + b_1u^{p_1}) \\ &\quad - u(a_1q_1u^{q_1-1} + b_1p_1u^{p_1-1})^2 \\ &= a_1b_1(p_1 - q_1)^2u^{p_1+q_1-1} > 0 \quad \text{on } (0,\infty).\end{aligned}$$

Thus (4.1) holds. Secondly, assume that, for $n = s \geq 1$, $f = f_{1,s}(u) = a_1u^{q_1} + \sum_{j=1}^{s} b_j u^{p_j}$ satisfies (4.1). Hence,

$$(uf'_{1,s}(u))'f_{1,s}(u) - u(f'_{1,s}(u))^2 \geq 0 \quad \text{on } (0,\infty). \tag{4.2}$$

Then for $n = s+1$, $f = f_{1,s+1}(u) = f_{1,s}(u) + b_{s+1}u^{p_{s+1}}$, by (4.2) and (1.6), we compute that

$$\begin{aligned}&(uf'_{1,s+1}(u))'f_{1,s+1}(u) - u(f'_{1,s+1}(u))^2 \\ &\quad = [uf'_{1,s}(u) + b_{s+1}p_{s+1}u^{p_{s+1}}]'[f_{1,s}(u) + b_{s+1}u^{p_{s+1}}] \\ &\qquad - u[f'_{1,s}(u) + b_{s+1}p_{s+1}u^{p_{s+1}-1}]^2 \\ &\quad = [(uf'_{1,s}(u))'f_{1,s}(u) - u(f'_{1,s}(u))^2] \\ &\qquad + b_{s+1}u^{p_{s+1}-1}\left[a_1(p_{s+1} - q_1)^2u^{q_1} + \sum_{j=1}^{s} b_j(p_{s+1} - p_j)^2u^{p_j}\right] \\ &\quad \geq b_{s+1}u^{p_{s+1}-1}\left[a_1(p_{s+1} - q_1)^2u^{q_1} + \sum_{j=1}^{s} b_j(p_{s+1} - p_j)^2u^{p_j}\right] > 0\end{aligned}$$

on $(0,\infty)$. Thus (4.1) holds for $f = f_{1,n}(u)$, $n = s+1$. So by mathematical induction, for any positive integer n, (4.1) holds for $f = f_{1,n}(u)$.

We next prove (4.1) for $f = f_{m,n}(u) := \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{p_j}$ for any fixed $n \geq 1$ by induction on m. First, by the above, for $m = 1$, $f = f_{1,n}(u) = a_1 u^{q_1} + \sum_{j=1}^{n} b_j u^{p_j}$ satisfies (4.1) for any fixed $n \geq 1$. Secondly, assume that, for $m = t \geq 1$, $f = f_{t,n}(u) = \sum_{i=1}^{t} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{p_j}$ satisfies (4.1) for any fixed $n \geq 1$. Hence,

$$(uf'_{t,n}(u))' f_{t,n}(u) - u(f'_{t,n}(u))^2 \geq 0 \quad \text{on } (0,\infty). \tag{4.3}$$

Then, for $m = t+1$, $f = f_{t+1,n}(u) = f_{t,n}(u) + a_{t+1}u^{q_{t+1}}$, by (4.3) and (1.6), we compute that

$$\begin{aligned}
&(uf'_{t+1,n}(u))' f_{t+1,n}(u) - u(f'_{t+1,n}(u))^2 \\
&\quad = [uf'_{t,n}(u) + a_{t+1}q_{t+1}u^{q_{t+1}}]'[f_{t,n}(u) + a_{t+1}u^{q_{t+1}}] \\
&\qquad - u[f'_{t,n}(u) + a_{t+1}q_{t+1}u^{q_{t+1}-1}]^2 \\
&\quad = [(uf'_{t,n}(u))' f_{t,n}(u) - u(f'_{t,n}(u))^2] \\
&\qquad + a_{t+1}u^{q_{t+1}-1}\left[\sum_{i=1}^{t} a_i (q_{t+1}-q_i)^2 u^{q_i} + \sum_{j=1}^{n} b_j (q_{t+1}-p_j)^2 u^{p_j}\right] \\
&\quad \geq a_{t+1}u^{q_{t+1}-1}\left[\sum_{i=1}^{t} a_i (q_{t+1}-q_i)^2 u^{q_i} + \sum_{j=1}^{n} b_j (q_{t+1}-p_j)^2 u^{p_j}\right] > 0
\end{aligned}$$

on $(0,\infty)$. Thus, (4.1) holds for $f = f_{m,n}(u)$ for any fixed $n \geq 1$, $m = t+1$. So, by mathematical induction, for any fixed $n \geq 1$ and any positive integer m, (4.1) holds for $f = f_{m,n}(u)$. Hence, $f_{m,n}$ satisfies (H4) for any positive integers m, n.

So, by (1.6) and Lemma 3.2,

$$\lim_{\alpha \to 0^+} T(\alpha) = 0 = \lim_{\alpha \to \infty} T(\alpha) \tag{4.4}$$

and $T(\alpha)$ has exactly one critical point, a maximum, on $(0,\infty)$. Thus, there exists $\lambda^* := (T(\alpha^*))^p = (\max_{\alpha \in (0,\infty)} T(\alpha))^p > 0$ for some $\alpha^* \in (A,B)$ such that (1.5) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $0 < \lambda < \lambda^*$ (the ordering of u_λ, v_λ can be proved easily), exactly one positive solution u_{λ^*} for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Moreover, if $0 < \lambda_1 < \lambda_2 \leq \lambda^*$, then

(a) $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ and $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$,

(b) $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ for $-1 < x < 1$ by Lemma 3.3 (i),

(c) $v_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p} v_{\lambda_2}(x)$ for $-1 < x < 1$ by Lemma 3.3 (ii).

(ii) It is easy to see that $\|u_\lambda\|_\infty < \|u_{\lambda^*}\|_\infty < \|v_\lambda\|_\infty$ for $0 < \lambda < \lambda^*$, $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = \infty$. Equations (2.2) and (2.3) follow immediately by (3.3) and (3.4).

The proof of Theorem 2.2 is complete. □

Proof of Theorem 2.1. For fixed $\lambda > 0$, suppose that $u_\lambda(x)$ is a positive solution of (1.1) with $||u_\lambda||_\infty = \alpha$. We write

$$f_\lambda(u) = \lambda \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{r_j} = \lambda \bigg[\sum_{i=1}^{m} a_i u^{q_i} + \frac{1}{\lambda} \sum_{j=1}^{n} b_j u^{r_j} \bigg].$$

Then, by (3.2) and (3.8), it is easy to see that

$$\begin{aligned}
\lambda^{1/p} &= \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \left[\int_u^\alpha \sum_{i=1}^{m} a_i s^{q_i} + \frac{1}{\lambda} \sum_{j=1}^{n} b_j s^{r_j} \, ds \right]^{-1/p} du \\
&= \left(\frac{p-1}{p}\right)^{1/p} \lambda^{1/p} \int_0^\alpha \left[\int_u^\alpha \lambda \sum_{i=1}^{m} a_i s^{q_i} + \sum_{j=1}^{n} b_j s^{r_j} \, ds \right]^{-1/p} du \\
&= \left(\frac{p-1}{p}\right)^{1/p} \lambda^{1/p} \int_0^\alpha \Big[F_\lambda(\alpha) - F_\lambda(u) \Big]^{-1/p} du.
\end{aligned}$$

This and (3.9) imply that the positive solution $u_\lambda(x)$ of (1.1) corresponds to $||u_\lambda||_\infty = \alpha$ and

$$T_\lambda(\alpha) = \left(\frac{p-1}{p}\right)^{1/p} \int_0^\alpha \Big[F_\lambda(\alpha) - F_\lambda(u) \Big]^{-1/p} du = 1. \tag{4.5}$$

It is easy to check that (4.5) holds for any $\lambda \geq 0$.

Suppose that $f_\lambda(u) = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfies (1.2). First, for $\lambda = 0$, $f_0(u) = \sum_{j=1}^n b_j u^{r_j}$ and $F_0(u) = \sum_{j=1}^n b_j/(r_j+1) u^{r_j+1}$. We first show some properties of $T_0(\alpha)$ and $T_\lambda(\alpha)$. We have

(1)

$$\begin{cases} \lim_{\alpha\to 0^+} T_0(\alpha) = ((p-1)/b_1)^{1/p} (\pi/p) \csc(\pi/p) > 1 & \text{if } r_1 = p-1, \\ \lim_{\alpha\to 0^+} T_0(\alpha) = \infty & \text{if } r_1 > p-1, \end{cases}$$

by (1.2) and Lemma 3.1(i).

(2) $\lim_{\alpha\to\infty} T_0(\alpha) = 0$ by (1.2) and Lemma 3.1 (i).

(3) $T_0(\alpha)$ is a strictly decreasing function of $\alpha > 0$ as it is easy to see that

$$T_0'(\alpha) = \left(\frac{p-1}{p^{p+1}}\right)^{1/p} \frac{1}{\alpha} \int_0^\alpha \frac{\theta_{f_0}(\alpha) - \theta_{f_0}(u)}{[F_0(\alpha) - F_0(u)]^{(p+1)/p}}\, du < 0 \quad \text{for} \quad \alpha > 0,$$

since

$$\theta_{f_0}(\alpha) - \theta_{f_0}(u) = \sum_{j=1}^{n} \frac{b_j(p-1-r_j)}{r_j+1}\left(\alpha^{r_j+1} - u^{r_j+1}\right) < 0 \quad \text{for} \quad 0 < u < \alpha.$$

By the above, there exists a unique positive number B^* satisfying

$$T_0(B^*) = 1. \tag{4.6}$$

(4) For each fixed $\alpha > 0$, $T_\lambda(\alpha)$ is a continuous function of $\lambda \geq 0$, $\lim_{\lambda\to 0^+} T_\lambda(\alpha) = T_0(\alpha)$ and $\lim_{\lambda\to\infty} T_\lambda(\alpha) = 0$ by Lemma 3.4.

(5) For $0 \leq \lambda_1 < \lambda_2$,

$$\begin{aligned} f_{\lambda_1}(u) &= \lambda_1 \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{r_j} < \lambda_2 \sum_{i=1}^{m} a_i u^{q_i} + \sum_{j=1}^{n} b_j u^{r_j} \\ &= f_{\lambda_2}(u), \quad u > 0. \end{aligned}$$

So we obtain $T_{\lambda_1}(\alpha) > T_{\lambda_2}(\alpha)$ for $\alpha > 0$ by (3.8) and (3.9).

(6) For each fixed $\lambda > 0$, by (4.4), $\lim_{\alpha\to 0^+} T_\lambda(\alpha) = 0 = \lim_{\alpha\to\infty} T_\lambda(\alpha)$. In addition, $T_\lambda(\alpha)$ has exactly one critical point, a maximum at some α_λ^*, on $(0,\infty)$.

By the above, there exist two positive numbers $\lambda_1 < \lambda_2$ such that $M(\lambda_2) < 1 < M(\lambda_1)$. In the proof of Theorem 2.2, we know that, for fixed $\lambda > 0$, the nonlinearity $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfies (H1)–(H4); then $A_\lambda < \alpha_\lambda^* < B_\lambda$, where A_λ and B_λ satisfy $\theta_{f_\lambda}'(A_\lambda) = \theta_{f_\lambda}(B_\lambda) = 0$. Since the functions

$$\begin{aligned} \theta_{f_\lambda}(u) &= pF_\lambda(u) - uf_\lambda(u) \\ &= \lambda \sum_{i=1}^{m} \frac{a_i(p-1-q_i)}{q_i+1}\, u^{q_i+1} + \sum_{j=1}^{n} \frac{b_j(p-1-r_j)}{r_j+1}\, u^{r_j+1} \end{aligned}$$

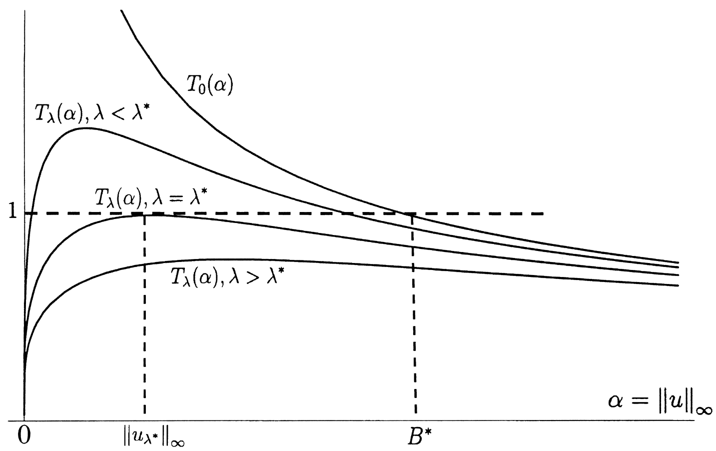

FIGURE 3. Graph of $T_\lambda(\alpha)$ for different λ's of (1.1).

and

$$\begin{aligned}\theta'_{f_\lambda}(u) &= (p-1)f_\lambda(u) - uf'_\lambda(u)\\ &= \lambda\sum_{i=1}^{m} a_i(p-1-q_i)u^{q_i} + \sum_{j=1}^{n} b_j(p-1-r_j)u^{r_j}\end{aligned}$$

are both strictly increasing in $\lambda > 0$. Thus, positive numbers A_λ and B_λ are both strictly increasing in $\lambda > 0$ by (3.5) and (3.6). Hence

$$\begin{aligned}0 < A_{\lambda_1} &= \inf\{A_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\} \le \inf\{\alpha^*_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\}\\ &\le \sup\{\alpha^*_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\} \le \sup\{B_\lambda \mid \lambda \in [\lambda_1, \lambda_2]\} = B_{\lambda_2} < \infty.\end{aligned}$$

By the above and by Lemma 3.5, we obtain

(7) There exists a unique number $\lambda^* > 0$ such that

$$M(\lambda^*) = \max_{\alpha\in(0,\infty)} T_{\lambda^*}(\alpha) = 1. \tag{4.7}$$

So by the above, we obtain graphs of $T_\lambda(\alpha)$ of (1.1) for different λ's as in Figure 3. It follows that

(i) Problem (1.1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $0 < \lambda < \lambda^*$ (the ordering of u_λ, v_λ can be proved easily), exactly one positive solution u_{λ^*} for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Moreover, if $0 < \lambda_1 < \lambda_2 \leq \lambda^*$, then we obtain that

(a) $\|u_{\lambda_1}\|_\infty < \|u_{\lambda_2}\|_\infty$ and $\|v_{\lambda_1}\|_\infty > \|v_{\lambda_2}\|_\infty$,

(b) $u_{\lambda_1}(x) < u_{\lambda_2}(x)$ for $-1 < x < 1$ by Lemma 3.6 (i),

(c) $v_{\lambda_1}(x) > (\lambda_1/\lambda_2)^{1/p} v_{\lambda_2}(x)$ for $-1 < x < 1$ by Lemma 3.6 (ii).

(ii) Let u be a positive solution of (1.1). Then $\|u\|_\infty < B^*$. In addition, if $n = 1$, by (4.5), we compute that

$$\begin{aligned}
T_0(\alpha) &= \left(\frac{(p-1)(r_1+1)}{pb_1}\right)^{1/p} \int_0^\alpha (\alpha^{r_1+1} - u^{r_1+1})^{-1/p}\, du \\
&= \left(\frac{(p-1)(r_1+1)}{pb_1}\right)^{1/p} \alpha^{(p-1-r_1)/p} \\
&\quad \times \int_0^1 (1 - w^{r_1+1})^{-1/p}\, dw \qquad (\text{let } u = \alpha w) \\
&= \left(\frac{(p-1)}{pb_1(r_1+1)^{p-1}}\right)^{1/p} \alpha^{(p-1-r_1)/p} \\
&\quad \times \int_0^1 t^{-1/p}(1-t)^{-r_1/(r_1+1)}\, dt \qquad (\text{let } t = 1 - w^{r_1+1}) \\
&= \left(\frac{(p-1)}{pb_1(r_1+1)^{p-1}}\right)^{1/p} \alpha^{(p-1-r_1)/p} B\left(\frac{p-1}{p}, \frac{1}{r_1+1}\right).
\end{aligned}$$

By (4.6), we solve that

$$B^* = \left[\left(\frac{p-1}{pb_1\,(r_1+1)^{p-1}}\right)^{1/p} B\left(\frac{p-1}{p}, \frac{1}{r_1+1}\right)\right]^{p/(r_1-p+1)}.$$

(iii) It is easy to see that $\|u_\lambda\|_\infty < \|u_{\lambda^*}\|_\infty < \|v_\lambda\|_\infty$ for $0 < \lambda < \lambda^*$. The proofs of $\lim_{\lambda\to 0^+} \|u_\lambda\|_\infty = 0$ and $\lim_{\lambda\to 0^+} \|v_\lambda\|_\infty = B^*$ are easy but tedious; we omit them.

(iv) Suppose that $r_1 = p-1$; then for any fixed $\lambda \geq 0$ and a_i, b_j, q_i, r_j, $1 \leq i \leq m$ and $2 \leq j \leq n$, $f_\lambda = f_{\lambda,b_1} = \lambda \sum_{i=1}^m a_i u^{q_i} + b_1 u^{p-1} + \sum_{j=2}^n b_j u^{r_j}$ is strictly increasing in $b_1 \in$

$(0, (p-1)((\pi/p)\csc(\pi/p))^p)$. So $T_\lambda(\alpha) = T_{\lambda,b_1}(\alpha)$ is strictly decreasing in $b_1 \in (0, (p-1)((\pi/p)\csc(\pi/p))^p)$ by (4.5)–(4.7), and hence positive numbers $\lambda^* = \lambda^*(b_1)$ and $B^* = B^*(b_1)$ are both strictly decreasing in $b_1 \in (0, (p-1)((\pi/p)\csc(\pi/p))^p)$. The proof of (2.1) is easy but tedious; we omit it.

The proof of Theorem 2.1 is complete. $\square$

5. An example with some negative coefficient. Actually, to Theorem 2.1, we give an example to demonstrate that the hypotheses of positive coefficients a_i and b_j for $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ in (1.2) can be weakened. Our time-map techniques as in the proof of Theorem 2.1 can be adapted such that the same exact multiplicity results in Theorem 2.1 hold for some nonlinearities $f_\lambda = \lambda \sum_{i=1}^m a_i u^{q_i} + \sum_{j=1}^n b_j u^{r_j}$ satisfying either $a_i < 0$ or $b_j < 0$ for some $1 < i < m$, $1 < j < n$.

An example with some negative coefficient (See Figure 4). For (1.1), take $p = 3$ and $f_\lambda(u) = \lambda u^{-1/4} + u^2 - u^3 + u^4$. It can be proved that

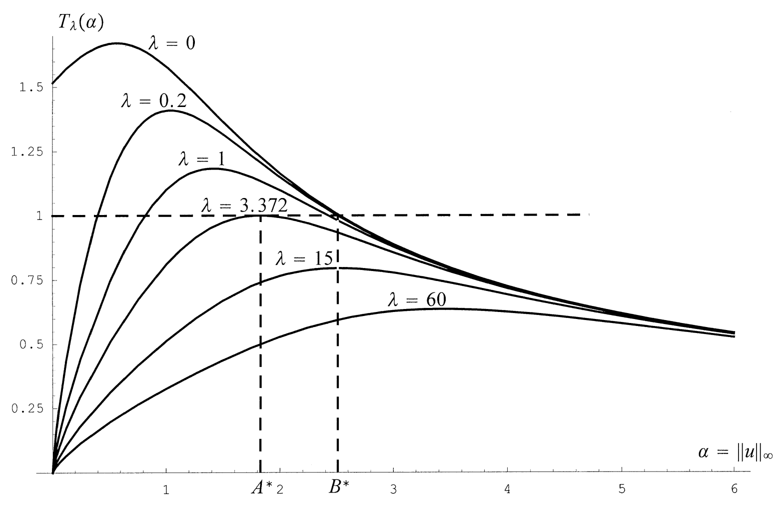

FIGURE 4. Numerical simulations of $T_\lambda(\alpha) : f_\lambda(u) = \lambda u^{-1/4} + u^2 - u^3 + u^4$, $\lambda = 0,\ 0.2,\ 1,\ 3.372,\ 15,\ 60$ for $p = 3$. $\lambda^* \approx 3.372$, $B^* \approx 2.528$, $A^* \approx 1.82$.

(i) for fixed λ with $0 \le \lambda \le 5$, $f_\lambda(u)$ satisfies all hypotheses in Lemma 3.2 and $T_\lambda(\alpha)$ has exactly one critical point, a maximum, on $(0,\infty)$. In addition,

$$\lim_{\alpha\to 0^+} T_\lambda(\alpha) = \begin{cases} (2^{4/3}\pi/3^{3/2}) \approx 1.524 > 1 & \text{if } \lambda = 0, \\ 0 & \text{if } 0 < \lambda \le 5, \end{cases}$$

and

$$\lim_{\alpha\to\infty} T_\lambda(\alpha) = 0,$$

(ii) for $\lambda > 5$, $T_\lambda(\alpha) < 1$ for all $\alpha > 0$.

Then applying the same arguments as in the proof of Theorem 2.1, we obtain that there exists $\lambda^* > 0$ such that (1.1) has exactly two positive solutions u_λ, v_λ with $u_\lambda < v_\lambda$ for $0 < \lambda < \lambda^*$, exactly one positive solution u_{λ^*} for $\lambda = \lambda^*$, and no positive solution for $\lambda > \lambda^*$. Actually, numerical simulations as given in Figure 4 show that $\lambda^* \approx 3.372$, $B^* \approx 2.528$ and $A^* \approx 1.824$.

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