ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 2, 2007

NONOSCILLATORY CRITERIA FOR SECOND-ORDER NONLINEAR DIFFERENCE EQUATIONS

JIQIN DENG

ABSTRACT. In this paper, we obtain some nonoscillatory theories of the second-order nonlinear difference equation

$$\triangle(r_n(\triangle x_n)^{\alpha}) + f(n+1,\ x_{n+1}) = 0, \quad n \in \mathbf{N}$$

where α is a quotient of positive odd integers, $r_n > 0$ for $n \in \mathbf{N}$ and $f \in C(\mathbf{N} \times \mathbf{R},\ \mathbf{R})$.

1. Introduction. Consider the following second-order difference equation

$$\triangle(r_n(\triangle x_n)^{\alpha}) + f(n+1,\ x_{n+1}) = 0, \quad n \in \mathbf{N} \tag{1}$$

where α is a quotient of positive odd integers, $\triangle x_n = x_{n+1} - x_n$, $r_n > 0$ for $n \in \mathbf{N}$ and $f \in C(\mathbf{N} \times \mathbf{R},\ \mathbf{R})$.

A solution of (1) is called nonoscillatory if it is either eventually positive or eventually negative; otherwise, it is called oscillatory.

In [**6**–**10**], many good results for nonoscillatory solutions of differential equations corresponding to (1) were obtained, but in the results the condition where $f(t,\ x)$ is either linear or quasi-linear was adopted. So far, very few results for nonoscillation of (1) with generally nonlinear term have been obtained. In this paper, by using the methods in the proof of [**1**], we discuss nonoscillatory solutions of (1) and obtain the following results.

2000 AMS *Mathematics Subject Classification.* Primary 34C10.
Key words and phrases. Nonoscillatory solution, nonlinear term, condensing operators.
Research supported by Science and Research Fund of Hunan Provincial Education Department and HPNSF (No. 05C101).
Received by the editors on April 22, 2004, and in revised form on Oct. 9, 2004.

Theorem 1. *Take a fixed positive number* K. *If, for any* $\varepsilon > 0$, *there exists* $n_0 \in \mathbf{N}$ *such that for each* $\{x_i\}_{i=n_0}^{\infty}$ *with* $K/2 \le x_{n_0} \le x_{n_0+1} \le \cdots \le K$,

$$\sum_{j=n}^{\infty} f(j+1,\, x_{j+1}) \ge 0, \quad n \ge n_0 \tag{2}$$

and

$$\sum_{k=n_0}^{\infty} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha} < \varepsilon, \tag{3}$$

then, (1) *has a bounded nonoscillatory solution and the solution is eventually nondecreasing.*

Theorem 2. *Take a fixed positive number* K. *If, for any* $\varepsilon > 0$, *there exists* $n_0 \in \mathbf{N}$ *such that for each* $\{x_i\}_{i=n_0}^{\infty}$ *with* $K \ge x_{n_0} \ge x_{n_0+1} \ge \cdots \ge K/2$,

$$\sum_{j=n}^{\infty} f(j+1,\, x_{j+1}) \le 0, \quad n \ge n_0 \tag{4}$$

and

$$\sum_{k=n_0}^{\infty} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha} > -\varepsilon, \tag{5}$$

then, (1) *has a bounded nonoscillatory solution and the solution is eventually nonincreasing.*

Theorem 3. *Take a fixed positive number* K, *a fixed nonnegative sequence* $\{\lambda_n\}$ *with* $\lambda_n \to 0$ *as* $n \to \infty$ *and a fixed mapping* $m : \mathbf{N} \to \mathbf{N}$. *If for any* $\varepsilon > 0$, *there exist* $n_0 \in \mathbf{N}$ *such that for each* $\{x_n\}$ *with* $K/2 \le x_n \le K$ *and* $|x_{n+m(n)} - x_n| \le \lambda_{n+m(n)}$ *for* $n \ge n_0$,

$$\left| \sum_{k=n}^{n+m(n)-1} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\; x_{j+1}) \right)^{1/\alpha} \right| \le \lambda_{n+m(n)}, \quad n \ge n_0 \tag{6}$$

and

$$\left|\sum_{k=n_0}^{n-1}\left(\frac{1}{r_k}\sum_{j=k}^{\infty}f(j+1,\ x_{j+1})\right)^{1/\alpha}\right|\le\varepsilon,\quad n\ge n_0+1, \tag{7}$$

then, (1) *has a bounded nonoscillatory solution.*

Define N_0 as follows:

$$N_0=\{n_0,\ n_0+1,\ n_0+2,\dots\}.$$

As in [**1**], the following theorems and notations shall be used. $B(N_0)$ is the Banach space of all bounded mappings from N_0 (discrete topology) to $\mathbf{R}$ with the norm: $|\{x_n\}|_\infty=\sup_{i\in N_0}|x_i|$.

Theorem A (see [**4**]). *Let C be a closed, convex subset of a Banach space E and U an open subset of C with $\{p^\star\}\in U$. Also $T:\ \overline{U}\to C$ is a continuous, condensing map with $T(\overline{U})$ bounded. Then one of the following conclusions holds*:

(A_1) *T has a fixed point in $\overline{U}$; or*

(A_2) *there is an $x\in\partial U$ and $\lambda\in(0,\,1)$ with $x=(1-\lambda)\,p^\star+\lambda Tx$.*

Theorem B (see [**1–5**]). *Let E be a uniformly bounded subset of the Banach space $B(\mathbf{N})$. If E is equiconvergent at ∞, it is also relatively compact.*

2. Proofs of theorems.

Proof of Theorem 1. For any $0<\varepsilon<K/8$, take $n_0\in\mathbf{N}$ sufficiently large so that (2) holds and for each $\{x_i\}_{i=n_0}^{\infty}$ with $K/2\le x_{n_0}\le x_{n_0+1}\le\cdots\le K$,

$$0\le\sum_{k=n_0}^{\infty}\left(\frac{1}{r_k}\sum_{j=k}^{\infty}f(j+1,\,x_{j+1})\right)^{1/\alpha}<(K/4)-\varepsilon. \tag{8}$$

Let

$$E = (B(N_0),\ |.|_\infty),$$
$$C = \left\{ \{x_i\} \in B(N_0) :\ x_{i+1} \geq x_i \geq \frac{K}{2}, \quad i \in N_0 \right\},$$
$$U = \{x = \{x_i\} \in C :\ |x|_\infty < K\}$$

and $p^\star = K - \varepsilon$. Then, $\{p^\star\} \in U$.

Define operators T_1 and T_2 as follows:

$$T_1 x_n = \frac{3}{8}\,K + \frac{1}{2}\,x_n, \quad n \in N_0$$
$$T_2 x_{n_0} = 0, \quad T_2 x_n = \frac{1}{2} \sum_{k=n_0}^{n-1} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha},$$

$n \geq n_0 + 1$.

Set $T = T_1 + T_2$. First, for any $\{x_n\} \in \overline{U}$, from (8), it is easy to see that

$$Tx_n \geq \frac{3}{8}\,K + \frac{1}{4}\,K \geq \frac{K}{2}$$

and $\{Tx_n\}$ is nondecreasing on N_0. Thus,

(9) $$T :\ \overline{U} \to C.$$

Next, The continuity of T_2 is obvious and clearly, $T_2\overline{U} = \{T_2 x :\ x \in \overline{U}\}$ is a uniformly bounded subset of $B(N_0)$. Also, for any $\{x_n\} \in \overline{U}$, we have

$$|T_2 x_\infty - T_2 x_n| \leq \sum_{k=n}^{\infty} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha}.$$

Hence, $T_2\overline{U}$ is equiconvergent at ∞. From Theorem B, it is easy to see that $T_2\overline{U}$ is a relatively compact subset of $B(N_0)$. Therefore,

(10) $T_2 :\ \overline{U} \longrightarrow E$ is a continuous, relatively compact map.

Next, if $\{x_n\},\ \{y_n\} \in \overline{U}$, then we have

$$|T_1 x_n - T_1 y_n| = \frac{1}{2}\,|x_n - y_n| \le \frac{1}{2}\,|\{x_n\} - \{y_n\}|_\infty$$

which, together with (10), yields

$$T:\ \overline{U} \longrightarrow C \quad \text{is a continuous, condensing map.} \tag{11}$$

Next, we show that operator T does not satisfy condition (A_2). Assume that there exists $\{x_n\} \in \partial U$ such that, for some $0 < \lambda < 1$,

$$x_n = (1-\lambda)\,p^\star + \lambda T x_n.$$

Then,

$$\begin{aligned} x_n &= (1-\lambda)\,p^\star + \lambda T x_n \\ &= (1-\lambda)(K-\epsilon) + \lambda\left[\frac{3}{8}\,K + \frac{1}{2}\,x_n + \frac{1}{2}\sum_{k=n_0}^{n-1}\left(\frac{1}{r_k}\sum_{j=k}^{\infty} f(j+1,\, x_{j+1})\right)^{1/\alpha}\right], \\ &\qquad n \ge n_0 \end{aligned}$$

which, together with (8), yields

$$\begin{aligned} \sup_{n\in N_0} |x_n| &\le (1-\lambda)(K-\varepsilon) + \lambda\left[\frac{3}{8}\,K + \frac{1}{2}\,K + \frac{1}{8}\,K - (\varepsilon/2)\right] \\ &\le K - (\varepsilon/2) < K \end{aligned}$$

which gives a contradiction since $K = |\{x_n\}|_\infty = \sup_{n\in N_0} |x_n|$. From Theorem A, it is easy to see that there exists $\{x_n\} \in \overline{U}$ with $x_n = T x_n$, i.e.,

$$x_n = \frac{3}{4}\,K + \sum_{k=n_0}^{n-1}\left(\frac{1}{r_k}\sum_{j=k}^{\infty} f(j+1,\, x_{j+1})\right)^{1/\alpha} \quad \text{for} \quad n \ge n_0 + 1.$$

Clearly, x_n for $n \ge n_0 + 1$ is a bounded nonoscillatory solution of (1) and the solution is eventually nondecreasing. The proof is complete.

Proof of Theorem 2. For any $0 < \varepsilon < K/8$, take $n_0 \in \mathbf{N}$ sufficiently large so that (4) holds and for each $\{x_i\}_{i=n_0}^{\infty}$ with $K \geq x_{n_0} \geq x_{n_0+1} \geq \cdots \geq K/2$,

$$0 \geq \sum_{k=n_0}^{\infty} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha} > -((K/4) - \varepsilon).$$

Let N and E be as in the proof of Theorem 1 and

$$C = \Big\{ \{x_i\} \in B(N_0) :\ x_i \geq x_{i+1} \geq \frac{K}{2},\ i \in N_0 \Big\},$$
$$U = \{ x = \{x_i\} \in C :\ |x|_\infty < K \}$$

and $p^\star = K - \varepsilon$. Then, $\{p^\star\} \in U$.

The rest is similar to the proof of Theorem 1. Thus, we omit it.

Proof of Theorem 3. For any $0 < \varepsilon < K/8$, take $n_0 \in \mathbf{N}$ sufficiently large so that (6) holds and for each $\{x_n\}$ with $K/2 \leq x_n \leq K$ and $|x_{n+m(n)} - x_n| \leq \lambda_{n+m(n)}$ for $n \geq n_0$,

$$\left| \sum_{k=n_0}^{n-1} \left(\frac{1}{r_k} \sum_{j=k}^{\infty} f(j+1,\, x_{j+1}) \right)^{1/\alpha} \right| < (K/4) - \varepsilon, \quad n \geq n_0 + 1.$$

Let N, E and $p^\star$ be as in the proof of Theorem 1 and

$$C = \Big\{ \{x_i\} \in B(N_0) :\ x_i \geq \frac{K}{2} \text{ and } |x_{i+m(i)} - x_i| \leq \lambda_{i+m(i)},\ i \in N_0 \Big\},$$
$$U = \big\{ x = \{x_i\} \in C :\ |x|_\infty < K \big\}.$$

The rest is similar to the proof of Theorem 1. Thus, we omit it.

Example 1. Consider the following equation

$$\triangle(\triangle x_n)^{1/3} + p_{n+1}(1 + x_{n+1}) = 0, \quad n \in \mathbf{N}. \tag{12}$$

If

$$p_{2k} = 1/(2k)^{(1+(1/2))/3} = 1/\sqrt{2k}, \quad k = 1, 2, \dots,$$
$$p_{2k+1} = -1/\sqrt{2k+2}, \quad k = 0, 1, 2, \dots$$

and taking $K = 1$, then we have

$$\begin{aligned}
&\sum_{j=2k+1}^{2(k+m+1)} f(j+1,\, x_{j+1}) \\
&= \sum_{j=2k+1}^{2(k+m+1)} p_{j+1}(1+x_{j+1}) \\
&= \sum_{j=2k+1}^{2(k+m+1)} (-1)^j(1+x_{j+1})\Big/\sqrt{2j+1-(-1)^j} \qquad (13) \\
&= \sum_{j=k+1}^{k+1+m} \Big[x_{j+1}\big(1/\sqrt{2j}-\sqrt{2(j+1)}\,\big)+(x_{j+2}-x_{j+1})\Big/\sqrt{2j}\,\Big] \\
&\le 2\Big/\sqrt{2(k+1)}, \quad m \in \mathbf{N},
\end{aligned}$$

$$\begin{aligned}
&\sum_{j=2k+1}^{2(k+m+1)+1} f(j+1,\, x_{j+1}) \\
&= \sum_{j=2k+1}^{2(k+m+1)+1} p_{j+1}(1+x_{j+1}) \\
&= \sum_{j=2k+1}^{2(k+m+1)+1} (-1)^j(1+x_{j+1})\Big/\sqrt{2j+1-(-1)^j} \qquad (14) \\
&= \sum_{j=k+1}^{k+1+m} \Big[x_{j+1}\big(1/\sqrt{2j}-\sqrt{2(j+1)}\,\big)+(x_{j+2}-x_{j+1})/\sqrt{2j}\,\Big] \\
&\qquad - x_{2(k+m+1)+2}\Big/\sqrt{2(k+m+1)+2}, \quad m \in \mathbf{N}
\end{aligned}$$

which, together with (13), yields that, for each $m \in \mathbf{N}$ with $m \ge 2$,

$$\begin{aligned}
&\sum_{j=k+1}^{k+[m/2]} \big[x_{j+1}\big(1/\sqrt{2j}-\sqrt{2(j+1)}\,\big)+(x_{j+2}-x_{j+1})/\sqrt{2j}\,\big] \\
&\qquad - x_{2(k+2[m/2])}\Big/\sqrt{2(k+2[m/2])}
\end{aligned}$$

$$\leq \sum_{j=2k+1}^{2k+1+m} f(j+1,\, x_{j+1})$$
$$\leq \sum_{j=k+1}^{k+[m/2]+1} \left[x_{j+1}\left(1/\sqrt{2j}-\sqrt{2(j+1)}\,\right)+(x_{j+2}-x_{j+1})/\sqrt{2j}\,\right].$$

It follows that

(15)
$$0 \leq \sum_{j=2k+1}^{\infty} f(j+1,\ x_{j+1})$$
$$= \lim_{m\to\infty} \sum_{j=k+1}^{k+[m/2]+1} \left[x_{j+1}\left(1/\sqrt{2j}-\sqrt{2(j+1)}\,\right)+(x_{j+2}-x_{j+1})/\sqrt{2j}\,\right]$$
$$\leq 2/\sqrt{2(k+1)}, \quad k\in \mathbf{N}.$$

Similarly, we have

(16)
$$0 \leq \sum_{j=2k}^{\infty} f(j+1,\ x_{j+1})$$
$$= x_{2k+1}/\sqrt{2k}$$
$$+ \lim_{m\to\infty} \sum_{j=k+1}^{k+[m/2]+1} \left[x_{j+1}\left(1/\sqrt{2j}-\sqrt{2(j+1)}\,\right)+(x_{j+2}-x_{j+1})/\sqrt{2j}\,\right]$$
$$\leq 3/\sqrt{2k}, \quad k\in \mathbf{N}$$

which, together with (15), yields that (2) and (3) hold. And, from Theorem 1, it is easy to see that (12) has a bounded nondecreasing nonoscillatory solution. And, if $p_{2k} = 1/\sqrt{2k+2}$, $k = 1, 2, \dots$, $p_{2k+1} = -1/\sqrt{2k}$, $k = 0, 1, 2, \dots$ and taking $K = 1$, similar to the above discussion and from Theorem 2, it is easy to see that (12) has a bounded nonincreasing nonoscillatory solution.

Example 2. Consider the following equation

$$\triangle(\triangle x_n)^{\alpha} + (-1)^n\left(\frac{a}{n^{\alpha(1+\tau)}}+\frac{a}{(n+1)^{\alpha(1+\tau)}}\right)x_{n+1} = 0, \quad n\in \mathbf{N} \tag{17}$$

where α is a quotient of positive odd integers, $a > 0$ and $\tau > 0$. Take $K = 1$, $m(n) \equiv 1$, $\lambda_n = 1/n$. Then for any $0 < \varepsilon < 1/8$, it is easy to see that there exists integer n_0 with $n_0 \geq 2^{1/\tau}(a + a/[\alpha(1+\tau)])^{1/\alpha\tau}$ such that

$$(a + a/[\alpha(1+\tau)])^{1/\alpha} \sum_{j=n_0}^{\infty} \frac{1}{j^{1+\tau}} < \varepsilon. \tag{18}$$

Then, for each sequence $\{x_n\}_{n=1}^{\infty}$ with $K/2 \leq x_n \leq K$ and $|x_{n+1} - x_n| \leq \lambda_{n+1}$ for $n \geq n_0$, we have

$$\begin{aligned} &\bigg|\sum_{j=n}^{\infty}(-1)^j\bigg(\frac{a}{j^{\alpha(1+\tau)}} + \frac{a}{(j+1)^{\alpha(1+\tau)}}\bigg)x_{j+1}\bigg|^{1/\alpha} \\ &= \bigg|\sum_{j=n}^{\infty}(-1)^j\bigg(\frac{a}{j^{\alpha(1+\tau)}}\,x_{j+1} + \frac{a}{(j+1)^{\alpha(1+\tau)}}\,x_{j+2} \\ &\qquad\qquad + \frac{a}{(j+1)^{\alpha(1+\tau)}}\,(x_{j+1} - x_{j+2})\bigg)\bigg|^{1/\alpha} \\ &= \bigg|(-1)^n\frac{a}{n^{\alpha(1+\tau)}}\,x_{n+1} + \sum_{j=n+1}^{\infty}(-1)^j\frac{a}{j^{\alpha(1+\tau)}}\,(x_{j+1} - x_j)\bigg|^{1/\alpha} \\ &\leq \bigg(\frac{a}{n^{\alpha(1+\tau)}} + \sum_{j=n+1}^{\infty}\frac{a}{j^{\alpha(1+\tau)+1}}\bigg)^{1/\alpha} \\ &\leq \bigg(\frac{a + a/[\alpha(1+\tau)]}{n^{\alpha(1+\tau)}}\bigg)^{1/\alpha} \leq \frac{1}{n+1} \quad \text{for} \quad n \geq n_0. \end{aligned} \tag{19}$$

And, consequently, for each sequence $\{x_n\}_{n=1}^{\infty}$ with $K/2 \leq x_n \leq K$ and $|x_{n+1} - x_n| \leq \lambda_{n+1}$ for $n \geq n_0$, from (18), we have

$$\begin{aligned} &\bigg|\sum_{k=n_0}^{n-1}\bigg(\sum_{j=k}^{\infty}(-1)^j\bigg(\frac{a}{j^{\alpha(1+\tau)}} + \frac{a}{(j+1)^{\alpha(1+\tau)}}\bigg)x_{j+1}\bigg)^{1/\alpha}\bigg| \\ &\qquad\qquad \leq (a + a/[\alpha(1+\tau)])^{1/\alpha}\sum_{k=n_0}^{\infty}\frac{1}{k^{1+\tau}} < \varepsilon, \quad n \geq n_0 + 1, \end{aligned}$$

which, together with Theorem 3 and (19), yields that equation (17) has a bounded nonoscillatory solution.

REFERENCES

1. R.P. Agarwal, S.R. Grace and D. O'Regan, *Nonoscillatory solutions for discrete equations*, Comp. Math. Appl. **45** (2003), 1297–1302.

2. R.P. Agarwal and D. O'Regan, *Existence principles for continuous and discrete equations on infinite intervals in Banach spaces*, Math. Nachr. **207** (1999), 5–19.

3. R.P. Agarwal and P.J.Y. Wong, *Advanced topics in difference equations*, Kluwer, Academic, Dordrecht, 1997.

4. J. Dugundji and A. Granas, *Fixed point theory*, Monogr. Math. **61**, PWN, Warszawa, (1982).

5. N. Dunford and J. Schwartz, *Linear operators*, Interscience, New York, 1958.

6. A. Elbert, *Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations*, in *Ordinary and partial differential equations*, Lecture Notes in Math., vol. 964, Springer-Verlag, New York, 1982, pp. 187–212.

7. T. Kusano and N. Yoshida, *Nonoscillation theorems for a class of quasi-linear differential equations of second order*, J. Math. Anal. Appl. **189** (1995), 115–127.

8. W.T. Li, *Positive solutions of second order nonlinear differential equations*, J. Math. Anal. Appl. **221** (1998), 326–337.

9. P.J.Y. Wong and R.P. Agarwal, *Oscillation behavior of solutions of certain second order nonlinear differential equations*, J. Math. Anal. Appl. **198** (1996), 337–354.

10. X.J. Yang, *Nonoscillation criteria for second-order nonlinear differential equations*, Appl. Math. Comput. **131** (2002), 125–131.

Department of Mathematics, Xiangtan University, Hunan 411105, P.R. China
E-mail address: dengjiqin@yahoo.com.cn