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CONTINUITY OF HOMOMORPHISMS AND DERIVATIONS ON NORMED ALGEBRAS WHICH ARE TENSOR PRODUCTS OF ALGEBRAS WITH INVOLUTION

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ABSTRACT. We prove that, if A is a normed $*$-algebra of the form $B \otimes C$ for some central simple finite-dimensional algebra B with involution different from $\pm I_B$ and some algebra C with involution and a unit, then homomorphisms from A to normed algebras and derivations from A to normed A-bimodules are continuous whenever they are continuous on the hermitian part of A. When A is associative, some additional information is given.

1. Introduction. The aim of this paper is to study the automatic continuity of some homomorphisms and derivations with "arbitrary range" and whose domains are normed $*$-algebras over $\mathbf{K}$ ($= \mathbf{R}$ or $\mathbf{C}$) of the type $B\otimes C$. Here $B\otimes C$ stands for the algebraic tensor product of algebras B and C, each of them endowed with a (linear) involution. Our achievements in this line are collected in two independent results of the same flavor, namely Theorems 3 and 5, and are derived from Theorem 2, which is the main result in this paper. In the last quoted theorem we show that, if A is a normed $*$-algebra of the form $B\otimes C$ for some central simple finite-dimensional algebra B with involution different from $\pm I_B$, and some algebra C with involution and a unit, then two algebra norms on A making the tensor involution continuous are equivalent whenever they are equivalent on the hermitian part of A. As a consequence, if $n \geq 2$, if C is an algebra over $\mathbf{K}$ with involution and a unit, if $M_n(C)$ denotes the algebra of all $n \times n$ matrices with entries in C, and if we endow $M_n(C)$ with the standard involution (consisting in transposing a given matrix and applying the involution of C to each entry), then two algebra norms on $M_n(C)$ making its involution continuous are equivalent whenever they are equivalent on the hermitian part of $M_n(C)$. The fact just reviewed can be reformulated as follows. If A is a normed $*$-algebra over $\mathbf{K}$ which, algebraically regarded, is of

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the form $M_n(C)$ for some $n \geq 2$ and some algebra C with involution and a unit, then homomorphisms (respectively, derivations) from A to arbitrary normed algebras (respectively, normed A-bimodules) are continuous whenever they are continuous on the hermitian part of A. If, in addition, the algebra C above is associative, then the norms of continuous homomorphisms and derivations on A are "almost determined" by the norms of their restrictions to the hermitian part of A (see assertion (iii) in Corollary 4). An extension of this result to more general tensor algebras is shown in Theorem 5. Results of the same flavor were obtained in [**5**]. There it was shown that if A is a $*$-simple (associative) Banach algebra with an involution and a unit, then homomorphisms from A to normed algebras and derivations from A to normal A-modules are continuous whenever they are continuous on the hermitian part of A.

2. The results. Throughout this paper, $\mathbf{K}$ will denote either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. Let X and Y be normed spaces over $\mathbf{K}$ and, as usual, let $X \otimes Y$ denote the algebraic tensor product of X and Y. The *projective tensor norm* $\|\cdot\|_\pi$ on $X \otimes Y$ is defined by

$$\|a\|_\pi := \inf\left\{\sum_{j=1}^{n} \|x_j\|\|y_j\| : a = \sum_{j=1}^{n} x_j \otimes y_j\right\},$$

for every $a \in X \otimes Y$. Sometimes it will be convenient to denote $\|\cdot\|_\pi$ by $\|\cdot\|_X \otimes \|\cdot\|_Y$ where $\|\cdot\|_X$ (respectively, $\|\cdot\|_Y$) is the norm of X (respectively, of Y).

If B and C are (nonassociative) algebras over $\mathbf{K}$, then $B \otimes C$ will be considered, without notice, as an algebra over $\mathbf{K}$ under the product determined on elementary tensors by

$$(b_1 \otimes c_1)(b_2 \otimes c_2) := b_1b_2 \otimes c_1c_2.$$

Moreover, when B and C are normed algebras, the projective norm on $B \otimes C$ becomes an algebra norm.

An *involution* on an algebra B over a field $\mathbf{F}$ is an $\mathbf{F}$-linear involutive mapping $* : B \to B$ satisfying $(b_1b_2)^* = b_2^*b_1^*$ for all b_1, b_2 in B. If B and C are algebras with involution over $\mathbf{F}$, then $B \otimes C$ will be regarded without notice as an algebra with involution, namely, the *tensor involution*, which is determined on elementary tensors by $(b_1 \otimes c_1)^* := b^* \otimes c^*$ for all b in B and c in C.

We recall that an algebra B over a field $\mathbf{F}$ is called *central* (over $\mathbf{F}$) if the $\mathbf{F}$-multiples of the identity operator on B are the unique linear mappings $f : B \to B$ satisfying $f(b_1 b_2) = b_1 f(b_2) = f(b_1) b_2$ for all b_1, b_2 in B. It is well-known and easy to see that every finite-dimensional simple algebra over an algebraically closed field $\mathbf{F}$ is automatically central over $\mathbf{F}$.

In [**4**, Theorem 1.4], it is shown that, if B and C are algebras over $\mathbf{K}$, if B is finite-dimensional and central simple, and if C has a unit, then every algebra norm on $B \otimes C$ is equivalent to the projective tensor norm of suitable algebra norms on the factors. A non difficult consequence of this result is the following lemma, which will be crucial in our approach.

Lemma 1 [**4**, Corollary 2.1]. *Let B and C be algebras with involution over $\mathbf{K}$. Assume that B is finite-dimensional and central simple and that C has a unit. Then every algebra norm on the tensor product $B \otimes C$ making the tensor involution continuous is equivalent to the projective tensor norm of suitable algebra norms on the factors, making their involutions continuous.*

Let B be an algebra with involution $*$. We consider the *hermitian* part $H(B, *)$ and the *skew-hermitian* part $S(B, *)$ of B given by

$$H(B, *) := \{c \in C : c^* = c\} \quad \text{and} \quad S(C, *) = \{c \in C : c^* = -c\}.$$

On the other hand, I_B will denote the identity of B.

Theorem 2. *Let B and C be algebras with involution over $\mathbf{K}$. Assume that B is finite-dimensional and central simple, that the involution of B is different from $\pm I_B$ and that C has a unit. Let $\|\cdot\|$ and $\cdot\|\cdot\|'$ be algebra norms on $B \otimes C$ making the tensor involution continuous. Then $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if* (*and only if*) *they are equivalent on the hermitian part of $B \otimes C$.*

Proof. Put $A := B \otimes C$. Let $\|\cdot\|$ and $\|\cdot\|'$ be algebra norms on A which are equivalent on $H(A, *)$. By the finite dimensionality of B

and the above lemma, we can assume that $\|\cdot\| = \|\cdot\|_B \otimes \|\cdot\|_C$ and $\|\cdot\|' = \|\cdot\|_B \otimes \|\cdot\|'_C$ for suitable algebra norms $\|\cdot\|_B$, $\|\cdot\|_C$ and $\|\cdot\|'_C$ which make continuous the respective involutions. Moreover, it is not restrictive to assume that the involution of B (respectively, of C) is actually an isometry for the norm $\|\cdot\|_B$ (respectively, for both the norms $\|\cdot\|_C$ and $\|\cdot\|'_C$).

Since the involution of B is not $\pm I_B$, we can fix $u \in H(B,*)$ such that $\|u\|_B = 1$ and $v \in S(B,*)$ with $\|v\|_B = 1$. Then, for every b in $S(B,*)$ we have

$$1 = \frac{1}{2}\|(u+b)+(u+b)^*\|_B \leq \|u+b\|_B.$$

Therefore $\|u+S(B,*)\|_B = 1$ and hence, by the Hahn-Banach theorem, there exists a linear function f on B vanishing on $S(B,*)$ and satisfying $f(u) = \|f\|_B = 1$. Consider the mapping $F : B \to B$ given by $F(b) := f(b)u$ for every b in B. It follows from the equality $\|F\|_B = 1$ that

$$\|F \otimes I_C\| = \|F\|_B \|I_C\|_C = 1.$$

Now, let h be in $H(C,*)$ and s in $S(C,*)$. Then we have

$$\|h\|_C = \|u \otimes h\| = \|(F \otimes I_C)(u \otimes h + v \otimes s)\| \leq \|u \otimes h + v \otimes s\|.$$

A similar argument shows that $\|s\|_C \leq \|u \otimes h + v \otimes s\|$. Therefore,

$$\|h+s\|_C \leq \|h\|_C + \|s\|_C \leq 2\|u \otimes h + v \otimes s\|.$$

Let $k > 0$ be such that $\|\cdot\| \leq k\|\cdot\|'$ on $H(A,*)$. Since $u \otimes h + v \otimes s$ lies in $H(A,*)$, we obtain

$$\|u \otimes h + v \otimes s\| \leq k\|u \otimes h + v \otimes s\|' \leq 2k \max\{\|h\|'_C, \|s\|'_C\}.$$

But

$$\|h\|'_C = \frac{1}{2}\|(h+s)+(h+s)^*\|'_C \leq \|h+s\|'_C$$

and, similarly $\|s\|'_C \leq \|h+s\|'_C$, so that $\max\{\|h\|'_C, \|s\|'_C\} \leq \|h+s\|'_C$. It follows

$$\|h+s\|_C \leq 4k\|h+s\|'_C.$$

By symmetry, there exists $k' > 0$ such that

$$\|h + s\|'_C \leq 4k'\|h + s\|_C.$$

Since $C = H(C, *) + S(C, *)$, this proves the equivalence of $\| \cdot \|_C$ and $\| \cdot \|'_C$ on C and, consequently, the equivalence of $\| \cdot \|$ and $\| \cdot \|'$ in A. $\square$

Given an algebra A over $\mathbf{K}$, an A-*bimodule* is a vector space (say X) over $\mathbf{K}$ together with two bilinear mappings $(a, x) \to ax$ and $(a, x) \to xa$ from $A \times X$ to X. Since we are dealing with general nonassociative algebras, no rules of good behavior of the above bilinear mappings are required (see for instance [**3**, Section II.5]). If A is a normed algebra, if X is a normed space and also an A-bimodule, and if the inequalities $\|ax\| \leq \|a\|\|x\|$ and $\|xa\| \leq \|x\|\|a\|$ hold for every a in A and x in X, then we say that X is a *normed* A-*bimodule*. Given an algebra A and an A-bimodule X, a derivation from A to X is a linear map $\delta : A \to X$ such that

$$\delta(ab) = \delta(a)b + a\delta(b),$$

for all a, b in A.

By a *normed* $*$-*algebra* over $\mathbf{K}$ we mean a normed algebra over $\mathbf{K}$, endowed with a continuous involution.

Theorem 3. *Let A be a normed $*$-algebra over $\mathbf{K}$ which, algebraically regarded, is of the form $B \otimes C$ for some central simple finite-dimensional algebra B with involution different from $\pm I_B$ and some algebra C with involution and a unit. Then we have*

(i) *Homomorphisms from A to arbitrary normed algebras over $\mathbf{K}$ are continuous whenever they are continuous on $H(A, *)$.*

(ii) *Derivations from A to arbitrary normed A-bimodules over $\mathbf{K}$ are continuous whenever they are continuous on $H(A, *)$.*

Proof. (i) Let φ be a homomorphism from A to a normed algebra such that $\varphi_{\prime H(A,*)}$ is continuous. Consider the algebra norm $\| \cdot \|'$ on A given by

$$\|a\|' := \max\{\|a\|, \|\varphi(a)\|, \|\varphi(a^*)\|\}.$$

It follows from the $\|\cdot\|$-continuity of $*$ that $*$ is also $\|\cdot\|'$-continuous. On the other hand, the continuity of $\varphi_{/H(A,*)}$ leads to the equivalence of $\|\cdot\|$ and $\|\cdot\|'$ on $H(A,*)$. Therefore, by Theorem 2, we have that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent on A, which proves the continuity of φ.

(ii) Given a derivation δ from A to a normed A-bimodule, we can consider the algebra norm $\|\cdot\|'$ on A given by

$$\|a\|' := \max\{\|a\| + \|\delta(a)\|, \|a^*\| + \|\delta(a^*)\|\},$$

and argue as above to obtain the continuity of δ. $\square$

For an algebra C over $\mathbf{K}$, and a natural number n, we denote by $M_n(C)$ the algebra over $\mathbf{K}$ of all $n \times n$ matrices with entries in C. When C has an involution, $M_n(C)$ will be provided with the so-called *standard involution*, namely, the one consisting in transposing a given matrix and applying the involution of C to each entry.

Corollary 4. *Let A be a normed $*$-algebra over $\mathbf{K}$ which, algebraically regarded, is of the form $M_n(C)$ for some algebra C with involution and a unit and some $n \geq 2$. Then we have*

(i) *Homomorphisms from A to arbitrary normed algebras are continuous whenever they are continuous on the hermitian part of A.*

(ii) *Derivations from A to arbitrary normed A-bimodules are continuous whenever they are continuous on the hermitian part of A.*

(iii) *If, in addition, C is associative, then there exists a positive constant K (only depending on A) such that, for every continuous homomorphism φ from A to a normed algebra, and for every continuous derivation δ from A to a normed A-bimodule, we have*

$$\|\varphi\| \leq K\|\varphi_{/H(A,*)}\|^3 \quad \text{and} \quad \|\delta\| \leq K\|\delta_{/H(A,*)}\|.$$

Proof. Under the identification $M_n(C) \cong M_n(\mathbf{K}) \otimes C$, the involution of $M_n(C)$ is nothing but the tensor involution of the transpose involution on $M_n(\mathbf{K})$ and the given involution on C. Since $M_n(\mathbf{K})$ is central simple and the transpose involution on $M_n(\mathbf{K})$ is different from $\pm I_{M_n(\mathbf{K})}$ for $n \geq 2$, assertions (i) and (ii) follow from Theorem 3.

To prove (iii), we begin by pointing out that, by Lemma 1, the normed $*$-algebra $\hat{A}$, obtained by completing A, is algebraically of the form $M_n(\hat{C})$, where $\hat{C}$ is an algebra with involution and a unit. Then, since continuous homomorphisms (respectively, derivations) from A to normed algebras (respectively, to normed A-bimodules) can be extended to continuous homomorphisms (respectively, derivations) from $\hat{A}$ to complete normed algebras (respectively, to complete normed $\hat{A}$-bimodules) we can and will assume that A is complete. On the other hand, since $n \geq 2$, we can apply [**4**, Proposition 5.1] to obtain that A is (algebraically) generated by its hermitian part. Now assume that C is associative. Then it is folklore that $A = (H(A,*))^3$ (see for instance [**2**, p. 602]). Let $U := \{h_1h_2h_3 : h_1, h_2, h_3 \in B_H\}$ where B_H denotes the closed unit ball of $H(A,*)$, and let D be the closed convex hull of U. By keeping in mind that U is balanced, from the equality $A = (H(A,*))^3$ it easily follows that $A = \cup_{n\in\mathbf{N}} nD$. The completeness of A and the Baire category theorem give that the interior of D is not empty. Actually, since D is convex and symmetric, we obtain that $\varepsilon B_A \subseteq C$ for some $\varepsilon > 0$, where B_A denotes the closed unit ball of A. Consequently, if φ is a continuous homomorphism from A to a normed algebra we have

$$\begin{aligned}\varepsilon\|\varphi\| &= \varepsilon \sup_{a\in B_A} \{\|\varphi(a)\|\} \leq \sup_{b\in D}\{\|\varphi(b)\|\} \\ &= \sup_{c\in U}\{\|\varphi(c)\|\} \leq \|\varphi_{\prime H(A,*)}\|^3.\end{aligned}$$

Similarly, for a derivation δ from A to a normed A-bimodule we obtain, by applying the derivation rule, that

$$\varepsilon\|\delta\| = \varepsilon \sup_{a\in B_A} \{\|\delta(a)\|\} \leq \sup_{b\in D}\{\|\delta(b)\|\} = \sup_{c\in U}\{\|\delta(c)\|\} \leq 3\|\delta_{\prime H(A,*)}\|.$$

This proves the existence of a positive constant K such that

$$\|\varphi\| \leq K\|\varphi_{\prime H(A,*)}\|^3 \quad \text{and} \quad \|\delta\| \leq K\|\delta_{\prime H(A,*)}\|,$$

as desired. $\square$

The last part of the proof of assertion (iii) in the above theorem is strongly inspired by an argument in Theorem 1.6.2 of [**1**] (see also Proposition 5.3 of [**2**]).

The concluding result in this paper will be an extension of the above corollary as well as a variant of Theorem 3. Actually we will prove that, when in Theorem 3 the algebra B is associative and its hermitian part has degree ≥ 2 over its center, then that theorem remains true if the centrality of B is avoided and the simplicity of B is relaxed to the $*$-simplicity. Moreover, for such an algebra B, the assumption that its involution is different from $\pm I_B$ becomes superfluous.

Let A be an algebra. If a natural number n exists such that the dimensions of all one-generated subalgebras of A are less than or equal to n, then we define the *degree* of A as the smallest such n. Otherwise, we say that A is of *infinite degree*. By a $*$-*simple algebra* we mean an algebra A with involution $*$, nonzero product, and without $*$-invariant (two sided) ideals different from $\{0\}$ and A. Given an algebra A with involution $*$, the self-adjoint part of A, $H(A,*)$, will be considered without notice as an algebra under the product

$$a.b = \frac{1}{2}(ab + ba).$$

The *center* $Z(A)$ of an algebra A is defined as the set of all elements in A which commute with every element of A and associate with each two elements of A. The next theorem improves [**4**, Theorem 5.3] in several directions.

Theorem 5. *Let A be a normed $*$-algebra over $\mathbf{K}$ which, algebraically regarded, is of the form $B \otimes C$ for some $*$-simple finite-dimensional associative algebra B whose hermitian part $H(B,*)$ is of degree ≥ 2 over its center, and some algebra C with involution and a unit. Then we have*

(i) *Homomorphisms from A to arbitrary normed algebras are continuous whenever they are continuous on $H(A,*)$.*

(ii) *Derivations from A to arbitrary normed algebras are continuous whenever they are continuous on $H(A,*)$.*

(iii) *If, in addition, C is associative, then there exists a positive constant K (only depending on A) such that, for every continuous homomorphism φ from A to a normed algebra, and for every continuous derivation δ from A to a normed A-bimodule, we have*

$$\|\varphi\| \leq K\|\varphi_{\prime H(A,*)}\|^3 \quad \textit{and} \quad \|\delta\| \leq K\|\delta_{\prime H(A,*)}\|.$$

Proof. First assume that $\mathbf{K} = \mathbf{C}$. By [**4**, Lemma 3.4], there exist $n \geq 2$ and a complex composition associative algebra D such that $B = M_n(D)$ and $*$ is the standard involution relative to the Cayley involution on D. Therefore,

$$A = B \otimes C = M_n(D) \otimes C = M_n(\mathbf{C}) \otimes D \otimes C = M_n(D \otimes C)$$

with involution equal to the standard involution on $M_n(D \otimes C)$ relative to a suitable involution on $D \otimes C$. The proof in this case concludes by applying Corollary 4.

To prove the theorem in the case $\mathbf{K} = \mathbf{R}$, we being by considering the $*$-center $Z(B, *)$ of B defined by the equality $Z(B, *) := Z(B) \cap H(B, *)$. It is well known (see for instance [**3**]) that the assumptions on B imply that $Z(B, *) = Z(H(B, *))$ is a field isomoprhic to $\mathbf{R}$ or $\mathbf{C}$ and that B can be regarded as a $*$-simple algebra over $Z(B, *)$.

Now assume that $\mathbf{K} = \mathbf{R}$ and $Z(B, *) = \mathbf{C}$. Then B is the realification $E_{\mathbf{R}}$ of a complex finite dimensional $*$-simple associative algebra E. As in the first paragraph of the proof, there exist $n \geq 2$ and a complex composition associative algebra D such that $B = (M_n(D))_{\mathbf{R}}$ and $*$ is the standard involution relative to the Cayley involution on D. Now

$$\begin{aligned} A = B \otimes C &= (M_n(D))_{\mathbf{R}} \otimes C = M_n(D_{\mathbf{R}}) \otimes C \\ &= M_n(\mathbf{R}) \otimes D_{\mathbf{R}} \otimes C = M_n(D_{\mathbf{R}} \otimes C), \end{aligned}$$

with involution equal to the standard involution on $M_n(D_{\mathbf{R}} \otimes C)$ relative to a suitable involution on $D_{\mathbf{R}} \otimes C$. Again Corollary 4 concludes the proof in this case.

To study the remaining case, namely that $\mathbf{K} = \mathbf{R}$ and $Z(B, *) = \mathbf{R}$, we introduce some terminology and facts. For every real vector space X we denote by $X_{\mathbf{C}}$ its complexification, $X_{\mathbf{C}} = \mathbf{C} \otimes X$ and, by σ_X, the conjugate-linear involutive operator on $X_{\mathbf{C}}$ determined on elementary tensors by

$$\sigma_X(\alpha \otimes x) = \bar{\alpha} \otimes x.$$

If X and Y are real vector spaces, and if $\varphi : X \to Y$ is a linear map, then $\varphi_{\mathbf{C}} := I_{\mathbf{C}} \otimes \varphi$ is a complex-linear map from $X_{\mathbf{C}}$ to $Y_{\mathbf{C}}$. If X is a real normed space, then $X_{\mathbf{C}}$ will be regarded as a complex normed space relative to the projective tensor norm of the absolute value of $\mathbf{C}$ and the norm of X. In this regard, σ_X becomes an isometry. Now let X and Y be real normed spaces and $\varphi : X \to Y$ a linear map. If φ is continuous, then $\varphi_{\mathbf{C}}$ is also continuous and satisfies $\|\varphi_{\mathbf{C}}\| = \|\varphi\|$. If W is any σ_X-invariant subspace of $X_{\mathbf{C}}$, if we put

$$W^{\sigma_X} := \{w \in W : \sigma_X(w) = w\} \subseteq X,$$

and if $\varphi_{\prime W_X^{\sigma}}$ is continuous, then $(\varphi_{\mathbf{C}})_{\prime W}$ is continuous and we have

$$\|(\varphi_{\mathbf{C}})_{\prime W}\| \leq 2\|\varphi_{\prime W^{\sigma_X}}\|.$$

Indeed, for $w \in W$, we have that $x := [(w + \sigma_X(w))/2]$, $y := [(w - \sigma_X(w))/(2i)]$ are elements of W^{σ_X}, so that

$$\begin{aligned}\|\varphi_{\mathbf{C}}(w)\| &= \|\varphi_{\mathbf{C}}(x + iy)\| \\ &= \|\varphi(x) + i\varphi(y)\| \\ &\leq \|\varphi_{\prime W_X'}\|(\|x\| + \|y\|) \\ &\leq 2\|\varphi_{\prime W^{\sigma_X}}\|\|w\|.\end{aligned}$$

If X, Y, Z are real normed spaces and if $f : X \times Y \to Z$ is a continuous bilinear map, then the complex-bilinear mapping $f_{\mathbf{C}} : X_{\mathbf{C}} \times Y_{\mathbf{C}} \to Z_{\mathbf{C}}$ which extends f is also continuous with $\|f_{\mathbf{C}}\| = \|f\|$. This allows us to see complexifications of real normed algebras (respectively, normed bimodules) as complex normed algebras (respectively, complex normed bimodules).

Now assume that $\mathbf{K} = \mathbf{R}$ and that $Z(B, *) = \mathbf{R}$. Let φ be a homomorphism from A to a normed algebra E such that $\varphi_{\prime H(A,*)}$ is continuous. Then $A_{\mathbf{C}} := \mathbf{C} \otimes A$ is a normed $*$-algebra relative to the involution $I_{\mathbf{C}} \otimes *$, and $\varphi_{\mathbf{C}}$ is a homomorphism from $A_{\mathbf{C}}$ to $E_{\mathbf{C}}$ whose restriction to the hermitian part of $A_{\mathbf{C}}$ is continuous (indeed, $H(A_{\mathbf{C}}, *)$ is a σ_A invariant subspace of $A_{\mathbf{C}}$ and $H(A, *) = (H(A_{\mathbf{C}}, *))^{\sigma_A}$). Since the complex algebras $B_{\mathbf{C}}$ and $C_{\mathbf{C}}$ are algebras with involution in a natural manner, and $B_{\mathbf{C}}$ is finite-dimensional and $*$-simple ([**3**, p. 208]), and $C_{\mathbf{C}}$ has a unit, and $A_{\mathbf{C}} = B_{\mathbf{C}} \otimes C_{\mathbf{C}}$ (algebraically), by the first paragraph of the proof (when $A_{\mathbf{C}}, B_{\mathbf{C}}$ and $C_{\mathbf{C}}$ replace A, B and C, respectively) we conclude that $\varphi_{\mathbf{C}}$ is continuous on $A_{\mathbf{C}}$ and

$$\|\varphi_{\mathbf{C}}\| \leq K_{\mathbf{C}}\|(\varphi_{\mathbf{C}})_{\prime H(A_{\mathbf{C}},*)}\|^3,$$

where $K_{\mathbf{C}}$ is a positive constant only depending on A. Therefore, φ is continuous and

$$\|\varphi\| \leq \|\varphi_{\mathbf{C}}\| \leq K_{\mathbf{C}}\|(\varphi_{\mathbf{C}})_{/H(A_{\mathbf{C}},*)}\|^3 \leq 8K_{\mathbf{C}}\|\varphi_{/H(A,*)}\|^3.$$

The case of derivations from A to normed A-bimodules is handled in a similar way. □

The assumption in the above theorem that the degree of $H(B,*)$ over its center is ≥ 2 cannot be relaxed. Indeed, it is easy to find suitable normed $*$-algebras C over $\mathbf{K}$ with a unit and such that there exist discontinuous homomorphisms (respectively, derivations) from C to certain normed algebras (respectively normed C-bimodules) and therefore, by taking $B = \mathbf{K}$ with involution the identity, assertions (i) and (ii) in the theorem fail. On the other hand, we can take B equal to the realification of $\mathbf{C}$ with involution the conjugation, and $C = \mathbf{R}$ with involution the identity, to realize that assertion (iii) in the theorem fails (see for instance [**5**, Remark 2.5]). In these examples, all assumptions in the theorem are fulfilled except the one concerning the degree of $H(B,*)$ over its center.

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