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WEAKLY COMPACT COMPOSITION OPERATORS ON VMO

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ABSTRACT. A holomorphic map ϕ of the unit disk into itself induces an operator C_ϕ on holomorphic or harmonic functions by composition. The operator C_ϕ takes VMOA into itself if and only if the symbol ϕ belongs to VMOA. A number of necessary and sufficient conditions for C_ϕ to be weakly compact on VMOA are given. In particular, C_ϕ is weakly compact on VMOA if and only if $C_\phi(1_E) \in$ VMO for every Borel subset E of the unit circle. A hyperbolic version of Bloch's theorem is used to give a geometric characterization of those ϕ which induce compact composition operators on the little Bloch space $\mathfrak{B}_0$, and this is used to prove that if C_ϕ is weakly compact on VMOA, then C_ϕ is compact on $\mathfrak{B}_0$.

1. Introduction. Let ϕ be a holomorphic function mapping the unit disk $\mathbf{D} = \{z \mid |z| < 1\}$ into itself. Clearly, $f \circ \phi$ is analytic in the unit disk if f is, and similarly, $f \circ \phi$ is harmonic if f is. When restricted to various Banach spaces of analytic or harmonic functions, the operation of composition with ϕ, usually denoted C_ϕ, has been the object of intense study in recent years, especially the problem of relating operator-theoretic properties of C_ϕ to function-theoretic properties of ϕ. Certain aspects of this problem will be considered here for Banach spaces of functions of bounded and vanishing mean oscillation, particularly the question of which symbols yield weakly compact operators on VMOA and the question of when a weakly compact composition operator on VMOA will be compact.

Boundedness of composition on BMO follows easily from the fact that any function f on the unit circle $\mathbf{T}$ is of bounded mean oscillation if and only if it admits a representation

$$f = u + \tilde{v},$$

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where u and v are bounded functions and $\tilde{v}$ denotes the harmonic conjugate of v, together with the fact that composition commutes with conjugation. This seems to have been first noticed by Stephenson [**17**] and independently by Arazy, Fisher and Peetre [**1**]. On the other hand, in order for C_ϕ to map VMO into itself, it is clearly necessary that ϕ belong to VMOA, and it is not too difficult to prove that this is also sufficient [**3**]. It should be noted that here and throughout this paper a function $f \in L^1(\mathbf{T})$ will be freely identified with its harmonic extension to the unit disk $\mathbf{D}$. Thus,

$$f(a) = \mathcal{P}_a f = \int_{\mathbf{T}} f(\zeta) P_a(\zeta)\, dm(\zeta),$$

where m denotes normalized Lebesgue measure on $\mathbf{T}$ and $P_a(\zeta) = (1-|a|^2)/(|a-\zeta|^2)$ is the Poisson kernel for $a \in \mathbf{D}$.

There are a number of equivalent norms on BMO, but for the purposes of this paper the Garsia norm will be the most useful. Let

$$\|f\|_*^2 = \sup_{a\in\mathbf{D}} \{\mathcal{P}_a|f|^2 - |\mathcal{P}_a f|^2\}.$$

If $\sigma_a(z) = (a-z)/(1-\bar{a}z)$, then an easy calculation shows that

$$\|f\|_*^2 = \sup_{a\in\mathbf{D}} \int_{\mathbf{T}} |f(\sigma_a(\zeta)) - f(a)|^2\, dm(\zeta),$$

and so $\|f\|_*$ is clearly Möbius invariant. In order to distinguish constants, BMO will be normed by $\|f\|_G = |f(0)| + \|f\|_*$. It will be important to know that a function $f \in$ BMO belongs to VMO if and only if

$$\lim_{|a|\to 1} \{\mathcal{P}_a|f|^2 - |\mathcal{P}_a f|^2\} = 0,$$

and also if and only if f admits a representation $f = u + \tilde{v}$, where u and v are continuous on $\mathbf{T}$. All of this can be found in [**5**] or [**11**].

The question of which symbols ϕ induce compact composition operators on BMOA has been the object of several recent investigations. Various aspects of this problem have been studied by Bourdon, Cima and Matheson [**3**], Smith [**14**] and Tjani [**18**], with the first two papers giving (different) characterizations of compactness for composition operators on BMOA and VMOA.

The main purpose of this paper is to investigate the qeustion as to which symbols generate weakly compact composition operators on VMO and VMOA. According to Gantmakher's theorem, for Banach spaces X and Y, an operator $T : X \to Y$ is weakly compact if and only if $T^{**}(X^{**}) \subset Y$. Since BMO is isomorphic to the second dual of VMO, and C_ϕ on BMO is the second adjoint of C_ϕ on VMO, it follows that C_ϕ is weakly compact on VMO if and only if $f \circ \phi \in \text{VMO}$ for all $f \in \text{BMO}$, and a similar statement holds for VMOA.

The authors were unable to find a reference for the fact that the second adjoint of C_ϕ acting on VMO is C_ϕ acting on BMO, so the following proof is provided. If $T : \mathcal{X} \to \mathcal{Y}$ is any bounded operator from the Banach space $\mathcal{X}$ to the Banach space $\mathcal{Y}$, the second adjoint T^{**} of T can be computed as follows. For each x^{**} in the unit ball of X^{**}, there is, by Goldstine's theorem, [**4**, Theorem V.4.5], a net (x_α) in the unit ball of $\mathcal{X}$ which converges weak* to x^{**}. Then $T^{**}(x^{**})$ is the weak* limit of the net $(T(x_\alpha))$. Now every function in BMO has a harmonic extension to the open unit disk, and the unit ball of BMO is compact in the topology of uniform convergence on compact subsets of $\mathbf{D}$. Since the unit ball is also compact in the weak* topology, the two topologies coincide. If $f \in \text{BMO}$ is identified with its harmonic extension and $f_r(z) = f(rz)$ for $0 < r < 1$, then clearly $f_r \in \text{VMO}$ and $f_r \to f$ weak* as $r \to 1$. Evidently, $f_r \circ \phi$ also converges uniformly on compact subsets of $\mathbf{D}$ to $f \circ \phi$, and the result follows.

The question of the weak compactness of composition operators on VMO should be compared to the analogous question for the little Bloch space $\mathfrak{B}_0$ of functions f analytic in the unit disk for which

$$\lim_{|z|\to 1} (1 - |z|^2)|f'(z)| = 0. \tag{1}$$

In this case the second dual of $\mathfrak{B}_0$ is isomorphic to the Bloch space $\mathfrak{B}$ of functions for which the above quantity remains bounded. The questions of compactness and weak compactness for composition operators on $\mathfrak{B}_0$ were answered by Madigan and Matheson [**9**] who showed that C_ϕ is compact on $\mathfrak{B}_0$ if and only if

$$\lim_{|z|\to 1} \frac{1 - |z|^2}{1 - |\phi(z)|^2} |\phi'(z)| = 0, \tag{2}$$

and that every weakly compact composition operator on $\mathfrak{B}_0$ is actually compact. It should be remarked that the second statement follows abstractly from the known fact that $\mathfrak{B}_0$ is isomorphic to the Banach space c_0.

The class of functions satisfying condition (2) above has been called the hyperbolic little Bloch class [**15**], denoted $\mathfrak{B}_0^h$. The principal result of this paper is the following theorem.

Theorem 1. *If ϕ is a function which maps the unit disk into itself, and C_ϕ is a weakly compact operator on* VMOA*, then $\phi \in \mathfrak{B}_0^h$.*

Since the condition $\phi \in$ VMOA is needed to guarantee that C_ϕ map VMOA into itself, the appropriate converse to this theorem would be the statement that $\phi \in \text{VMOA} \cap \mathfrak{B}_0^h$ implies that C_ϕ is weakly compact on VMOA. As of this writing this is not known to be true. On the other hand, Smith [**14**] has shown that if ϕ is univalent, then C_ϕ is compact on VMOA if and only if $\phi \in \mathfrak{B}_0^h$. Combined with the above theorem, this shows that for univalent ϕ the composition operator C_ϕ is compact on VMOA if it is weakly compact on VMOA.

The proof of Theorem 1 will proceed in two steps. In the next section it will be shown that if C_ϕ is weakly compact on VMOA, then ϕ satisfies a certain geometric condition related to Bloch's theorem. Then in Section 3 a hyperbolic version of Bloch's theorem will be proved showing that the geometric condition is equivalent to membership in $\mathfrak{B}_0^h$.

Further connections between composition operators on $\mathfrak{B}$ and on BMOA will be explored in Section 4. Finally a number of equivalences with weak compactness on VMOA will be presented in the last section. An example of an operator which is weakly compact on VMOA but not compact will be presented along with a number of open questions.

2. A geometric condition. The pseudohyperbolic metric on the unit disk $\mathbf{D}$ is given by $\rho(a,b) = |(a-b)/(1-\bar{a}b)|$. For $a \in \mathbf{D}$ and $0 < r < 1$, let $\Delta(w,r)$ denote the pseudohyperbolic disk of center a and radius r, so that

$$\Delta(w,r) = \{z \in \mathbf{D} \mid \rho(z,w) < r\}.$$

If $\tau(z) = \lambda(z-a)/(1-\bar{a}z)$ is a Möbius transformation, then τ is an isometry for the pseudohyperbolic metric. In particular, τ maps $\Delta(a,r)$

onto $\Delta(\tau(a), r)$. Moreover, each pseudohyperbolic disk $\Delta(a, r)$ is also a Euclidean disk $D(c, \rho)$ with center

$$c = a\,\frac{1 - r^2}{1 - r^2|a|^2}$$

and radius

$$\rho = r\,\frac{1 - |a|^2}{1 - r^2|a|^2}.$$

The pseudohyperbolic disk $\Delta(a, \eta)$ is an *unramified* disk for the holomorphic function $\phi : \mathbf{D} \to \mathbf{D}$ at $a \in \mathbf{D}$ if $\Delta(a, \eta) \subset \phi(\mathbf{D})$, and there is an open set $G \subset \mathbf{D}$ such that ϕ restricted to G is a homeomorphism of G onto $\Delta(a, \eta)$. Let $\beta_\phi(r)$ be the least upper bound of the numbers η for which there is an unramified disk $\Delta(a, \eta)$ for ϕ with $|a| \geq r$. Clearly, $\beta_\phi(r)$ is decreasing as r increases. Let $\beta_\phi = \lim_{r \to 1} \beta_\phi(r)$. Since $\Delta(a, 1) = \mathbf{D}$ for each $a \in \mathbf{D}$, it follows that $\beta = 1$ for the identity function.

Proposition 1. *If* $\phi \in$ VMOA *and* C_ϕ *is weakly compact on* VMOA*, then* $B_\phi = 0$*.*

Proof. Assume that $\phi \in$ VMOA but $\beta_\phi > 0$. In order to show that C_ϕ is not weakly compact it will be enough to find a function $f \in$ BMOA such that $f \circ \phi \notin$ VMOA. Since $H^\infty \subset$ BMOA, it will suffice to find a Blaschke product b such that $b \circ \phi \notin$ VMOA.

The condition on β_ϕ guarantees the existence of a sequence $(a_n)_{n=1}^\infty$ in $\mathbf{D}$ and a positive number $\eta < \beta_\phi$ such that $|a_n| \to 1$ and the pseudohyperbolic disks $\Delta(a_n, \eta) \subset \phi(\mathbf{D})$ are unramified for ϕ. Passing to a subsequence, if necessary it may be assumed that $(a_n)_{n=1}^\infty$ is an interpolating sequence. Let b be the Blaschke product with zeros $(a_n)_{n=1}^\infty$.

An application of Lemma 1.4 in Chapter X of [**5**] shows that there exist λ, $0 < \lambda < \eta$, and δ, $0 < \delta < 1$, such that the set $\{z \mid |b(z)| < \delta\}$ is the union of pairwise disjoint domains V_n, with $a_n \in V_n$, and

$$V_n \subset \Delta(a_n, \lambda)$$

for each n. In particular, $|b(z)| \geq \delta$ on the boundary $\Gamma(a_n, \lambda)$ of $\Delta(a_n, \lambda)$. Let $f = b \circ \phi$. Since the disks $\Delta(a_n, \lambda)$ are unramified for ϕ, there exist disjoint domains G_n in $\mathbf{D}$ such that ϕ restricted to G_n is a univalent map of G_n onto $\Delta(a_n, \lambda)$ for each n. Let $b_n \in G_n$ be such that $\phi(b_n) = a_n$. Then, evidently, $|b_n| \to 1$ as $n \to \infty$. In order to show that $f \notin$ VMOA it will be enough to find a positive lower bound for the expressions

$$\mathcal{P}_{b_n}|f|^2 - |\mathcal{P}_{b_n} f|^2.$$

But, by construction, $f(b_n) = b(a_n) = 0$, so it will suffice to find a positive lower bound for

$$\mathcal{P}_{b_n}|f|^2 = \mathcal{P}_{b_n}|b \circ \phi|^2.$$

Let

$$V(z) = \mathcal{P}_z|b \circ \phi|^2$$

be the harmonic function in the unit disk with boundary values $|b \circ \phi|^2$. Since $|b \circ \phi|^2$ is subharmonic in $\mathbf{D}$, it follows that

$$V(z) \geq |b \circ \phi(z)|^2$$

for all $z \in \mathbf{D}$. But $|b \circ \phi| \geq \delta$ on ∂G_n, so $V(z) \geq \delta^2$ on ∂G_n. By the maximum principle $\mathcal{P}_{b_n}|f|^2 = V(b_n) \geq \delta^2$ for each n. Hence, $f \notin$ VMOA and that completes the proof. $\square$

In order to understand the geometric condition of the preceding section it will be useful to consider some examples from [**9**]. First, the function $\phi(z) = 1 - (1/2)\sqrt{1-z}$ maps the unit disk univalently onto a region G which behaves at 1 like a Stolz angle of opening $\pi/2$. An easy calculation shows that there is an $\eta > 0$ such that G contains pseudohyperbolic disks $\Delta(a, \eta)$ with a real and arbitrarily close to one. In particular $\beta_\phi > 0$ and so C_ϕ is not weakly compact on VMOA. It is easy to show that $\phi \notin \mathfrak{B}_0^h$.

Next let ψ be a univalent map of the unit disk $\mathbf{D}$ onto a region $G \subset \mathbf{D}$ such that $\overline{G} \cap \mathbf{T} = \{1\}$. The region G has a cusp at 1 if

$$\operatorname{dist}(w, \partial G) = o(|1 - w|)$$

as $w \to 1$ in G. The cusp is nontangential if G lies inside a Stolz angle near 1, i.e., there exist $r, M > 0$ such that

$$|1 - w| \leq M(1 - |w|^2)$$

if $|1-w|<R$, $w \in G$. Again it is easy to see that $\beta_\psi = 0$. As in [**9**] it is an easy consequence of Koebe's distortion theorem that $\psi \in \mathfrak{B}_0^h$.

Even though the examples above are univalent, the next proposition shows that this phenomenon is completely general.

Proposition 2. *Let ϕ be a holomorphic mapping of the unit disk into itself. Then $\phi \in \mathfrak{B}_0^h$ if and only if $\beta_\phi = 0$.*

This proposition will be a consequence of the following two lemmas, the first of which appears as the first exercise in Chapter I of [**5**]. Let r_ϕ be the largest r such that $\{|w|<r\}$ is an unramified disk for ϕ. If $\phi(0) = 0$ and $|\phi'(0)| = \delta > 0$, then a standard application of the Schwarz lemma to the inverse function shows that $r_\phi \le \delta$. Finally, let b_δ be the greatest lower bound of the r_ϕ as ϕ ranges over all functions ϕ with $\phi(0) = 0$ and $|\phi'(0)| = \delta$.

Lemma 2. *Let ϕ be a holomorphic map of the unit disk into itself with $\phi(0) = 0$ and $|\phi'(0)| = \delta$. If $|z| < \eta < \delta$, then*

$$|\phi(z)| \ge \left(\frac{\delta-\eta}{1-\eta\delta}\right)|z|.$$

Furthermore, ϕ takes each value w,

$$|w| < \left(\frac{\delta-\eta}{1-\eta\delta}\right)\eta,$$

exactly once in the disk $\{|z|<\eta\}$.

Proof. Let $g(z) = \phi(z)/z$, so $\|g\|_\infty \le 1$ and $g(0) = \phi'(0)$. By the Schwarz-Pick lemma,

$$\left|\frac{g(z)-\phi'(0)}{1-\overline{\phi'(0)}g(z)}\right| \le |z|.$$

Hence $g(z)$ lies in the pseudohyperbolic disk $\Delta(\phi'(0), |z|)$. This is the Euclidean disk with diameter

$$\left[\frac{|\phi'(0)|-|z|}{1-|\phi'(0)|\,|z|}\,\frac{\phi'(0)}{|\phi'(0)|},\ \frac{|\phi'(0)|+|z|}{1+|\phi'(0)|\,|z|}\,\frac{\phi'(0)}{|\phi'(0)|}\right].$$

In particular, since $|z| < \eta < \delta = |\phi'(0)|$,

$$|g(z)| \geq \frac{\delta - |z|}{1 - \delta|z|} \geq \frac{\delta - \eta}{1 - \eta\delta},$$

or

$$|\phi(z)| \geq \frac{\delta - \eta}{1 - \eta\delta}\,|z|.$$

Thus ϕ has exactly one zero in the disk $\{|z| < \eta\}$. Hence, by the argument principle, the image Γ of the circle $\{|z| = \eta\}$ has winding number $n(\Gamma, 0) = 1$ at 0. But Γ lies outside the disk $\{|w| < (\delta - \eta)/(1 - \eta\delta)\eta\}$, so $n(\Gamma, a) = 1$ for any a inside this disk. A second application of the argument principle then yields the last assertion. □

Lemma 3. *As* $\delta \to 0$,

$$\frac{\delta^2}{4} + O(\delta^4) \leq b_\delta \leq \frac{\delta^2}{4} + O(\delta^3).$$

Proof. The lower estimate follows from Lemma 2 as follows. Evidently,

$$r_\phi \geq \sup_{0 \leq \eta \leq \delta} \left(\frac{\delta - \eta}{1 - \eta\delta}\right)\eta,$$

since $\{|w| < (\delta - \eta)/(1 - \eta\delta)\eta\}$ is an unramified disk for ϕ if $0 < \eta < \delta$. Now the supremum is attained at

$$\eta_* = \frac{1 - (1 - \delta^2)^{1/2}}{\delta} = \frac{1}{2}\,\delta + O(\delta^3).$$

Hence, for small δ,

$$\begin{aligned} b_\delta &\geq \eta_* \left(\frac{\delta - \eta_*}{1 - \eta_*\delta}\right) \geq \eta_*(\delta - \eta_*) \\ &\geq \left(\frac{\delta}{2} + O(\delta^3)\right)\left(\frac{\delta}{2} + O(\delta^3)\right) \\ &= \frac{\delta^2}{4} + O(\delta^4). \end{aligned}$$

For the upper estimate consider the quadratic map

$$\phi(z) = \delta z - \varepsilon z^2,$$

where $\delta + \varepsilon = 1$. This has a branch point at $z = (\delta/2\varepsilon)$, and

$$\phi\left(\frac{\delta}{2\varepsilon}\right) = \frac{\delta^2}{4\varepsilon} = \frac{\delta^2}{4(1-\delta)} = \frac{\delta^2}{4} + O(\delta^3).$$

Hence $r_\phi \leq (\delta^2/4) + O(\delta^3)$ for this map. □

Proof of Proposition 2. For $a \in \mathbf{D}$, let $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$ be the Möbius transformation which interchanges a and 0. Note that σ_a maps the disk $\Delta(0, r)$ onto the disk $\Delta(a, r)$ and vice versa, since σ_a is its own inverse. Consider the change of variables

$$\psi(z) = \sigma_{\phi(a)} \circ \phi \circ \sigma_a(z).$$

Then $\psi(0) = 0$ and a calculation shows that

$$\psi'(0) = -\frac{1 - |a|^2}{1 - |\phi(a)|^2}\, \phi'(a),$$

so that $|\psi'(0)| = \tau_\phi(a)$, where

$$\tau_\phi(z) = \frac{1 - |z|^2}{1 - |\phi(z)|^2}\, |\phi'(z)|.$$

Now $\Delta(\phi(a), r)$ is an unramified disk for ϕ at a if and only if $\Delta(0, r)$ is an unramified disk for ψ at 0. If $r(a)$ is the radius of the largest unramified disk for ϕ at a, then

$$\frac{1}{4}\, \tau_\phi(a)^2 + O(\tau_\phi^4(a)) \leq r(a) \leq \tau_\phi(a)$$

by Lemma 3 and the estimate preceding Lemma 2. Hence, $\lim_{|a| \to 1} r(a) = 0$ if and only if $\lim_{|a| \to 1} \tau_\phi(a) = 0$. That completes the proof. □

3. Comparisons. It was shown by Bourdon, Cima and Matheson that if C_ϕ is compact on BMOA, then it must also be compact on H^2.

It is easy to find a function $\psi \in$ VMOA such that C_ψ is compact on H^2 but not on BMOA. In order to do this, let $\psi(z) = 1 - (1/2)\sqrt{1-z}$. As noted in the last section, ψ is a univalent map of the unit disk onto a region G which lies in a Stolz angle with vertex at 1 and opening angle $\pi/2$. Indeed, the boundary Γ of G is tangent to the two rays $1 + te^{\pm i(3\pi/4)}$. Since ψ is univalent and has no angular derivatives, C_ψ is compact on H^2 by Shapiro's univalent compactness condition [**13**, p. 39]. This particular function is continuous up to **T**, so certainly belongs to VMOA. Since G lies in a polygon, a result of Tjani [**18**] guarantees that C_ψ is compact on BMOA if and only if it is compact on $\mathfrak{B}$. But Madigan and Matheson [**9**] showed that C_ψ is not compact on $\mathfrak{B}_0$.

Using the fact that no inner function which is not a finite Blaschke product belongs to VMOA, it is possible to construct a $\phi \notin$ VMOA such that C_ϕ is compact on H^2 but not on BMOA. The authors would like to thank an anonymous referee for simplifying the original argument, which used ideas from [**12**].

Indeed, let $\phi = \psi \circ b$, where b is an infinite Blaschke product. Since $C_\phi = C_b \circ C_\psi$ and C_ψ is compact on H^2, it follows that C_ϕ is compact on H^2. Now

$$b = \psi^{-1} \circ \phi = 1 - 4(1-\phi)^2.$$

Since $L^\infty \cap$ VMOA is an algebra, it follows that $\phi \notin$ VMOA.

The collection of functions $s_r(z) - (r-z)/(1-rz)$ for $0 < r < 1$ satisfies $\|s_r\|_\infty = 1$, and so forms a bounded family in BMOA. Also $s_r(z) \to 1$ uniformly on compact subsets of **D** as $r \to 1$, and consequently so does $s_r \circ \phi$. It will be enough to show that there exists $\delta > 0$ such that $\|s_r \circ \phi\|_* \geq \delta$ for most $r, 0 \leq r < 1$, since $\|1\|_* = 0$.

The Frostman set for b is the set of points $a \in$ **D** such that $(a - b(z))/(1 - \bar{a}b(z))$ is not a Blaschke product. As the Frostman set has capacity zero, $\psi^{-1}(r)$ will not belong to the Frostman set for most values of r. For any such r, choose x such that $b(x) = \psi^{-1}(r)$. Then $\phi(x) = r$, and so $s_r \circ \phi(x) = 0$. On the other hand, there is a $\delta > 0$ such that $|\psi(e^{i\theta})| \geq \delta$ for all θ, and hence $|\phi(e^{i\theta})| \geq \delta$ almost everywhere. Thus,

$$\|s_r \circ \phi\|_*^2 \geq \int_{-\pi}^{\pi} \left| \frac{r - \phi(e^{i\theta})}{1 - r\phi(e^{i\theta})} \right|^2 P_x(e^{i\theta})\, \frac{d\theta}{2\pi} \geq \delta^2$$

for any such r and x. Since r can be chosen arbitrarily close to 1, C_ϕ is not compact on BMOA. Choosing b to be an interpolating Blaschke product, one can in fact guarantee that $\phi \notin \mathfrak{B}_0$.

As mentioned previously, since BMOA is isomorphic to the second dual of VMOA, it follows from Gantmakher's theorem that C_ϕ is weakly compact on VMOA if and only if $C_\phi(\text{BMOA}) \subset \text{VMOA}$. As the next theorem shows, it is possible to consider BMO and VMO in place of BMOA and VMOA. In the last condition of the theorem below, 1_E denotes the characteristic function of the set E.

Theorem 4. *The following are equivalent:*

(i) C_ϕ *is weakly compact on* VMOA;

(ii) $C_\phi(\text{BMOA}) \subset \text{VMOA}$;

(iii) $C_\phi(\text{BMO}) \subset \text{VMO}$;

(iv) $C_\phi(L^\infty) \subset \text{VMO}$;

(v) $C_\phi(1_E) \in \text{VMO}$ *for every Borel subset E of the unit circle.*

Proof. Clearly (i) and (ii) are equivalent, (iv) implies (v) and (iii) implies (ii) and (iv). To see that (iv) implies (iii), it is enough to note that any function $f \in \text{BMO}$ can be written in the form $f = u + \tilde{v}$ where $u, v \in L^\infty$, composition commutes with conjugation, and BMO is closed under conjugation.

To see that (ii) implies (iii), it is enough to consider real valued $u \in L^\infty$. Then $f = u + i\tilde{u} \in \text{BMOA}$, and so by (ii) $f \circ \phi \in \text{VMOA}$. Hence $u \circ \phi = \Re f \circ \phi \in \text{VMO}$.

Finally the implication (v) implies (iv) follows from the fact that $\text{QC} = L^\infty \cap \text{VMO}$ is uniformly closed and the fact that simple functions are dense in L^∞. □

A result of P. Jones [**8**] leads to the following curious corollary. Note that in general $\mathcal{P}_a(u\bar{v}) \neq \mathcal{P}_a(u)\mathcal{P}_a(\bar{v})$.

Corollary 1. *The composition operator C_ϕ is not weakly compact on* VMOA *if and only if there exist two interpolating Blaschke products*

u and v such that $C_\phi(u\bar{v}) \notin$ VMO*.*

Proof. According to the theorem of Jones, every unimodular function in L^∞ is a uniform limit of functions of the form $u\bar{v}$, where u and v are interpolating Blaschke products. If C_ϕ is not weakly compact on VMOA, then there is a Borel set E such that $C_\phi(1_E) \notin$ VMO. Now apply the theorem of Jones to the unimodular function $2 \cdot 1_E - 1$. The corollary again follows because QC is uniformly closed. □

Theorem 5. *The following are equivalent*:

(vi) $C_\phi(H^\infty + C) \subset$ VMO;

(vii) $C_\phi(H^\infty) \subset$ VMOA;

(viii) $u \circ \phi \in$ VMOA *for every interpolating Blaschke product u. Moreover, each of these conditions holds if C_ϕ is weakly compact on* VMOA.

Proof. Clearly (vi) implies (vii) which in turn implies (viii). In order to prove the implication that (viii) implies (vi), let $f = g + h$, where $g \in H^\infty$ and $h \in C$. Since $C \subset$ VMO, it follows that $C_\phi(h) \in$ VMO. According to a recent result of Garnett and Nicolau [**6, 10**], the interpolating Blaschke products generate H^∞. Hence there is a sequence (p_n) of linear combinations of finite products of interpolating Blaschke products which converge in H^∞ to g. Since VMO $\cap L^\infty$ is a closed subalgebra of L^∞, it follows that $C_\phi(p_n) \in$ VMOA for each n and that the limit function g also belongs to VMOA. The last remark follows from Theorem 2 as (iv) implies (vi). □

4. Weakly compact operators. Finally the question has been raised as to whether or not every weakly compact operator on VMOA is compact. A Banach space X is said to have the Schur property if every weakly null sequence in X converges to zero in norm. It is easy to see that if X^* has the Schur property, then every weakly compact operator $T : X \to Y$ is actually compact. In particular, since the dual of $\mathfrak{B}_0$ is isomorphic to the sequence space l_1, and the latter space is known to have the Schur property, every weakly compact operator on $\mathfrak{B}_0$ is compact. On the other hand, H^1, the dual of VMOA, is readily seen to not have the Schur property. Indeed, H^1 contains weak* closed complemented copies of Hilbert space, and this fact leads to a negative resolution of the above question. Indeed, the following theorem then guarantees that VMOA contains complemented copies of Hilbert space, and the associated projections are clearly weakly compact but not compact.

Theorem 6. *Let $X^* = Y \oplus Z$ be a direct sum decomposition of the dual space X^*, where each of the factors Y and Z is weak* closed. Then $Z_\perp = \{x \in X \mid z^*x = 0 \text{ for all } z \in Z\}$ is complemented in X.*

Proof. Throughout this proof, X will be identified with its canonical embedding in X^{**}. Let $P : X^* \to X^*$ be a bounded projection of X^* onto Z. Then the adjoint operator P^* is a projection of X^{**} onto $Z^\perp$, which is clearly the identity operator on $Z_\perp \subset Z^\perp$. To complete the proof it will suffice to show that $P^*x \in X$ for each $x \in X$. This will follow by showing that P^*x is weak* continuous. According to Theorem V.5.6 of [**4**], it is enough to show that P^*x is continuous for the bounded weak* topology on X^*. To this end, let (x^*_α) be a bounded net in X^* which converges to x^*. Then each x_α has a unique decomposition $x^*_\alpha = y_\alpha + z_\alpha$, where $y_\alpha \in Y$, $z_\alpha \in Z$ and $\|y_\alpha\|, \|z_\alpha\| \le K$ for some constant K. By the Banach-Alaoglu theorem, each subnet $(x^*_{\alpha'})$ of (x_α) has a further subnet $(x^*_{\alpha''})$ for which the nets $(y_{\alpha''})$ and $(z_{\alpha''})$ both converge, say, to y, and z, respectively. Clearly, $x^* = y + z$. Since Y and Z are weak* closed, it follows that $y \in Y$ and $z \in Z$. In particular, y and z are uniquely determined, and so the original nets (y_α) and (z_α) converge to y and z, respectively. It follows that $z = Px^*$. Now

$$\begin{aligned}\lim_\alpha P^*x(x^*_\alpha) &= \lim_\alpha P^*x(y_\alpha + z_\alpha) = \lim_\alpha P(y_\alpha + z_\alpha)(x) \\ &= \lim_\alpha y_\alpha(x) = y(x) = Px^*(x) = P^*x(x^*).\end{aligned}$$

Hence P^*x is bounded weak* continuous. That completes the proof. □

Now if E is a subset of the set of nonnegative integers, let H^1_E be the subspace of functions f in H^1 whose Taylor coefficients $\hat{f}(m)$ vanish unless $m \in E$. Since the evaluation of Taylor coefficients is weak* continuous on H^1, each H^1_E is weak* closed. If E is a gap sequence, i.e., $E = \{n_k\}$, where $(n_{k+1}/n_k) \geq \lambda > 1$ for each k, then an inequality of Paley [**19, 20**] shows that $\sum_{k=0}^{\infty} |\hat{f}(n_k)|^2 \leq C\|f\|_1^2$ for each $f \in H^1$. On the other hand, if $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, then the function $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ belongs to H^2 and hence a fortiori to H^1. It follows that H^1_E is a complemented subspace of H^1 isomorphic to l_2. If F is the complement of E in the set of nonnegative integers, then the above theorem shows that $\text{VMOA}_E = (H^1_F)_\perp$ is complemented in VMOA. Since its dual is isomoprhic to H^1_E, it follows that VMOA_E is a complemented copy of Hilbert space in VMOA.

In closing, there are several questions remaining. First, is it the case that every weakly compact composition operator on VMOA is compact? It follows from the results in this paper that $\phi \in \text{VMOA} \cap \mathfrak{B}_0^h$ is necessary for C_ϕ to be weakly compact on VMOA. Is this condition also sufficient? Finally, in light of the work of Garnett and Nicolau [**6**], and of Marshall and Stray [**10**], is it possible to close the gap between the corollary to Theorem 2 and condition (viii) of Theorem 3, i.e., is (viii) sufficient?

In another direction, one could ask for a characterization of functions $\phi \in \mathfrak{B}_0^h$ or functions $\phi \in \text{VMOA} \cap \mathfrak{B}_0^h$ similar to the characterization of bounded functions in $\mathfrak{B}_0$ given by Bishop [**2**]. Such characterizations could lead to the resolution of some of the questions above.

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