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TRANSMUTATION KERNELS FOR THE LITTLE q-JACOBI FUNCTION TRANSFORM

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ABSTRACT. The little q-Jacobi function transform depends on three parameters. An explicit expression as a sum of two very well-poised ${}_8W_7$-series is derived for the dual transmutation kernel relating little q-Jacobi function transforms for different parameter sets. A product formula for the dual transmutation kernel is obtained. For the inverse transform, the transmutation kernel is given as a ${}_3\varphi_2$-series, and a product formula as a finite sum is derived. The transmutation kernel gives rise to intertwining operators for the second order hypergeometric q-difference operator, which generalize the intertwining operators arising from a Darboux factorization.

1. Introduction. The Jacobi transform is an integral transform on the positive half-line with a hypergeometric ${}_2F_1$-series as its kernel. This transform is a two-parameter extension of the Fourier-cosine transform and the Mehler-Fock transform and also contains the Hankel transform as a limit case. The inversion formula for the Jacobi transform can be found explicitly in several ways, using asymptotics, spectral analysis, group theory or intertwining properties. This transform has a long history and we refer the reader to the survey paper [**13**] by Koornwinder.

There are several levels of q-analogues of the Jacobi function and of the corresponding transform pair (see [**9**] for an overview and references). Here we consider the so-called little q-Jacobi function and the corresponding transform. The little q-Jacobi function transform has been studied by Kakehi, Masuda and Ueno [**7, 6**] as the (spherical) Fourier transform on the quantum $SU(1,1)$ group using the interpretation of the little q-Jacobi functions on the quantum $SU(1,1)$ group using the interpretation of the little q-Jacobi functions as matrix elements of unitary irreducible representations of $U_q(\mathfrak{su}(1,1))$ (see [**14**]). On the other hand, the little q-Jacobi function transform occurs when

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studying the action of so-called twisted primitive elements in the principal unitary series representations of $U_q(\mathfrak{su}(1,1))$, and this has been the motivation for the study of this paper. However, the paper is completely analytic in nature, and the quantum group theoretic interpretation is only discussed briefly in Section 6.

The little q-Jacobi function transform can be obtained from the spectral analysis of the second order hypergeometric q-difference operator

$$L = L^{(a,b)} = a^2\left(1+\frac{1}{x}\right)(T_q - \mathrm{Id}\,) + \left(1+\frac{aq}{bx}\right)(T_q^{-1} - \mathrm{Id}\,), \tag{1.1}$$

where $T_q f(x) = f(qx)$ on a suitable Hilbert space (see Kakehi [**6**] or [**8**, Appendix A], where a slightly more general result is given). So we have eigenfunctions to L in terms of basic hypergeometric series (see [**4**, Chapter 1]). The little q-Jacobi function is defined as

$$\phi_\lambda(x;a,b;q) = {}_2\varphi_1\left(\begin{matrix} a\sigma, a/\sigma \\ ab \end{matrix}\; ; q, -\tfrac{bx}{a}\right), \quad \lambda = \frac{1}{2}(\sigma+\sigma^{-1}). \tag{1.2}$$

The notation for q-hypergeometric series follows Gasper and Rahman [**4**], and we assume $0<q<1$;

(1.3)
$$\begin{aligned} {}_{k+1}\varphi_k\left(\begin{matrix} a_1, a_2, \dots, a_{k+1} \\ b_1, \dots, b_k \end{matrix}\; ; q, z\right) &= \sum_{j=0}^\infty \frac{(a_1,a_2,\dots,a_{k+1};q)_j}{(q,b_1,\dots,b_k;q)_j} z^j, \\ (a_1,\dots,a_{k+1};q)_j &= (a_1;q)_j \dots (a_{k+1};q)_j, \\ (a;q)_j &= \prod_{i=0}^{j-1}(1-aq^i), \quad j \in \mathbf{Z}_{\geq 0}\cup\{\infty\}. \end{aligned}$$

The radius of convergence is 1 for generic parameters, but there exists a one-valued analytic continuation to $\mathbf{C}\setminus\mathbf{R}_{\geq 1}$ (see [**4**, Section 4.5]).

The little q-Jacobi function satisfies $L\phi_\lambda(\cdot;a,b;q) = (-1-a^2+2a\lambda)\phi_\lambda(\cdot;a,b;q)$. For later use, we note that the little q-Jacobi functions are eigenfunctions for the eigenvalue λ of

$$\begin{aligned}\mathcal{L}^{(a,b)} &= \frac{1}{2a}L^{(a,b)} + \frac{1}{2}(a+a^{-1}) \\ &= \frac{a}{2}\left(1+\frac{1}{x}\right)T_q - \left(\frac{a}{2x}+\frac{q}{2bx}\right)\mathrm{Id} \\ &\quad + \frac{1}{2a}\left(1+\frac{aq}{bx}\right)T_q^{-1}.\end{aligned} \tag{1.4}$$

For simplicity, we assume that $a, b > 0$, $ab < 1$ and $y < 0$, but the results hold, mutatis mutandis, for the more general range of the parameters as discussed in [**8**, Appendix A]. Then the operator L is an unbounded symmetric operator on the Hilbert space $\mathcal{H}(a,b;y)$ of square integrable sequences $u = (u_k)_{k\in\mathbf{Z}}$ with respect to the weights

$$\sum_{k=-\infty}^{\infty} |u_k|^2 (ab)^k \frac{(-byq^k/a;q)_\infty}{(-yq^k;q)_\infty}, \tag{1.5}$$

where the operator L is initially defined on the sequences with finitely many nonzero entries.

Note that (1.5) may be written as a q-integral. Indeed, by associating to u a function f on $yq^{\mathbf{Z}}$ by $f(yq^k) = u_k$ and using the notation (see [**4**, Section 1.11]),

$$\int_0^{\infty(y)} f(x)\, d_qx = y \sum_{k=-\infty}^{\infty} f(yq^k)q^k, \tag{1.6}$$

we see that for $a = q^{(\alpha+\beta+1)/2}$ and $b = q^{(\alpha-\beta+1)/2}$, the sum in (1.5) can be written as

$$y^\alpha \int_0^{\infty(y)} |f(x)|^2 x^\alpha \frac{(-xq^{-\beta};q)_\infty}{(-x;q)_\infty}\, d_qx, \quad \Re\alpha > -1. \tag{1.7}$$

Using the q-binomial theorem (see [**4**, Section 1.3]), we see that the quotient of q-shifted factorials in (1.7) tends to $(1+x)^\beta$ as q tends to 1. In the paper we will use the correspondence between $u \in \mathcal{H}(a,b;y)$ and functions f given by $f(yq^k) = u_k$ repeatedly.

The spectral analysis of L, or equivalently $\mathcal{L}^{(a,b)}$, on $\mathcal{H}(a,b;y)$ can be completely carried out, and this leads to corresponding transform

$$(1.8)\qquad \begin{aligned}(\mathcal{F}_{a,b,y}u)(\lambda) &= \sum_{k=-\infty}^{\infty} u_k \phi_\lambda(yq^k;a,b;q)(ab)^k \frac{(-byq^k/a;q)_\infty}{(-yq^k;q)_\infty},\\ u_k &= \int_{\mathbf{R}} (\mathcal{F}_{a,b,y}u)(\lambda)\phi_\lambda(yq^k;a,b;q)\,d\nu(\lambda;a,b;y;q),\end{aligned}$$

for an explicit measure $d\nu(\cdot;a,b;y;q)$ described in (1.11). Here we use the one-valued analytic continuation of the ${}_2\varphi_1$-series. Then $\mathcal{F} = \mathcal{F}_{a,b,y}$ extends to an isometric operator from $\mathcal{H}(a,b;y)$ onto $L^2(d\nu(\cdot;a,b;y;q))$.

The goal of this paper is to establish a number of links between two little q-Jacobi function transforms for different parameters (a,b,y). Our main interest lies with the dual transmutation kernels P satisfying

$$(1.9)\qquad (\mathcal{F}_{c,d,y}[\delta_t u])(\mu) = \int_{\mathbf{R}} (\mathcal{F}_{c,d,y}u)(\lambda)P_t(\lambda,\mu)\,d\nu(\lambda;a,b;y;q),$$

where $(\delta_t u)_k = t^k u_k$ for an extra parameter t. Note that the kernel P is analogous to a nonsymmetric Poisson kernel, and to a Poisson kernel for $(a,b) = (c,d)$ for a family of orthogonal polynomials. In our main result, Theorem 2.1, we present an explicit expression for the kernel in case $ad = bc$. The result is much inspired by Mizan Rahman's summation formulas [**8**, Appendix B]. We similarly study the transmutation kernel P for the inverse transform,

$$(\mathcal{F}^{-1}_{c,d,x}f)_l = \sum_{k=-\infty}^{\infty} (\mathcal{F}^{-1}_{a,b,y}f)_k P_{k,l}(ab)^k \frac{(-byq^k/a;q)_\infty}{(-yq^k;q)_\infty}.$$

An explicit expression for the transmutation kernel is obtained in case $dx = by$ in Theorem 2.2. For the transmutation kernels we show that these kernels do indeed satisfy the transmutation property, i.e., they intertwine the second order q-difference operator $\mathcal{L}^{(a,b)}$ for different parameters (a,b). This is closely related to results on q-analogues of Erdélyi's fractional integrals recently obtained by Gasper [**3**], and we give new proofs of some of his main results using the little q-Jacobi function transform (1.8). The main results are formulated in Section 2. The proofs of the statements are contained in Sections 3, 4 and 5, where in Section 3 we prove some summation formulas, amongst others an extension of Ramanujan's ${}_1\psi_1$-summation, that are of independent interest. In the final section, Section 6, we discuss some of the quantum group theoretic interpretations of these results.

The little q-Jacobi functions may also be considered as q-analogues of the Bessel function (see [**9**] for further references). It would be interesting to see if the (generalized) transmutation kernels can be evaluated also for other entries in the Askey-Wilson function scheme as described in [**9**].

We now give the precise form of the spectral measure $d\nu$ in (1.8). The measure can be obtained from the c-function expansion for the little q-Jacobi function (see Kakehi [**6**] or [**8**, Appendix A]), analogously to the case of the Jacobi transform. Explicitly,

$$
\begin{aligned}
\phi_\lambda(yq^k;a,b;q) &= c(\sigma;a,b;q)\Phi_\sigma(yq^k;a,b;q) \\
&\qquad + c(\sigma^{-1};a,b;q)\Phi_{\sigma^{-1}}(yq^k;a,b;q), \\
\Phi_\sigma(yq^k;a,b;q) &= (a\sigma)^{-k}{}_2\varphi_1\left(\begin{matrix} a\sigma, q\sigma/b \\ q\sigma^2 \end{matrix};q,-\frac{q^{1-k}}{y}\right), \\
c(\sigma;a,b,y;q) &= \frac{(b/\sigma,a/\sigma;q)_\infty}{(\sigma^{-2},ab;q)_\infty}\frac{(-by\sigma,-q/by\sigma;q)_\infty}{(-by/a,-qa/by;q)_\infty},
\end{aligned}
\tag{1.10}
$$

valid for $\sigma^2 \notin q^{\mathbf{Z}}$. Then Φ_σ is the asymptotically free solution; $L\Phi_\sigma(\cdot;a,b;q) = (-1-a^2+2a\lambda)\Phi_\sigma(\cdot;a,b;q)$ on $yq^{\mathbf{Z}}$ with, as before, $\lambda = (\sigma+\sigma^{-1})/2$. The measure $d\nu$ in (1.8) can be obtained from (1.10) (see [**6**], [**8**, Appendix A]). For this we now assume that $a > b$, which we can do without loss of generality, cf. Lemma 5.2 and (5.8). Explicitly, we have

$$
\begin{aligned}
&\frac{1}{C}\int_{\mathbf{R}} f(\lambda)\,d\nu(\lambda;a,b;y;q) \\
&\quad = \frac{1}{2\pi}\int_0^\pi f(\cos\theta)w(e^{i\theta})\,d\theta + \sum_{\substack{k\in\mathbf{Z}_{\geq 0}\\ |aq^k|>1}} f\left(\frac{1}{2}(aq^k+a^{-1}q^{-k})\right)w_k \\
&\qquad\qquad + \sum_{\substack{k\in\mathbf{Z}\\ |q^{1-k}/by|>1}} f\left(-\frac{1}{2}\left(\frac{q^{1-k}}{by}+byq^{k-1}\right)\right)v_k,
\end{aligned}
\tag{1.11}
$$

where

$$
\begin{aligned}
C &= (ab, ab, -by/a, -aq/by, -y, -q/y; q)_\infty, \\
w(z) &= \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, -byz, -q/byz, -by/z, -qz/by; q)_\infty}, \\
w_k &= \frac{1-a^2q^{2k}}{1-a^2} \frac{(a^2, ab; q)_k}{(q, aq/b; q)_k} (ab)^{-k} \\
&\quad \times \frac{(a^{-2}; q)_\infty}{(q, ab, b/a, -aby, -q/aby, -by/a, -aq/by; q)_\infty}, \\
v_k &= \frac{[(q^{2-2k}/b^2y^2) - 1] q^{-k(k-1)} (q^2/b^2y^2)^{k-1}}{(q, q, -q^{1-k}/y, -aq^{1-k}/by, -b^2yq^{k-1}, -abyq^{k-1}; q)_\infty}.
\end{aligned}
$$

Note that the integral plus the first sum and the sum over $-\mathbf{Z}_{\geq 0}$ in the second sum can be written as $\mathrm{dm}\,(\lambda; a, b, -by, -q/by \mid q)$, where $\mathrm{dm}\,(\cdot; a, b, c, d \mid q)$ denotes the standard (nonnormalized) Askey-Wilson measure (see [**2**], [**4**, Chapter 6]). So the measure in (1.11) has an absolutely continuous part supported on $[-1, 1]$, and for $a < 1$ there are no other discrete mass points in $d\nu$ apart from the infinite series tending to $-\infty$ (see [**6**], [**8**, Section 5], [**9**]). It should be observed that (1.8) can be formally obtained from the limit transition of the Askey-Wilson polynomials to the little q-Jacobi functions (see [**9**, Sections 2.3, 4.1, 6.1]).

Later in this paper, especially in Section 3, we use the notation for very well-poised series (cf. [**4**, Section 2.1]):

$$
\begin{aligned}
(1.12)\quad {}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) &= \sum_{j=0}^{\infty} \frac{1-a_1q^{2j}}{1-a_1} \frac{(a_1, a_4, \dots, a_{r+1}; q)_j z^j}{(q, qa_1/a_4, \dots, qa_1/a_{r+1}; q)_j} \\
&= {}_{r+1}\varphi_r \left(\begin{matrix} a_1, q\sqrt{a_1}, -q\sqrt{a_1}, a_4, \dots, a_{r+1} \\ \sqrt{a_1}, -\sqrt{a_1}, qa_1/a_r, \dots, qa_1/a_{r+1} \end{matrix} ; q, z \right).
\end{aligned}
$$

2. Statement of main results. In this section we describe the main results of the paper. We start with the dual transmutation kernel for the little q-Jacobi function, i.e., we want an explicit expression for

(2.1)
$$
\begin{aligned}
&\mathcal{F}_{a,b,y}[k \mapsto t^k \phi_\mu(xq^k; c, d; q)](\lambda) \\
&= \sum_{k=-\infty}^{\infty} (abt)^k \phi_\mu(xq^k; c, d; q) \phi_\lambda(yq^k; a, b; q) \frac{(-byq^k/a; q)_\infty}{(-yq^k; q)_\infty} \\
&= \sum_{k=-\infty}^{\infty} (abt)^k {}_2\varphi_1 \left(\begin{matrix} c\tau, c/\tau \\ cd \end{matrix} ; q, -q^k \tfrac{dx}{c} \right) {}_2\varphi_1 \left(\begin{matrix} b\sigma, b/\sigma \\ ab \end{matrix} ; q, -yq^k \right)
\end{aligned}
$$

using (1.8) and Heine's transformation formula [**4**, (1.4.6)]. Here $\lambda = (\sigma + \sigma^{-1})/2$ and $\mu = (\tau + \tau^{-1})/2$. In general, we do not have an explicit expression, but we have the following theorem, which will be proved in Section 3.

Theorem 2.1. *Let $a > b > 0$, $ab < 1$, $y > 0$. Define the dual transmutation kernel*

$$
P_t(\lambda, \mu; q^\alpha; a, b) = \mathcal{F}_{a,b,y}[k \longmapsto t^k \phi_\mu(yq^k; aq^\alpha, bq^\alpha; q)](\lambda).
$$

Then using the notation $\lambda = (\sigma + \sigma^{-1})/2$ and $\mu = (\tau + \tau^{-1})/2$ with $|\sigma|, |\tau| \geq 1$, the series defining P_t by (1.8) *is absolutely convergent for $|abq^\alpha \sigma\tau| < |abt| < 1$. For $|t| > q/ab$, $t \notin a^2 q^{2\alpha + \mathbf{Z}_{\geq 0}}$, $t \notin b^2 q^{2\alpha + \mathbf{Z}_{\geq 0}}$, P_t can be expressed explicitly as*

$$
\begin{aligned}
&P_t(\lambda, \mu; q^\alpha; a, b) \\
&= \frac{(b\sigma^{\pm 1}, bq^\alpha \tau^{\pm 1}, aq^{2\alpha}\sigma^{\pm 1}/t, aq^\alpha \tau^{\pm 1}/t, q, -yabt, -q/abyt; q)_\infty}{(ab, abq^{2\alpha}, b/a, q^\alpha \sigma^{\pm 1}\tau^{\pm 1}/t, a^2 q^{2\alpha}/t, abt, -y, -q/y; q)_\infty} \\
&\quad \times {}_8W_7(a^2 q^{2\alpha - 1}/t; aq^\alpha \tau^{\pm 1}, a\sigma^{\pm 1}, abq^{2\alpha - 1}/t; q, q/abt) \\
&\quad + \frac{(a\sigma^{\pm 1}, aq^\alpha \tau^{\pm 1}, bq^{2\alpha}\sigma^{\pm 1}/t, bq^\alpha \tau^{\pm 1}/t, q, -b^2 yt, -q/b^2 yt; q)_\infty}{(ab, abq^{2\alpha}, a/b, q^\alpha \sigma^{\pm 1}\tau^{\pm 1}/t, b^2 q^{2\alpha}/t, abt, -by/a, -qb/ay; q)_\infty} \\
&\quad \times {}_8W_7(b^2 q^{2\alpha - 1}/t; bq^\alpha \tau^{\pm 1}, b\sigma^{\pm 1}, abq^{2\alpha - 1}/t; q, q/abt).
\end{aligned}
$$

The expression for the dual transmutation kernel remains valid for λ a discrete mass point of the measure $d\nu(\cdot; a, b; y; q)$ as defined in (1.11).

Moreover, for $q^\alpha|\tau| < |t| < 1/\sqrt{ab}$ and $q^{\beta-\alpha}|t\rho| < |s| < tq^{-2\alpha}/\sqrt{ab}$, where $\nu = (\rho+\rho^{-1})/2$, $|\rho| \geq 1$, the product formula

$$\begin{aligned} &P_s(\mu,\nu;q^\beta;aq^\alpha,bq^\alpha) \\ &\qquad = \int_{\mathbf{R}} P_{q^{2\alpha}s/t}(\lambda,\nu;q^{\alpha+\beta};a,b)P_t(\lambda,\mu;q^\alpha;a,b)\,d\nu(\lambda;a,b;y;q) \end{aligned}$$

is valid.

As remarked in Section 1, the little q-Jacobi functions can be obtained as a limit of the Askey-Wilson polynomial. In this limit transition, one of the parameters tends to zero and another one of the parameters tends to ∞ exponentially. This case is not considered in Askey et al. [**1**] where the nonsymmetric Poisson kernel for the Askey-Wilson polynomials is derived. Note that the expression [**1**, (3.9)-(3.11)] is much more complicated than the expression in Theorem 2.1. Motivated by the quantum group theoretic interpretation (see Section 6), we should compare Theorem 2.1 to the nonsymmetric Poisson kernel for Al-Salam and Chihara polynomials, which consists of one very well-poised ${}_8W_7$-series; see Askey, Rahman and Suslov [**1**, (14.8)] and Ismail and Stanton [**5**, Theorem 4.2].

For the transmutation kernel we have the following result.

Theorem 2.2. *Define the transmutation kernel*

$$\begin{aligned} P_{k,l}(a,b,y;r,s) &= \mathcal{F}^{-1}_{a,b,y}[\lambda \longmapsto \phi_\lambda(yq^l/s;ar,bs;q)]_k \\ &= \int_{\mathbf{R}} \phi_\lambda(yq^l/s;ar,bs;q)\phi_\lambda(yq^k;a,b;q)\,d\nu(\lambda;a,b;y;q). \end{aligned}$$

For $r,s>0$, $rs<1$, the transmutation kernel is given by

$$\begin{aligned} P_{k,l}(a,b,y;r,s) = (ab)^{-l}&\frac{(ab,rs,q^{k-l+1},-yq^k;q)_\infty}{(q,abrs,rsq^{k-l},-byq^k/ar;q)_\infty} \\ &\times {}_3\varphi_2\left(\begin{matrix} q^{l-k},r,ar/b \\ rs,-arq^{1-k}/by \end{matrix};q,q\right) \end{aligned}$$

with the convention $P_{k,l}(a,b,y;r,s)=0$ for $k<l$. Moreover, the transmutation kernel satisfies the product formula for $r,s,t,u>0$,

$rs < 1$, $tu < 1$,

$$\sum_{l=k}^{p} P_{k,l}(a,b,y;r,s)P_{l,p}\left(ar,bs,\frac{y}{s};rt,su\right)(abrs)^l \frac{(-ybq^l/ar;q)_\infty}{(-yq^l/s;q)_\infty} = P_{k,p}(a,b,y;rt,su).$$

See (4.7) for the explicit expression for the product formula in terms of the ${}_3\varphi_2$-series. The resulting product formula is already contained in Gasper [**3**, (1.7)], and it can also be obtained from a general expansion formula [**4**, (3.7.9)] with $k=r=t=u=2, s=1$, due to Verma using transformation formulas for ${}_3\varphi_2$-series (see also [**3**, Section 4]).

Note that for $r=1$, or $s=1$, the transmutation kernel in Theorem 2.2 simplifies; for $k \geq l$ and $|s| < 1$,

$$P_{k,l}(a,b,y;1,s) = (ab)^{-l} \frac{(ab,-yq^k;q)_\infty}{(abs,-byq^k/a;q)_\infty} \frac{(s;q)_{k-l}}{(q;q)_{k-l}}$$

and letting $s \uparrow 1$ gives

$$\lim_{s\uparrow 1} P_{k,l}(a,b,y;1,s) = \delta_{k,l}(ab)^{-k} \frac{(-yq^k;q)_\infty}{(-byq^k/a;q)_\infty}$$

in accordance with (1.5) and (1.8).

The last main result deals with intertwining operators for the second order q-difference operator $\mathcal{L}^{(a,b)}$ as in (1.4). The intertwining properties are also known as transmutation properties and for the Jacobi function transform the intertwining operators are known as the Abel transform (see [**13**]).

Theorem 2.3. (i) *Let $a,b \in \mathbf{C}\setminus\{0\}$, $\nu,\mu \in \mathbf{C}$ with $|q^{\nu-\mu}b/a| < 1$. Define the operator*

$$\begin{aligned}(W_{\nu,\mu}(a,b)f)(x) &= \frac{(-x;q)_\infty}{(-xq^{-\mu};q)_\infty} q^{-\mu^2}\left(\frac{b}{a}\right)^{\mu} x^{\mu+\nu} \\ &\quad \times \sum_{p=0}^{\infty} f(xq^{-\mu-p})q^{-p\nu}\frac{(q^\nu;q)_p}{(q;q)_p} \\ &\quad \times {}_3\varphi_2\left(\begin{matrix} q^{-p}, q^{-\mu}, -q^{1+\mu-\nu}a/bx \\ q^{1-p-\nu}, -q^{\mu+1}/x \end{matrix}; q, q^{1-\mu}(b/a)\right)\end{aligned}$$

for any function f with $|f(xq^{-p})| = \mathcal{O}(q^{p(\varepsilon+\nu)})$ for some $\varepsilon > 0$. Then $W_{\nu,\mu}(a,b) \circ \mathcal{L}^{(a,b)} = \mathcal{L}^{(aq^{-\nu},bq^{-\mu})} \circ W_{\nu,\mu}(a,b)$ on the space of compactly supported functions and for $|a\sigma| < q^{\nu}$

$$\begin{aligned}(W_{\nu,\mu}(a,b)&\Phi_\sigma(\cdot;a,b;q))(yq^k)\\ &= y^{\mu+\nu}\frac{(a\sigma, b\sigma;q)_\infty}{(aq^{-\nu}\sigma, bq^{-\mu}\sigma;q)_\infty}\,\Phi_\sigma(yq^k;aq^{-\nu},bq^{-\mu};q).\end{aligned}$$

(ii) *Let $a,b > 0$, $ab < 1$, $\nu > 0$ and $\mu \in \mathbf{C}\setminus\mathbf{Z}_{\leq 0}$. Define the operator*

$$\begin{aligned}(A_{\nu,\mu}(a,b)f)(x) = &\frac{(-bxq^{\mu}/a;q)_\infty}{(-bxq^{\mu-\nu}/a;q)_\infty}\\ &\times\sum_{k=0}^{\infty} f(xq^{(\mu+k)})(ab)^k\frac{(q^{\nu},-xq^{\mu};q)_k}{(q,-bxq^{\mu}/;q)_k}\\ &\qquad\times {}_3\varphi_2\left(\begin{matrix}q^{-k}, q^{\mu}, -bxq^{\mu-\nu}/a\\ q^{1-\nu-k}, -xq^{\mu}\end{matrix};q,q\right)\end{aligned}$$

for any bounded function. Then $\mathcal{L}^{(aq^{\nu},bq^{\mu})} \circ A_{\nu,\mu}(a,b) = A_{\nu,\mu}(a,b) \circ \mathcal{L}^{(a,b)}$ on the space of functions compactly supported in $(0,\infty)$. Moreover,

$$(A_{\nu,\mu}(a,b)\phi_\lambda(\cdot;a,b;q))(x) = \frac{(abq^{\nu+\mu};q)_\infty}{(ab;q)_\infty}\phi_\lambda(x;aq^{\nu},bq^{\mu};q). \tag{2.2}$$

The operators $W_{\nu,\mu}(a,b)$ and $A_{\nu,\mu}(a,b)$ are q-analogues of the (generalized) Abel transform (see [**13**, Section 5]). Note that Theorem 2.3 gives q-integral representations for the little q-Jacobi function and the asymptotically free solution Φ_σ of (1.10).

The ${}_3\varphi_2$-kernel of $A_{\nu,\mu}(a,b)$ is the same as the transmutation kernel. In order to see this, we first invert the summation for the ${}_3\varphi_2$-series in (2.2) using [**4**, Ex. 1.4(ii)] and next transform it using [**4**, (III.13)]. Hence (2.2) is equivalent to (4.1). This shows that the transform with the transmutation kernel does indeed satisfy the transmutation property.

3. The dual transmutation kernel. The goal of this section is to prove Theorem 2.1. We start with proving some general results which are of independent interest and come back to the proof of Theorem 2.1 later.

We first formulate a general proposition generalizing Rahman's summation formulas in [**8**, Appendix B]. Note that in case the argument of the basic hypergeometric series has absolute value bigger than 1, we implicitly use the one-valued analytic continuation to $\mathbf{C} \setminus \mathbf{R}_{\geq 1}$.

Proposition 3.1. *Let $x, y \in \mathbf{C} \setminus \mathbf{R}_{\geq 0}$. Consider the sum*

$$S = \sum_{n=-\infty}^{\infty} z^n \, {}_2\varphi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, xq^n \right) {}_2\varphi_1 \left(\begin{matrix} d, e \\ f \end{matrix} ; q, yq^n \right),$$

which is absolutely convergent for $\max(|ad|, |ae|, |bd|, |be|) < |z| < 1$. *If, furthermore, $abde = cf$, $fx = dey$, $q < |z|$ and z/abf, $z/cde \notin q^{\mathbf{Z}_{\geq 0}}$, then S equals*

$$\begin{aligned} &\frac{(e, d, c/a, c/b, abd/z, abe/z, af/z, bf/z, q, yz, q/yz; q)_\infty}{(c, f, c/ab, ae/z, be/z, ad/z, bd/z, abf/z, z, y, q/y; q)_\infty} \\ &\quad \times {}_8W_7\left(\frac{abf}{qz}; a, b, \frac{f}{e}, \frac{f}{d}, \frac{cf}{qz}; q, \frac{q}{z}\right) \\ &\quad + \frac{(a, b, f/d, f/e, ade/z, bde/z, cd/z, ce/z, q, xz, q/xz; q)_\infty}{(c, f, f/de, ae/z, be/z, ad/z, bd/z, cde/z, z, x, q/x; q)_\infty} \\ &\quad \times {}_8W_7\left(\frac{cde}{qz}; d, e, \frac{c}{a}, \frac{c}{b}, \frac{cf}{qz}; q, \frac{q}{z}\right). \end{aligned}$$

Remark 3.2. Using the transformation formulas for very well-poised series we can find more expressions for the sum S, some of which are valid also if one of the conditions $q < |z|$, z/abf, $z/cde \notin q^{\mathbf{Z}_{\geq 0}}$ is violated (see (3.11), (3.12) and (3.13) below). Here we have chosen the expression as a sum of two very well-poised series that shows the symmetries $(a, b, c, x) \leftrightarrow (d, e, f, y)$, $a \leftrightarrow b$ and $d \leftrightarrow e$, which are obvious in the sum. (Note that the conditions $abde = cf$, $fx = dey$ also display this symmetry.) The expression also displays the symmetry $x \leftrightarrow y$,

$(a,b,d,e) \leftrightarrow (c/a, c/b, f/e, f/d)$. For the righthand sides, this follows from $abde = cf$ and $fx = dey$. For the sum S this follows from a double application of Heine's transformation [**4**, (III.3)], since $fx = dey$ and $abx = cy$.

The following lemma is of use in the proof of Proposition 3.1 and is of independent interest. Note that in case $k = 0$ the series in the summand can be summed by the q-binomial formula, and we obtain Ramanujan's ${}_1\psi_1$-summation formula (see [**4**, (5.2.1)]).

Lemma 3.3. *For* $\max(|a_1|, \dots, |a_{k+1}|) < |z| < 1$, *and for* $x \in \mathbf{C} \setminus \mathbf{R}_{\geq 0}$, *we have*

$$\sum_{n=-\infty}^{\infty} {}_{k+1}\varphi_k \left(\begin{matrix} a_1, \dots, a_{k+1} \\ b_1, \dots, b_k \end{matrix} ; q, xq^n \right) z^n = \frac{(a_1, \dots, a_{k+1}, b_1/z, \dots, b_k/z, q, xz, q/xz; q)_\infty}{(b_1, \dots, b_k, a_1/z, \dots, a_{k+1}/z, z, x, q/x; q)_\infty}.$$

Proof. By shifting the summation parameter we can assume, without loss of generality, that $1 \leq |x| < q^{-1}$. The series $\sum_{n=1}^{\infty}$ is absolutely convergent for $|z| < 1$. Let us assume for the moment that $|qb_1 \dots b_k| < |xq^n a_1 \dots a_{k+1}|$ for $n \leq 0$. Then the analytic continuation of the ${}_{r+1}\varphi_r$-series in the summand for $|xq^n| \geq 1$ is given by

$$\begin{aligned} {}_{k+1}\varphi_k &\left(\begin{matrix} a_1, \dots, a_{k+1} \\ b_1, \dots, b_k \end{matrix} ; q, xq^n \right) \\ &= \frac{(a_2, \dots, a_{k+1}, b_1/a_1, \dots, b_k/a_1, a_1xq^n, q^{1-n}/a_1x; q)_\infty}{(b_1, \dots, b_k, a_2/a_1, \dots, a_{k+1}/a_1, xq^n, q^{1-n}/x; q)_\infty} \\ &\quad \times {}_{k+1}\varphi_k \left(\begin{matrix} a_1, qa_1/b_1, \dots, qa_1/b_k \\ qa_1/a_2, \cdots, qa_1/a_{k+1} \end{matrix} ; q, \frac{q^{1-n}b_1 \dots b_k}{xa_1 \dots a_{k+1}} \right) \\ &\quad + \mathrm{idem}\,(a_1; a_2, \dots, a_{k+1}), \end{aligned} \tag{3.1}$$

where idem $(a_1; a_2, \dots, a_{k+1})$ after an expression stands for the sum of k terms obtained from the previous expression by interchanging a_1

with each a_i, $i = 2, 3, \dots, k+1$ (see [**4**, (4.5.2)]). Note that (1.10) is the case $k = 1$ of (3.1).

Using the theta product identity

$$(aq^n, q^{1-n}/a; q)_\infty = (-a)^{-n} q^{-n(n-1)/2} (a, q/a; q)_\infty \tag{3.2}$$

we see that the n-dependence in (3.1) simplifies. Indeed, since

$$\frac{(a_1 x q^n, q^{1-n}/a_1 x; q)_\infty}{(xq^n, q^{1-n}/x; q)_\infty} = a_1^{-n} \frac{(a_1 x, q/a_1 x; q)_\infty}{(x, q/x; q)_\infty} \tag{3.3}$$

the sum $\sum_{n=-\infty}^{0}$ of the first term in the righthand side of (3.1) times z^n is absolutely convergent for $|z/a_1| > 1$. Hence the sum is absolutely convergent for $\max(|a_1|, \dots, |a_{k+1}|) < |z| < 1$.

Next we split the sum $\sum_{n=1}^{\infty} + \sum_{n=-\infty}^{0}$, using the series for the ${}_{k+1}\varphi_k$-series in the first sum and (3.1) in the second sum. Interchanging summations we see from (3.3) that the sums over n are all geometric. So we see that the lefthand side of the lemma equals

$$\begin{aligned}
&z \sum_{j=0}^{\infty} \frac{(a_1, \dots, a_{k+1}; q)_j}{(q, b_1, \dots, b_k; q)_j} \frac{(qx)^j}{1 - q^j z} \\
&\quad + \frac{(a_2, \dots, a_{k+1}, b_1/a_1, \dots, b_k/a_1, a_1 x, q/a_1 x; q)_\infty}{(b_1, \dots, b_k, a_2/a_1, \dots, a_{k+1}/a_1, x, q/x; q)_\infty} \\
&\quad \times \sum_{j=0}^{\infty} \frac{(a_1, qa_1/b_1, \dots, qa_1/b_k; q)_j}{(q, qa_1/a_2, \dots, qa_1/a_{k+1}; q)_j} \\
&\qquad \times \left(\frac{qb_1 \dots b_k}{xa_1 \dots a_{k+1}} \right)^j \frac{1}{1 - a_1 q^j / z} \\
&\quad + \mathrm{idem}\,(a_1; a_2, \dots, a_{k+1}).
\end{aligned} \tag{3.4}$$

The sums in (3.4) can be written as ${}_{k+2}\varphi_{k+1}$-series. The first sum in

(3.4) equals

(3.5)

$$\frac{z}{z-1}\,{}_{k+2}\varphi_{k+1}\left(\begin{matrix} a_1,\dots,a_{k+1},z \\ b_1,\dots,b_k,qz \end{matrix};q,qx\right)$$
$$= \frac{z}{z-1}\frac{(a_1,\dots,a_{k+1},b_1/z,\dots,b_k/z,q,xqz,1/xz;q)_\infty}{(b_1,\dots,b_k,a_1/z,\dots,a_{k+1}/z,qz,xq,1/x;q)_\infty}$$
$$+ \frac{z}{z-1}\frac{(a_2,\dots,a_{k+1},b_1/a_1,\dots,b_k/a_1,a_1xq,1/a_1x,z,qz/a_1;q)_\infty}{(b_1,\dots,b_k,a_2/a_1,\dots,a_{k+1}/a_1,xq,1/x,qz,z/a_1;q)_\infty}$$
$$\times\,{}_{k+2}\varphi_{k+1}\left(\begin{matrix} a_1,qa_1/b_1,\dots,qa_1/b_k,a_1/z \\ qa_1/a_2,\dots,qa_1/a_{k+1},qa_1/z \end{matrix};q,\tfrac{qb_1\dots b_k}{xa_1\dots a_{k+1}}\right)$$
$$+ \text{idem}\,(a_1;a_2,\dots,a_{k+1}),$$

where we have used [**4**, (4.5.2)], cf. (3.1), once again. (The first ${}_{k+2}\varphi_{k+1}$-series reduces to 1, since an upper parameter is equal to 1.) The $k+1$ ${}_{k+2}\varphi_{k+1}$-series in (3.5) are the same as in (3.4), and a simple calculation reveals that they occur with opposite coefficients. Hence, using (3.5) in (3.4) leaves only the first term on the righthand side of (3.5). This proves the result for the condition $|qb_1\dots b_k| < |xq^n a_1\dots a_{k+1}|$ for $n \le 0$, and the general case follows by analytic continuation in the parameters of the ${}_{k+1}\varphi_k$-series. $\square$

Proof of Proposition 3.1. The sum $\sum_{n=0}^{\infty}$ in S is absolutely convergent for $|z| < 1$. As in the proof of Lemma 3.3, we can use (3.1) for $k=1$ twice to see that the sum $\sum_{n=-\infty}^{-1}$ in S is absolutely convergent for $|z| > \max(|ad|,|ae|,|bd|,|be|)$. For n large enough we have $|xq^n|, |yq^n| < 1$, so let us assume first $|x| < 1$ and $|y| < 1$. We can use the series representation (1.3) twice to write

$$(3.6)\quad {}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,x\right){}_2\varphi_1\left(\begin{matrix} d,e \\ f \end{matrix};q,y\right)$$
$$= \sum_{k=0}^{\infty}\frac{(d,e;q)_k}{(q,f;q)_k}y^k\,{}_4\varphi_3\left(\begin{matrix} a,b,q^{-k},q^{1-k}/f \\ c,q^{1-k}/d,q^{1-k}/e \end{matrix};q,\tfrac{qfx}{dey}\right),$$

cf. Rahman's proof in [**8**, Appendix B]. The terminating ${}_4\varphi_3$-series in the summand in (3.6) is balanced for $abde = cf$ and $dey = fx$, the assumptions in the proposition. Hence it can be transformed into a terminating very well-poised ${}_8W_7$-series; see (1.12) for the notation, by [**4**, (III.19)]. It follows that (3.6) equals

$$\sum_{k=0}^{\infty} \frac{(e, ab, bd; q)_k}{(q, f, abd; q)_k} y^k \quad {}_8W_7\left(\frac{abd}{q}; a, b, \frac{f}{e}, q^{-k}, q^{k-1}cf; q, \frac{qd}{f}\right). \tag{3.7}$$

The ${}_8W_7$-series in (3.7) can be rewritten as a sum of two nonterminating ${}_8W_7$-series using Bailey's three-term transformation [**4**, (III.37)] with (a, b, c, d, e, f) replaced by $(afq^k/e, adq^k, aq/c, q/e, a, f/e)$. Recalling that $abde = cf$, we find

$$\begin{aligned}
&\frac{(afq^k, q^{k+1}\frac{f}{e}, aq^{k+1}, eq^k, \frac{c}{a}, \frac{c}{b}, d; q)_\infty}{(afq^{k+1}/e; q)_\infty} \\
&\qquad \times {}_8W_7\left(\frac{afq^k}{e}; adq^k, \frac{aq}{c}, \frac{q}{e}, a, \frac{f}{e}; q, beq^k\right) \\
&\quad = \frac{(fq^k, aeq^k, q^{k+1}, c, bd, c/ab, ad; q)_\infty}{(abd; q)_\infty} \\
&\qquad \times {}_8W_7\left(\frac{abd}{q}; a, b, \frac{f}{e}, q^{-k}, q^{k-1}cf; q, \frac{qd}{f}\right) \\
&\qquad + \frac{ed}{f} \frac{(edq/f, cdq^k, adeq^k, dq^{k+1}, cq^{k+1}/b; q)_\infty}{(qf/de, fcq^k/e, qd/f, q^{1-k}/e, cdq^{k+1}/b; q)_\infty} \\
&\qquad \times (q/e, a, f/e, b, dq^{1-k}/f, q^k f/d; q)_\infty \\
&\qquad \times {}_8W_7\left(\frac{dcq^k}{b}; adq^k, \frac{dq}{f}, \frac{q}{b}, d, \frac{c}{b}; q, beq^k\right).
\end{aligned} \tag{3.8}$$

Note that the other two ${}_8W_7$-series in (3.8) are obtained from each other by interchanging (a, b, c, x) with (d, e, f, y). Using (3.8) in (3.7) and recalling that $abde = cf$ and $dey = fx$ leads to the following expression for (3.6)

$$\begin{aligned}
&\sum_{k=0}^{\infty} y^k \frac{(abdq^k, afq^k, q^{k+1}f/e, aq^{k+1}, e, c/a, c/b, d; q)_\infty}{(f, aeq^k, q, c, bdq^k, c/ab, adq^k, afq^{k+1}/e; q)_\infty} \\
&\times {}_8W_7\left(\frac{afq^k}{e}; adq^k, \frac{aq}{c}, \frac{q}{e}, a, \frac{f}{e}; q, beq^k\right) + \mathrm{Idem}\,((a, b, c, x); (d, e, f, y)),
\end{aligned} \tag{3.9}$$

where Idem $((a,b,c,x);(d,e,f,y))$ means that we have the same sum with the parameter sets (a,b,c,x) and (d,e,f,y) interchanged.

Since we assume $|y|<1$ and $|be|<1$, the double series in (3.9) is absolutely convergent. Interchanging summations and recalling $abde=cf$ shows that

$$
\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} a,b \\ c \end{matrix};q,x\right){}_2\varphi_1\left(\begin{matrix} d,e \\ f \end{matrix};q,y\right) \\
&= \sum_{j=0}^{\infty}(be)^j \frac{(e,c/a,c/b,d,afq^j,q^{j+1}f/e,aq^{j+1},abdq^j;q)_\infty}{(f,q,c,c/ab,ae,bd,afq^j/e,adq^j;q)_\infty} \\
&\quad\times \frac{(aq/c,q/e,a,f/e;q)_j}{(q,fq/de;q)_\infty}\left(1-\frac{afq^{2j}}{e}\right) \\
&\quad\times {}_6\varphi_5\left(\begin{matrix} q,ae,bd,afq^j/e,adq^j,afq^{1+2j}/e \\ abdq^j,afq^j,q^{j+1}f/e,aq^{j+1},afq^{2j}/e \end{matrix};q,yq^j\right) \\
&\quad+\mathrm{Idem}\,((a,b,c,x);(d,e,f,y)),
\end{aligned}
\tag{3.10}
$$

assuming that $abde=cf$, $dey=fx$, $|x|<1$, $|y|<1$ and $|be|<1$. As a function of y, the lefthand side has a unique analytic continuation to $\mathbf{C}\setminus(\mathbf{R}_{\geq 1}\cup(de/f)\mathbf{R}_{\geq 1})$. The y-dependence in the first sum on the righthand side is only at the argument spot of the ${}_6\varphi_5$-series, which has a unique analytic continuation to $\mathbf{C}\setminus\mathbf{R}_{\geq 1}$. As $j\to\infty$, the ${}_6\varphi_5$-series tends to 1, so the convergence with respect to y in the righthand side of (3.10) is uniform on compact sets. Similarly the ${}_6\varphi_5$-series in the other sum has a unique analytic extension to $\mathbf{C}\setminus[(de/f)\mathbf{R}_{\geq 1}]$. Hence (3.10) remains valid for $y\in\mathbf{C}\setminus(\mathbf{R}_{\geq 1}\cup(de/f)\mathbf{R}_{\geq 1})$ after using the analytic continuation of the ${}_2\varphi_1$- and ${}_6\varphi_5$-series. In particular, this means that (3.10) is valid for $abde=cf$, $dey=fx$, $|be|<1$ and $y\in\mathbf{C}\setminus(\mathbf{R}_{\geq 1}\cup(de/f)\mathbf{R}_{\geq 1})$.

To prove the result, we replace x and y by xq^n and yq^n in (3.10), multiply by z^n and sum over $n\in\mathbf{Z}$. If we assume for the moment that $\max(q,|ae|,|bd|,|ad|,|af/e|,|cd/b|)<|z|<1$, then we can interchange summations and use Lemma 3.3 twice to sum the inner sum. Some cancellation occurs, and after using the theta product identity (3.2) twice, we see that S equals

$$(3.11)\quad \frac{(e,c/a,c/b,d,yz,q/yz,q;q)_\infty}{(f,c,c/ab,z,q/z,ae/z,bd/z,y,q/y;q)_\infty}$$
$$\times \sum_{j=0}^{\infty}\left(\frac{be}{z}\right)^j\left(1-\frac{afq^{2j}}{ez}\right)\frac{(aq/c,q/e,a,f/e;q)_j}{(q,fq/de;q)_j}$$
$$\times \frac{(abdq^j/z,afq^j/z,fq^{j+1}/ez,aq^{j+1}/z;q)_\infty}{(afq^j/ez,adq^j/z;q)_\infty}$$
$$+\,\mathrm{Idem}\,((a,b,c,x);(d,e,f,y))$$
$$=\frac{(e,c/a,c/b,d,q,yz,q/yz,abd/z,af/z,qf/ez,aq/z;q)_\infty}{(f,c,c/ab,y,q/y,z,q/z,ae/z,bd/z,qaf/ze,ad/z;q)_\infty}$$
$$\times\, {}_8W_7\left(\frac{af}{ze};\frac{aq}{c},\frac{q}{e},a,\frac{f}{e},\frac{ad}{z};q,\frac{be}{z}\right)+\mathrm{Idem}\,((a,b,c,x);(d,e,f,y)).$$

For $q<|z|$ we can use [**4**, (III.23)] using $abde=cf$ to transform the ${}_8W_7$-series in the required form. Finally, use continuation in z to find the result. $\square$

Remark 3.4. (i) A proposition of this type has been proved first by Mizan Rahman for two special cases (see [**8**, Appendix B]). To see how the two special cases are contained in the result, we consider it in the form (3.11). To the first ${}_8W_7$-series, we apply [**4**, (III.37)] with (a,b,c,d,e,f) replaced by $(af/ze,ad/z,aq/c,f/e,a,q/e)$ to write it as a sum of two very well-poised ${}_8W_7$-series, of which the second is the same ${}_8W_7$-series as the second in (3.11). It turns out that we can add the coefficients using the theta product identity as in [**4**, Ex. 2.16] with (x,λ,μ,ν) replaced by $(\sqrt{xyz},\sqrt{xz/y},\sqrt{ef/dz},\sqrt{x/yz})$. This yields

$$(3.12)\quad S=\frac{(q,d,e,abd/z,f/z,adq/f,bdq/f,yz,q/yz;q)_\infty}{(f,z,ad/z,bd/z,e/z,qd/f,qc/e,y,q/y;q)_\infty}$$
$$\times\,{}_8W_7\left(\frac{abd}{f};a,b,\frac{q}{e},\frac{c}{z},\frac{qz}{f};q,d\right)$$
$$+\frac{(q,a,b,f/e,qde/f,qd/z,cd/z,ade/z;q)_\infty}{(c,f,ad/z,bd/z,ae/z,qd/f,cdq/bz,y,q/y,x,q/x,z/e,qe/z;q)_\infty},$$
$$\times\,(cq/bz,ey,q/ey,xz/e,qe/xz;q)_\infty\,{}_8W_7\left(\frac{cd}{bz};d,\frac{dq}{f},\frac{q}{b},\frac{c}{b},\frac{ad}{z};q,\frac{be}{z}\right)$$

valid for $|d| < 1$ and on any subregion of $\max(|ad|,|ae|,|bd|,|be|) < |z| < 1$ in the complex z-plane as long as the righthand side is analytic in this subregion. The first case proved by Mizan Rahman [**8**, Appendix B.1] corresponds to $f = qz$ in (3.12) so that the first ${}_8W_7$-series reduces to 1, and the second case proved by Mizan Rahman [**8**, Appendix B.3] corresponds to $ey \in q^{\mathbf{Z}}$ in (3.12) so that the second term vanishes.

(ii) The proof of Proposition 3.1 is inspired by Rahman's method as presented in [**8**, Appendix B.3] but is of a different nature. In Rahman's case, the balance ${}_4\varphi_3$-series is written as a q-integral using [**4**, (2.10.19)], which is a three-term transformation for balanced ${}_4\varphi_3$-series. In this paper, we use Bailey's three-term transformation [**4**, (2.11.1)] for very well-poised ${}_8\varphi_7$-series, which can be deduced from [**4**, (2.10.19)]. Note that the result as a sum of two very well-poised series cannot be simplified.

(iii) Note that (3.11) and (3.12) give alternative expressions for the sum S of Proposition 3.1 and together with the obvious symmetries in S we find more expressions for S in terms of a sum of two very well-poised ${}_8W_7$-series. We can also rewrite the result as a sum of three balanced ${}_4\varphi_3$-series as follows. Start with (3.12) with $(a,b,c,x) \leftrightarrow (d,e,f,y)$, apply [**4**, (III.36)] with (a,b,c,d,e,f) replaced by $(ade/c, q/b, f/z, d, e, qz/c)$ to the first ${}_8W_7$-series to write it as a sum of two balanced ${}_4\varphi_3$-series and apply [**4**, (III.36)] with (a,b,c,d,e,f) replaced by $(af/ez, q/e, f/e, aq/c, ad/z)$ to the second ${}_8W_7$-series to write it as a sum of two balanced ${}_4\varphi_3$-series of which one also occurs in the previous transformation. The balanced ${}_4\varphi_3$-series can be taken together using the theta product identity [**4**, Ex. 2.16] with (x,λ,μ,ν) replaced by $(\sqrt{bxz/a}, \sqrt{axz/b}, \sqrt{abx/z}, \sqrt{yz/c})$ resulting in the following expression:

$$
\begin{aligned}
S = {} & \frac{(q,a,b,c/z,xz,q/xz;q)_\infty}{(c,z,a/z,b/z,x,q/x;q)_\infty}\, {}_4\varphi_3\left(\begin{matrix} d,e,z,qz/c \\ f,qz/b,qz/a \end{matrix};q,q\right) \\
& + \frac{(q,d,b,c/a,af/z,e,ax,q/ax,yz/a,qa/yz;q)_\infty}{(c,f,b/a,z/a,ad/z,ae/z,x,q/x,y,q/y;q)_\infty} \\
& \times {}_4\varphi_3\left(\begin{matrix} a,qa/c,ad/z,ae/z \\ qa/b,qa/z,af/z \end{matrix};q,q\right)
\end{aligned}
\tag{3.13}
$$

$$+\frac{(q,d,a,c/b,bf/z,e,bx,q/bx,yz/b,qb/yz;q)_\infty}{(c,f,a/b,z/b,bd/z,be/z,x,q/x,y,q/y;q)_\infty}$$

$$\times {}_4\varphi_3\left(\begin{matrix} b,qb/c,bd/z,be/z \\ qb/a,qb/z,bf/z\end{matrix};q,q\right),$$

valid on any subregion of $\max(|ad|,|ae|,|bd|,|be|) < |z| < 1$ in the complex z-plane as long as the righthand side is analytic in this subregion.

As observed the case $ey \in q^{\mathbf{Z}}$ is special, and we see later that this case corresponds to the infinite set of discrete mass points in the spectral measure $d\nu$ of the little q-Jacobi function transform. However, $ey = q^{1-l}$ with $l \to \infty$ violates the conditions for absolute convergence of S as given in Proposition 3.1, so we have to deal with this case separately. Using Heine's transformation [**4**, (III.1)], we see in this case,

(3.14)

$$\begin{aligned}{}_2\varphi_1\left(\begin{matrix} d,e \\ f\end{matrix};q,yq^n\right) &= \frac{(d,eyq^n;q)_\infty}{(f,yq^n;q)_\infty}{}_2\varphi_1\left(\begin{matrix} f/d,yq^n \\ eyq^n\end{matrix};q,d\right) \\ &= \frac{(d,q^{1-l+n};q)_\infty}{(f,yq^n;q)_\infty}{}_2\varphi_1\left(\begin{matrix} f/d,yq^n \\ q^{1-l+n}\end{matrix};q,d\right),\end{aligned}$$

initially for $|yq^n| < 1$ and by analytic continuation in y to the general case for $n \in \mathbf{Z}$ and $y \in \mathbf{C}\setminus\mathbf{R}_{>0}$. Note that the righthand side of (3.14) displays q-Bessel coefficient behavior, which means that for $l > n$ the series on the righthand side of (3.14) starts at $l-n$, see the proof of Proposition 3.5.

We now prove the necessary result in some greater generality.

Proposition 3.5. *Let* $\max(|ya|,|yb|) < |z| < 1$, $|y| < 1$, $x \in \mathbf{C}\setminus\mathbf{R}_{>0}$

and $|qcd| < |xab|$, *then*

$$\sum_{k=-\infty}^{\infty} z^k {}_2\varphi_1\left(\begin{matrix} a, b \\ c \end{matrix} ; q, xq^k\right) \frac{(q^{k+1};q)_\infty}{(dq^k;q)_\infty} {}_2\varphi_1\left(\begin{matrix} dq^k, e \\ q^{k+1} \end{matrix} ; q, y\right)$$

$$= \frac{(q, b, c/a, dz/a, ax, q/ax, eya/z; q)_\infty}{(d, c, b/a, z/a, x, q/x, ya/z; q)_\infty}$$

$$\times {}_4\varphi_3\left(\begin{matrix} a, aq/c, aq/dz, ya/z \\ aq/b, aq/z, eya/z \end{matrix} ; q, \tfrac{qcd}{xab}\right)$$

$$+ \frac{(q, a, c/b, dz/b, bx, q/bx, eyb/z; q)_\infty}{(d, c, a/b, z/b, x, q/x, yb/z; q)_\infty}$$

$$\times {}_4\varphi_3\left(\begin{matrix} b, bq/c, bq/dz, yb/z \\ bq/a, bq/z, eyb/z \end{matrix} ; q, \tfrac{qcd}{xab}\right)$$

$$+ \frac{(q, a, b, c/z, zx, q/zx, ey; q)_\infty}{(z, c, a/z, b/z, x, q/x, y; q)_\infty} {}_4\varphi_3\left(\begin{matrix} z, zq/c, q/d, y \\ zq/a, zq/b, ey \end{matrix} ; q, \tfrac{qcd}{xab}\right).$$

Remark. The conditions for Proposition 3.5 are less severe than in Proposition 3.1. However, if we want the three ${}_4\varphi_3$-series to be balanced, we need the extra conditions $cd = xab$ and $abq = cde$.

Proof. As $k \to \infty$, the summand is $\mathcal{O}(1)$, so we need $|z| < 1$ for absolute convergence. As $k \to -\infty$ the first ${}_2\varphi_1$-series behaves like $C_1 a^{-k} + C_2 b^{-k}$ by (3.1) for $k = 1$ and for the second ${}_2\varphi_1$-series we use its q-Bessel coefficient behavior; for $k < 0$,

$$\frac{(q^{k+1};q)_\infty}{(dq^k;q)_\infty} {}_2\varphi_1\left(\begin{matrix} dq^k, e \\ q^{k+1} \end{matrix} ; q, y\right)$$

$$= \sum_{j=-k}^{\infty} \frac{(q^{1+k+j};q)_\infty (e;q)_j}{(dq^{k+j};q)_\infty (q;q)_j} y^j = \sum_{p=0}^{\infty} \frac{(q^{1+p};q)_\infty (e;q)_{p-k}}{(dq^p;q)_\infty (q;q)_{p-k}} y^{p-k}$$

$$= \frac{(q;q)_\infty (e;q)_{-k}}{(d;q)_\infty (q;q)_{-k}} y^{-k} {}_2\varphi_1\left(\begin{matrix} d, eq^{-k} \\ q^{1-k} \end{matrix} ; q, y\right),$$

which is $\mathcal{O}(y^{-k})$ as $k \to -\infty$. So the sum is absolutely convergent as $k \to -\infty$ for $|z| > |ya|$ and $|z| > |yb|$.

Let T be the sum; then interchanging summation over k and the summation for the series representation of the q-Bessel coefficient gives

$$T = \sum_{j=0}^{\infty} \left(\sum_{k=-j}^{\infty} z^k {}_2\varphi_1 \left(\begin{matrix} a, b \\ \\ c \end{matrix} ; q, xq^k \right) \frac{(q^{1+k+j};q)_\infty}{(dq^{k+j};q)_\infty} \right) \frac{(e;q)_j}{(q;q)_j} y^j.$$

Consider the inner sum, say T_j, for j fixed. Shifting the summation parameter and using the series representation for the ${}_2\varphi_1$-series gives

$$\begin{aligned} T_j &= \sum_{l=0}^{\infty} z^{-j} \frac{(a,b;q)_l}{(c,q;q)_l} x^l q^{-jl} \sum_{p=0}^{\infty} \frac{(q^{1+p};q)_\infty}{(dq^p;q)_\infty} z^p q^{pl} \\ &= z^{-j} \frac{(q,dz;q)_\infty}{(d,z;q)_\infty} {}_3\varphi_2 \left(\begin{matrix} a, b, z \\ \\ c, dz \end{matrix} ; q, xq^{-j} \right) \end{aligned}$$

using the q-binomial theorem. This is only valid for $|xq^{-j}| < 1$, but since the sum is uniformly convergent on compacta, T_j is analytic in $x \in \mathbf{C} \setminus \mathbf{R}_{>0}$, so the result remains valid using the unique analytic continuation of the ${}_3\varphi_2$-series to $\mathbf{C} \setminus \mathbf{R}_{>0}$.

So this gives

$$T = \frac{(g, dz/q)_\infty}{(d,z;q)_\infty} \sum_{j=0}^{\infty} \frac{(e;q)_j}{(q;q)_j} \left(\frac{y}{z}\right)^j {}_3\varphi_2 \left(\begin{matrix} a, b, z \\ \\ c, dz \end{matrix} ; q, xq^{-j} \right), \tag{3.15}$$

where we assume that we use the unique analytic continuation for the

${}_3\varphi_2$-series. Since $x \in \mathbf{C} \setminus \mathbf{R}_{>0}$ and $|qcd| < |xab|$, we have for all $j \geq 0$

$$
\begin{aligned}
&{}_3\varphi_2\left(\begin{matrix} a, b, z \\ c, dz \end{matrix}; q, xq^{-j}\right) \\
&\quad = z^j \frac{(a, b, \frac{c}{z}, d, zx, \frac{q}{zx}; q)_\infty}{(c, dz, \frac{a}{z}, \frac{b}{z}, x, \frac{q}{x}; q)_\infty} {}_3\varphi_2\left(\begin{matrix} z, \frac{zq}{c}, \frac{q}{d} \\ \frac{zq}{a}, \frac{zq}{b} \end{matrix}; q, \frac{q^{1+j}cd}{xab}\right) \\
&\qquad + a^j \frac{(b, z, \frac{c}{a}, \frac{dz}{a}, ax, \frac{q}{ax}; q)_\infty}{(c, dz, \frac{b}{a}, \frac{z}{a}, x, \frac{q}{x}; q)_\infty} {}_3\varphi_2\left(\begin{matrix} a, \frac{aq}{c}, \frac{aq}{dz} \\ \frac{aq}{b}, \frac{aq}{z} \end{matrix}; q, \frac{q^{1+j}cd}{xab}\right) \\
&\qquad + b^j \frac{(a, z, \frac{c}{b}, \frac{dz}{b}, bx, \frac{q}{bx}; q)_\infty}{(c, dz, \frac{a}{b}, \frac{z}{b}, x, \frac{q}{x}; q)_\infty} {}_3\varphi_2\left(\begin{matrix} b, \frac{bq}{c}, \frac{bq}{dz} \\ \frac{bq}{a}, \frac{bq}{z} \end{matrix}; q, \frac{q^{1+j}cd}{xab}\right),
\end{aligned}
$$

using (3.1) and the theta product identity (3.2). Using this in (3.15) and the q-binomial theorem three times gives the result. $\square$

Corollary 3.6. *Let S be the sum in Proposition* 3.1 *and assume $ey = q^{1-l}$ for $l \in \mathbf{Z}$. Then the series is absolutely convergent for $|da|, |db| < |z| < 1$. Assuming, moreover, $abde = cf$ and $fx = dey$, we have that S equals the sum of three balanced ${}_4\varphi_3$-series*

$$
\begin{aligned}
S &= \frac{(q, a, b, \frac{c}{z}, zx, \frac{q}{zx}; q)_\infty}{(z, c, \frac{a}{z}, \frac{b}{z}, x, \frac{q}{x}; q)_\infty} {}_4\varphi_3\left(\begin{matrix} z, \frac{zq}{c}, \frac{q^{1-l}}{y}, d \\ \frac{zq}{a}, \frac{zq}{b}, f \end{matrix}; q, q\right) \\
&\quad + \frac{(d, q, b, \frac{c}{a}, \frac{zyq^l}{a}, ax, \frac{q}{ax}, \frac{fa}{z}; q)_\infty}{(f, yq^l, c, \frac{b}{a}, \frac{z}{a}, x, \frac{q}{x}, \frac{da}{z}; q)_\infty} \left(\frac{z}{a}\right)^l \\
&\quad \times {}_4\varphi_3\left(\begin{matrix} a, \frac{aq}{c}, \frac{aq^{1-l}}{yz}, \frac{ad}{z} \\ \frac{aq}{b}, \frac{aq}{z}, \frac{af}{z} \end{matrix}; q, q\right) \\
&\quad + \frac{(d, q, a, \frac{c}{b}, \frac{zyq^l}{b}, bx, \frac{q}{bx}, \frac{fb}{z}; q)_\infty}{(f, yq^l, c, \frac{a}{b}, \frac{z}{b}, x, \frac{q}{x}, \frac{db}{z}; q)_\infty} \left(\frac{z}{b}\right)^l \\
&\quad \times {}_4\varphi_3\left(\begin{matrix} b, \frac{bq}{c}, \frac{bq^{1-l}}{yz}, \frac{bd}{z} \\ \frac{bq}{a}, \frac{bq}{z}, \frac{bf}{z} \end{matrix}; q, q\right).
\end{aligned}
$$

Remark. Note that putting $e = q^{1-l}/y$ in the expression (3.13) for S gives the same result using (3.2). Hence, the expressions for S derived in Proposition 3.1, (3.11), (3.12) and (3.13), remain valid in the case $ey \in q^{\mathbf{Z}}$.

Proof. Use (3.14) in S as in Proposition 3.1, shift the summation parameter $k = n - l$, and apply Proposition 3.5 with $x \mapsto xq^l$, $e \mapsto f/d$, $d \mapsto yq^l$, $y \mapsto d$ and use the theta product identity (3.2) to find an explicit expression for S as a sum of three ${}_4\varphi_3$-series for $|da|, |db| < |z| < 1$, $|d| < 1$, $x \in \mathbf{C} \setminus \mathbf{R}_{>0}$, $|qcy| < |xab|$. Under the conditions $abde = cf$ and $fx = dey$ we see that these ${}_4\varphi_3$-series are the balanced ${}_4\varphi_3$-series as stated. The condition $|d| < 1$ can be removed by analytic continuation in d. □

Proof of Theorem 2.1. We start off with the general dual transmutation kernel as in (2.1) assuming $|\sigma|, |\tau| \geq 1$. This sum is of the type as considered in Proposition 3.1, and the conditions in Proposition 3.1 lead to $ad = bc$ and $adx/c = by$ or $ad = bc$ and $x = y$. We put $c = aq^\alpha$ and $d = bq^\alpha$, and we apply Proposition 3.1 with $a \mapsto aq^\alpha/\tau$, $b \mapsto aq^\alpha\tau$, $c \mapsto abq^{2\alpha}$, $d \mapsto b/\sigma$, $e \mapsto b\sigma$, $f \mapsto ab$, $x \mapsto -bx/a$, $y \mapsto -y$ and $z \mapsto abt$. This gives the required expression for the dual transmutation kernel under the conditions $|abq^\alpha\sigma\tau| < |abt| < 1$ and $q < ab|t|$, $t \notin a^2q^{2\alpha+\mathbf{Z}_{\geq 0}}$, $t \notin b^2q^{2\alpha+\mathbf{Z}_{\geq 0}}$.

The previous paragraph deals with the case $\lambda \in [-1, 1]$ or $\sigma = e^{i\theta}$. For the discrete mass points, we first consider $\sigma_l = -q^{1-l}$ by $l \in \mathbf{Z}$ such that $|\sigma_l| > 1$. Corollary 3.6 then applies, since $b < 1$, $|\sigma_l| > 1$, $q^\alpha ab < |abt| < 1$. From the remark following Corollary 3.6 we see that the expression for the dual transmutation kernel remains valid for this set of discrete mass points of $d\nu(\cdot; a, b; y; q)$. The (possibly empty) set of discrete mass points of the form $\sigma_l = aq^l$, $l \in \mathbf{Z}_{\geq 0}$ such that $|\sigma_l| > 1$ make one of the ${}_2\varphi_1$-series terminating after using the symmetry described in Remark 3.2, and the result for the dual transmutation kernel remains valid. This proves the first statement of Theorem 2.1.

We now investigate when $\{t^k \phi_\mu(xq^k; c, d; q)\}_{k\in\mathbf{Z}} \in \mathcal{H}(a, b; y)$. From (1.2) and (1.5) we see that we need the condition $|t^2ab| < 1$ for convergence as $k \to \infty$. For $k \to -\infty$, we use the theta product identity (3.2) to find

(3.16)
$$(ab)^k \frac{(-byq^k/a;q)_\infty}{(-yq^k;q)_\infty} = a^{2k} \frac{(-q^{1-k}/y;q)_\infty}{(-q^{1-k}a/by;q)_\infty} \frac{(-by/a,-aq/by;q)_\infty}{(-y,-q/y;q)_\infty}$$

which is $\mathcal{O}(a^{2k})$ as $k \to -\infty$. The asymptotic behavior of the little q-Jacobi function as $|x| \to \infty$ on a q-grid follows from the expansion (1.10); for $k \to -\infty$,

$$(3.17)\quad t^k \varphi_\mu(xq^k;c,d;q) = \begin{cases} \mathcal{O}((t/c)^k) & |\tau| = 1, \\ \mathcal{O}((t/c\tau)^k) & |\tau| > 1,\ c(\tau;c,d,x;q) \neq 0, \\ \mathcal{O}((t\tau/c)^k) & |\tau| > 1,\ c(\tau;c,d,x;q) = 0. \end{cases}$$

This implies that for generic μ we need $|t^2\tau^{-2}c^{-2}a^2| > 1$ for convergence of (1.5) as $k \to -\infty$. We conclude

$$(3.18)\qquad \left|\frac{c\tau}{a}\right| < |t| < \frac{1}{\sqrt{|ab|}} \Longrightarrow \{t^k \phi_\mu(xq^k;c,d;q)\}_{k\in\mathbf{Z}} \in \mathcal{H}(a,b;y).$$

Under the assumption of (3.18), the L^2-theory for the little q-Jacobi function transform implies that the sum in (2.1) converges in $L^2(d\nu(\cdot;a,b;y;q))$. In the special case $c = aq^\alpha$, $d = bq^\alpha$, $x = -bx/a$, the conditions (3.18) are implied by $abq^\alpha|\tau| < ab|t| < 1$, since $0 < ab < 1$. Hence, $P_t(\cdot,\mu;q^\alpha;a,b) \in L^2(d\nu(\cdot;a,b;y;q))$ for $q^\alpha|\tau| < |t| < 1/\sqrt{ab}$.

Assuming $|q^\alpha\tau| < |t| < 1/\sqrt{ab}$, we find from (1.8)

$$\begin{aligned} &t^k \phi_\mu(yq^k;aq^\alpha,bq^\alpha;q) \\ &\qquad = \int_{\mathbf{R}} \phi_\lambda(yq^k;a,b;q) P_t(\lambda,\mu;q^\alpha;a,b)\, d\nu(\lambda;a,b;y;q). \end{aligned}$$

Taking linear combinations shows that, for $v = \{v_k\}_{k\in\mathbf{Z}}$ with only finitely many nonzero coefficients, we have

$$\begin{aligned} (3.19)\quad &\mathcal{F}_{aq^\alpha,bq^\alpha,y}[\delta_t v](\mu) \\ &\qquad = \int_{\mathbf{R}} \mathcal{F}_{a,b,y}[k \mapsto q^{2\alpha k} v_k](\lambda) P_t(\lambda,\mu;q^\alpha;a,b)\, d\nu(\lambda;a,b;y;q), \end{aligned}$$

using the notation as in (1.9). Again, using the L^2-theory of (1.8), (3.19) remains valid for $\{q^{2k\alpha}v_k\}_k \in \mathcal{H}(a,b;y)$, since this makes the integrand integrable with respect to $d\nu(\cdot;a,b;y;q)$. Now we take $v_k = s^k t^{-k}\phi_\nu(yq^k;aq^{\alpha+\beta},bq^{\alpha+\beta};q)$, which gives the product formula. This is valid for $|q^{\alpha+\beta}\rho|<|q^{2\alpha}s/t|<1/\sqrt{ab}$ by (3.18) where $\nu=(\rho+\rho^{-1})/2$ with $|\rho|\geq 1$. Note that these two conditions on s and t imply $|abq^{2\alpha+\beta}\rho\tau|<|abq^{2\alpha}s|<1$, which are precisely the conditions for the absolute convergence of the lefthand side of (3.19), cf. (2.1). □

4. The transmutation kernel. In this section we prove Theorem 2.2. The results in this section give another point of view to Gasper's results [**3**] on q-analogues of Erdélyi's fractional integrals, see also Section 5.

Rewriting Gasper's q-analogue [**3**, (1.8)] of Erdélyi's fractional integral gives

(4.1)
$$\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} ar\sigma, ar/\sigma \\ abrs \end{matrix};q,-\frac{byq^l}{ar}\right) \\
&\quad= \frac{(ab,rs;q)_\infty}{(q,abrs;q)_\infty}\sum_{k=0}^\infty (ab)^k\,\frac{(q^{k+1},-byq^{k+l}/a;q)_\infty}{(rsq^k,-byq^{k+l}/ar;q)_\infty} \\
&\qquad\times {}_3\varphi_2\left(\begin{matrix} q^{-k},r,ar/b \\ rs,-arq^{1-l-k}/by \end{matrix};q,q\right){}_2\varphi_1\left(\begin{matrix} a\sigma,a/\sigma \\ ab \end{matrix};q,-\frac{byq^{l+k}}{a}\right),
\end{aligned}$$

for $|rs|<1$, $|ab|<1$. In Section 5 (see Theorem 2.3(ii)), we give an alternative derivation of (4.1) using intertwining operators for the second order q-difference operator L as in (1.1). Indeed, as remarked in Section 2, (2.2) is equivalent to (4.1). We can also prove (4.1) using the connection coefficient formula

(4.2)
$$(b\sigma,b/\sigma;q)_m=\left(ab,\frac{b}{a};q\right)_m\sum_{k=0}^m\frac{(q^{-m};q)_k q^k}{(q,ab,q^{1-m}a/b;q)_k}(a\sigma,a/\sigma;q)_k,$$

which is the q-Saalschütz summation formula [**4**, (II.12)]. So (4.1) links two little q-Jacobi functions with different parameter sets but with related argument. Without this condition on the arguments, we have not been able to find a simple expression for the kernel, even if we allow the summation parameter to run over $\mathbf{Z}$ instead of $\mathbf{Z}_{\geq 0}$.

Note that (4.1) may be viewed as a connection coefficient problem for two sets of orthogonal functions. Having in mind the limit transition of Askey-Wilson polynomials to little q-Jacobi functions (see [**9**, Sections 2.3, 4.1, 6.1]), we see that (4.1) can be viewed as the limit case of the connection coefficient problem for Askey-Wilson polynomials (see [**2**, Section 6]).

In case $s = 1$ the ${}_3\varphi_2$-series in (4.1) reduces to a terminating ${}_2\varphi_1$-series that can be summed by the q-Chu-Vandermonde summation [**4**, (II.6)] yielding

(4.3)

$$\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} ar\sigma, ar/\sigma \\ abr \end{matrix}; q, -\tfrac{byq^l}{ar}\right) \\
&\quad = \frac{(ab, r, -yq^l; q)_\infty}{(q, abr, -byq^l/ar; q)_\infty} \sum_{k=0}^{\infty} (ab)^k \frac{(q^{k+1}, -byq^{k+l}/a; q)_\infty}{(rq^k, -yq^{k+l}; q)_\infty} \\
&\qquad\qquad \times {}_2\varphi_1\left(\begin{matrix} a\sigma, a/\sigma \\ ab \end{matrix}; q, -\tfrac{byq^{l+k}}{a}\right),
\end{aligned}$$

which is equivalent, by [**4**, (1.4.6)], to

(4.4)

$$\begin{aligned}
&{}_2\varphi_1\left(\begin{matrix} b\sigma, \frac{b}{\sigma} \\ abr \end{matrix}; q, -yq^l\right) \\
&\quad = \frac{(ab, r; q)_\infty}{(q, abr; q)_\infty} \sum_{k=0}^{\infty} (ab)^k \frac{(q^{k+1}; q)_\infty}{(rq^k; q)_\infty} \\
&\qquad\qquad \times {}_2\varphi_1\left(\begin{matrix} b\sigma, \frac{b}{\sigma} \\ ab \end{matrix}; q, -yq^{l+k},\right),
\end{aligned}$$

valid for $y \in \mathbf{C} \setminus \mathbf{R}_{>0}$. This can be proved easily using the q-binomial theorem. See also (5.7) and (5.13) for similar results. Another interesting case of (4.1) is $r = 1$, which reduces the ${}_3\varphi_2$-series to 1. This gives (4.4) after renaming. In Section 5 (see Theorem 2.3) we show that (4.1) can be derived from (4.4).

Proof of Theorem 2.2. Fix y and define $u = \{u_k\}_{k\in\mathbf{Z}}$ by

$$u_k = (ab)^{-l}\frac{(ab,rs;q)_\infty}{(q,abrs;q)_\infty}\frac{(q^{k-l+1},-yq^k;q)_\infty}{(rsq^{k-l},-byq^k/ar;q)_\infty} \times {}_3\varphi_2\left(\begin{matrix} q^{l-k},r,ar/b \\ rs,-arq^{1-k}/by\end{matrix};q,q\right)$$

with the convention $u_k = 0$ for $k < l$. Using [**4**, (III.12)] the ${}_3\varphi_2$-series can be written as

$${}_3\varphi_2\left(\begin{matrix} q^{l-k},r,ar/b \\ rs,-arq^{1-k}/by\end{matrix};q,q\right) = \frac{(-yq^l;q)_{k-l}}{(-byq^l/ar;q)_{k-l}}\,{}_3\varphi_2\left(\begin{matrix} q^{l-k},ar/b,s \\ rs,-yq^l\end{matrix};q,-\frac{byq^k}{a}\right),$$

so that $u_k = \mathcal{O}(1)$ as $k\to\infty$. It follows that $u\in\mathcal{H}(a,b;y)$. Then (4.1) states that the little q-Jacobi transform of u (see (1.8)) is

$$(\mathcal{F}_{a,b,y}u)(\lambda) = \phi_\lambda(yq^l/s;ar,bs;q) = {}_2\varphi_1\left(\begin{matrix} ar\sigma,ar/\sigma \\ abrs\end{matrix};q,-\frac{byq^l}{ar}\right),$$
$$\lambda = \frac{1}{2}(\sigma+\sigma^{-1}).$$

By the inversion formula of (1.8), we find the result for the transmutation kernel as in Theorem 2.2.

Note that we can write

$$\sum_{l=-\infty}^{\infty} u_l(abrs)^l\frac{(-byq^l/ar;q)_\infty}{(-yq^l/s;q)_\infty}P_{k,l}(a,b,y;r,s) = (\mathcal{F}^{-1}_{a,b,y}\mathcal{F}_{ar,bs,y/s}u)_k \tag{4.5}$$

valid for $u = \{u_k\}_k$ having only finitely many nonzero coefficients. In particular, choosing $u_l = P_{l,p}(ar, bs, y/s; t, u)$ for $l \le k$ and $u_l = 0$ for $l > k$ under the assumptions $u, t > 0$, $|ut| < 1$, we find from (4.5)

$$
\begin{aligned}
(4.6) \quad & \sum_{l=-\infty}^{\infty} P_{k,l}(a,b,y;r,s) P_{l,p}(ar,bs,y/s;t,u)(abrs)^l \frac{(-byq^l/ar;q)_\infty}{(-yq^l/s;q)_\infty} \\
&= \left(\mathcal{F}_{a,b,y}^{-1}\mathcal{F}_{ar,bs,y/s}\left[l \mapsto \left(\mathcal{F}_{ar,bs,y/s}^{-1}\left[\lambda \mapsto \phi_\lambda\left(\frac{yq^p}{su}; art, bsu; q\right)\right]_l\right)\right]_k\right) \\
&= \left(\mathcal{F}_{a,b,y}^{-1}\left[\lambda \mapsto \phi_\lambda\left(\frac{yq^p}{su}; art, bsu; q\right)\right]\right)_k \\
&= P_{k,p}(a,b,y;rt,su),
\end{aligned}
$$

which is the product formula. $\square$

If we plug the explicit expression for the transmutation kernel into the product formula, we obtain, after relabeling, the following expression. For $k \in \mathbf{Z}$, $p \in \mathbf{Z}_{\ge 0}$, $r, s, t, u > 0$ with $rs < 1$, $tu < 1$, we have

$$
\begin{aligned}
(4.7) \quad & \sum_{l=0}^{p} \frac{(tu;q)_{p-l}}{(q;q)_{p-l}} {}_3\varphi_2\left(\begin{matrix} q^{l-p}, t, art/bs \\ tu, -artq^{1+l-k}/by \end{matrix}; q, q\right) \frac{(rs;q)_l}{(q;q)_l} \\
& \times {}_3\varphi_2\left(\begin{matrix} q^{-l}, r, ar/b \\ rs, -arq^{1-k}/by \end{matrix}; q, q\right) (rs)^{p-l} t^l \frac{(-arq^{1-k}/by;q)_l}{(-artq^{1-k}/by;q)_l} \\
& \qquad = \frac{(rstu;q)_p}{(q;q)_p} {}_3\varphi_2\left(\begin{matrix} q^{-p}, rt, art/b \\ rstu, -artq^{1-k}/by \end{matrix}; q, q\right).
\end{aligned}
$$

This product formula is equivalent to Gasper [**3**, (1.7)] and, as remarked in [**3**], (4.7) implies (4.1) in the limit $p \to \infty$. Here we have shown that the converse is also valid, (4.1) implies (4.7) using the little q-Jacobi function transform.

5. Intertwining properties. In this section we prove Theorem 2.3, and as a motivation we start by giving a Darboux factorization of the second order q-difference operator $L^{(a,b)}$ or $\mathcal{L}^{(a,b)}$.

The backward q-derivative operator is $B_q = M_{1/x}(1-T_q^{-1})$, where M_g is the operator of multiplication by g; $(M_g f)(x) = g(x)f(x)$, and $T_q f(x) = f(qx)$ as introduced in Section 1. It is straightforward to check that

$$(5.1)\qquad (B_q\phi_\lambda(\cdot;a,b;q))(x) = \frac{b(1-a\sigma)(1-a/\sigma)}{qa(1-ab)}\phi_\lambda(x;aq,b;q).$$

Considering $\mathcal{H}(a,b;y)$ as an L^2-space with discrete weights $(ab)^k \times (-byq^k/a;q)_\infty/(-yq^k;q)_\infty$ at the point yq^k, $k\in\mathbf{Z}$, we look at B_q as (densely defined unbounded) operator from $\mathcal{H}(a,b;y)$ to $\mathcal{H}(aq,b;y)$. Its adjoint, up to a constant depending only on y, is given by

$$(5.2)\qquad A(a,b) = M_{1+bx/aq} - abM_{1+x}T_q,$$

and it's a straightforward calculation to show that

$$(5.3)\qquad (A(a,b)\phi_\lambda(\cdot;aq,b;q))(x) = (1-ab)\phi_\lambda(x;a,b;q)$$

and that $-bL^{(a,b)} = aqA(a,b)\circ B_q$, with the notation as in (1.1). Since B_q and $A(a,b)$ are triangular with respect to the standard orthogonal basis of Dirac deltas at yq^k of $\mathcal{H}(a,b;y)$, this means that we have a Darboux factorization of $L^{(a,b)}$. Also, $-b(L^{(aq,b)} + (1-q)(1-qa^2)) = aq^2B_q\circ A(a,b)$, from which we deduce $B_q\circ L^{(a,b)} = L^{(aq,b)}\circ B_q$ and $L^{(a,b)}\circ A(a,b) = A(a,b)\circ L^{(aq,b)}$. It is the purpose of this section to generalize these intertwining properties to arbitrary complex powers of B_q.

Introduce the operator W_ν, $\nu\in\mathbf{C}$, acting on functions defined on $[0,\infty)$ by

$$(5.4)\qquad (W_\nu f)(x) = x^\nu\sum_{l=0}^\infty f(xq^{-l})q^{-l\nu}\frac{(q^\nu;q)_l}{(q;q)_l},\quad x\in[0,\infty),$$

assuming that the infinite sum is absolutely convergent if $\nu\notin -\mathbf{Z}_{\geq 0}$. So we want f sufficiently decreasing on a q-grid tending to infinity, e.g., $f(xq^{-l}) = \mathcal{O}(q^{l(\nu+\varepsilon)})$ for some $\varepsilon>0$. Note that, for $\nu\in\mathbf{Z}_{\leq 0}$, the sum in (5.4) is finite and $W_0 = \mathrm{Id}$ and $W_{-1} = B_q$.

This operator is a q-analogue of the Weyl fractional integral operator as used in [**12**, Section 3], [**13**, Section 5.3], for the Abel transform. With the notation

$$\int_a^\infty f(t)\,d_qt = a\sum_{k=0}^\infty f(xq^{-k})q^{-k}$$

for the q-integral, cf. (1.6), we see that for $n \in \mathbf{N}$ the operator W_n is an iterated q-integral

$$(W_n f)(x) = \int_x^\infty \int_{x_1}^\infty \cdots \int_{x_{n-1}}^\infty f(x_n) d_q x_n d_q x_{n-1} \dots d_q x_1. \tag{5.5}$$

In the following lemma we collect some results on W_ν, where we use the function space

$$\mathcal{F}_\rho = \{ f : [0,\infty) \to \mathbf{C} \,|\, |f(xq^{-l})| = \mathcal{O}(q^{l\rho}),\ l \to \infty,\ \forall x \in (q,1] \}, \quad \rho > 0. \tag{5.6}$$

Recall that $\mathcal{L}^{(a,b)}$ is defined in (1.4).

Lemma 5.1. *Let $\nu, \mu \in \mathbf{C} \setminus \mathbf{Z}_{\leq 0}$.*

(i) *W_ν preserves the space of compactly supported functions,*

(ii) *$W_\nu : \mathcal{F}_\rho \to \mathcal{F}_{\rho - \Re\nu}$ for $\rho > \Re\nu > 0$,*

(iii) *$W_\nu \circ W_\mu = W_{\nu+\mu}$ on $\mathcal{F}_\rho$ for $\rho > \Re(\mu+\nu) > 0$,*

(iv) *$W_\nu \circ B_q = B_q \circ W_\nu = W_{\nu-1}$ on $\mathcal{F}_\rho$ for $\rho > \Re\nu - 1 > 0$, and $B_q^n \circ W_n = \mathrm{id}$ for $n \in \mathbf{N}$ on $\mathcal{F}_\rho$ for $\rho > n$,*

(v) *$\mathcal{L}^{(aq^{-\nu},b)} \circ W_\nu = W_\nu \circ \mathcal{L}^{(a,b)}$, valid for compactly supported functions.*

Remark. It follows from (iii) that $W_{-n} = B_q^n$, $n \in \mathbf{N}$ and $W_0 = \mathrm{id}$.

Proof. The first statement is immediate from (5.4). For (ii) we use that, for $f \in \mathcal{F}_\rho$, and $x \in (q,1]$, we have

$$\begin{aligned} |W_\nu f(xq^{-k})| &\leq M \sum_{l=0}^\infty q^{(k+l)\rho} q^{-(k+l)\Re\nu} \frac{(q^{\Re\nu};q)_l}{(q;q)_l} \\ &= M q^{k(\rho-\Re\nu)} \frac{(q^\rho;q)_\infty}{(q^{\rho-\Re\nu};q)_\infty} \end{aligned}$$

by the q-binomial theorem for $\rho > \Re\nu$. The third statement is a consequence of interchanging summations, valid for $f \in \mathcal{F}_\rho$, $\rho >$

$\Re(\mu+\nu)$, and

$$\sum_{k+l=p} \frac{(q^\mu;q)_k(q^\nu;q)_l}{(q;q)_k(q;q)_l} q^{-(l+k)\mu-l\nu} = q^{-p(\mu+\nu)} \frac{(q^{\mu+\nu};q)_p}{(q;q)_p},$$

which is the q-Chu-Vandermonde summation formula [**4**, (1.5.2)]. For (iv), we note that $B_q : \mathcal{F}_\rho \to \mathcal{F}_{\rho+1}$. Then the first statement of (iv) is a simple calculation involving q-shifted factorials, which reduces the second statement of (iv) to verifying the easy case $n = 1$. For (v) recall (1.4), so that $\mathcal{L}^{(aq^{-\nu},b)}(W_\nu f)(x)$ and $W_\nu(\mathcal{L}^{(a,b)}f)(x)$ involve the values $f(xq^{-k})$, $k+1 \in \mathbf{Z}_{\geq 0}$. A straightforward calculation using q-shifted factorials shows that the coefficients of $f(xq^{-k})$ in $\mathcal{L}^{(aq^{-\nu},b)}(W_\nu f)(x)$ and $W_\nu(\mathcal{L}^{(a,b)}f)(x)$ are equal. $\square$

The asymptotically free solution $\Phi_\sigma(yq^k;a,b;q) \in \mathcal{F}_\rho$ for $q^\rho > |a\sigma|$ as follows from (1.10). A calculation using the q-binomial formula gives, cf. (4.4),

$$(W_\nu \Phi_\sigma(\cdot;a,b;q))(yq^k) = y^\nu \frac{(a\sigma;q)_\infty}{(aq^{-\nu}\sigma;q)_\infty} \Phi_\sigma(yq^k;aq^{-\nu},b;q), \tag{5.7}$$

for $|a\sigma| < q^\nu$ in accordance with Lemma 5.1 (v). Note that (5.7) is a q-analogue of Bateman's formula, cf. [**3, 12**].

Lemma 5.2. *Define the operator*

$$S(a,b) = M_{(-x;q)_\infty/(-bx/a;q)_\infty} \circ T_{b/a}, \quad T_{b/a}f(x) = f\left(\frac{b}{a}x\right).$$

Then $S(a,b)^{-1} \circ \mathcal{L}^{(a,b)} \circ S(a,b) = \mathcal{L}^{(b,a)}$. *In particular,* $\tilde{W}_\nu^{(a,b)} = S(a,bq^{-\nu}) \circ W_\nu \circ S(a,b)^{-1}$ *satisfies the intertwining property* $\mathcal{L}^{(a,bq^{-\nu})} \circ \tilde{W}_\nu^{(a,b)} = \tilde{W}_\nu^{(a,b)} \circ \mathcal{L}^{(a,b)}$.

Note that $S(a,b)^{-1} = S(b,a)$ and that $S(a,b) : \mathcal{H}(b,a;yb/a) \to \mathcal{H}(a,b;y)$ is an isometric isomorphism. For $f \in \mathcal{F}_\rho$ we see that $(S(a,b)f)(xq^{-l}) = \mathcal{O}(|a/b|^l q^{l\rho})$ from (3.2), so that $S(a,b)f \in \mathcal{F}_{\rho+\ln(|a/b|)/\ln q}$.

Proof. It follows from (1.4) that

$$\mathcal{L}^{(a,b)}\left(x \mapsto \frac{(-x;q)_\infty}{(-bx/a;q)_\infty} f(x)\right)(x)$$
$$= \frac{(-x;q)_\infty}{(-bx/a;q)_\infty}\left(\frac{b}{2}\left(1+\frac{a}{bx}\right)f(qx) + \frac{1}{2b}\left(1+\frac{q}{x}\right)f(xq^{-1}) - \frac{1}{2}\left(\frac{a}{x}+\frac{q}{bx}\right)f(x)\right),$$

and the term in parentheses can be written as $T_{b/a}\circ\mathcal{L}^{(b,a)}\circ T_{a/b}$ applied to f. The second statement then follows from Lemma 5.1 (v). ◻

It follows directly from (1.2), (1.10) and [**4**, (1.4.6)] that

$$\begin{aligned}(S(a,b)\phi_\lambda(\cdot;b,a;q))(x) &= \phi_\lambda(x;a,b;q),\\ (S(a,b)\Phi_\sigma(\cdot;b,a;q))(x) &= \Phi_\sigma(x;a,b;q).\end{aligned} \tag{5.8}$$

Proof of the first statement of Theorem 2.3. It follows from Lemma 5.1 (v) and Lemma 5.2 that the operator

$$\begin{aligned}W_{\nu,\mu}(a,b) &= \tilde{W}_\mu^{(aq^{-\nu},b)}\circ W_\nu\\ &= S(aq^{-\nu},bq^{-\mu})\circ W_\mu\circ S(b,aq^{-\nu})\circ W_\nu\end{aligned}$$

satisfies the required intertwining property. For $f\in\mathcal{F}_\rho$ with $\rho>\Re\nu$ we can interchange summations, which leads to the sum with a terminating ${}_3\varphi_2$ as kernel. Note that the ${}_3\varphi_2$-series in the kernel of $W_{\nu,\mu}(a,b)$ behaves as

$${}_2\varphi_1\left(\begin{matrix}q^{-\mu},-q^{1+\mu-\nu}a/bx\\ -q^{1+\mu}/x\end{matrix};q,q^{\nu-\mu}\frac{b}{a}\right)$$

as $p\to\infty$.

The statement for the action on $\Phi_\sigma(\cdot;a,b;q)$ follows immediately from (5.7) and (5.8). ◻

In order to prove the remaining half of Theorem 2.3, we take appropriate adjoints of the previous construction. Consider W_ν, $\nu\in\mathbf{C}\setminus\mathbf{Z}_{\le 0}$, as a densely defined unbounded operator from $\mathcal{H}(aq^\nu,b;y)$ to $\mathcal{H}(a,b;y)$ and define $R_\nu^{(a,b)}$ as its adjoint, so

$$\langle R_\nu^{(a,b)} f, g\rangle_{\mathcal{H}(aq^\nu,b;y)} = \langle f, W_\nu g\rangle_{\mathcal{H}(a,b;y)} \tag{5.9}$$

for all compactly supported functions g, cf. Lemma 5.1 (i). Here we use the identification of $\mathcal{H}(a,b;y)$ as a weighted L^2-space on a discrete set (see Section 1). A q-integration by parts shows

$$(R_\nu^{(a,b)} f)(yq^p) = y^\nu \frac{(-byq^p/a;q)_\infty}{(-byq^{p-\nu}/a;q)_\infty} \sum_{l=0}^\infty f(yq^{p+l})(ab)^l \frac{(q^\nu,-yq^p;q)_l}{(q,-byq^p/a;q)_l}. \tag{5.10}$$

Now define, for functions f, the operator

$$(A_\nu^{(a,b)} f)(x) = \frac{(-bx/a;q)_\infty}{(-bxq^{-\nu}/a;q)_\infty} \sum_{l=0}^\infty f(xq^l)(ab)^l \frac{(q^\nu,-x;q)_l}{(q,-bx/a;q)_l}, \tag{5.11}$$

so that $A_\nu^{(a,b)}|_{\mathcal{H}(a,b;y)} = y^{-\nu} R_\nu^{(a,b)}$. Note that $A_\nu^{(a,b)}$ is well-defined for bounded functions assuming $|ab|<1$. Recall that the dense domain of finite linear combinations of the basis vectors for $\mathcal{L}^{(a,b)}$ corresponds to the functions compactly supported in $(0,\infty)$.

Lemma 5.3. *$\mathcal{L}^{(aq^\nu,b)} \circ A_\nu^{(a,b)} = A_\nu^{(a,b)} \circ \mathcal{L}^{(a,b)}$ on the space of functions compactly supported in $(0,\infty)$. Moreover,*

$$(A_\nu^{(a,b)} \phi_\lambda(\cdot;a,b;q))(x) = \frac{(abq^\nu;q)_\infty}{(ab;q)_\infty} \phi_\lambda(x;aq^\nu,b;q).$$

Defining $\tilde{A}_\nu^{(a,b)} = S(a,bq^\nu) \circ A_\nu^{(b,a)} \circ S(b,a)$, we have $\mathcal{L}^{(a,bq^\nu)} \circ \tilde{A}_\nu^{(a,b)} = \tilde{A}_\nu^{(a,b)} \circ \mathcal{L}^{(a,b)}$, and

$$(\tilde{A}_\nu^{(a,b)} \phi_\lambda(\cdot;a,b;q))(x) = \frac{(abq^\nu;q)_\infty}{(ab;q)_\infty} \phi_\lambda(x;a,bq^\nu;q).$$

Proof. Note that (5.10) and (5.11) show that the operators $R_\nu^{(a,b)}$ and $A_\nu^{(a,b)}$ preserve the space of functions compactly supported in $(0,\infty)$.

The intertwining property for $R_\nu^{(a,b)}$ follows from (5.9) and Lemma 5.1 and hence for $A_\nu^{(a,b)}$.

To calculate the action of $A_\nu^{(a,b)}$ on the little q-Jacobi function, we use [**4**, (1.4.6)] to write

$$\phi_\lambda(x;a,b;q) = \frac{(-x;q)_\infty}{(-bx/a;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} b\sigma, b/\sigma \\ ab \end{matrix}\; ; q, -x\right). \tag{5.12}$$

Using this in (5.11), interchanging summations, which is easily justified for $|x|<1$, and using the q-binomial theorem gives

(5.13)
$$(A_\nu^{(a,b)}\phi_\lambda(\cdot;a,b;q))(x) = \frac{(abq^\nu, -x;q)_\infty}{(ab, -bxq^{-\nu}/a;q)_\infty}\, {}_2\varphi_1\left(\begin{matrix} b\sigma, b/\sigma \\ abq^\nu \end{matrix}\; ; q, -x\right)$$

and using (5.12) again gives the result for $|x| < 1$. The general case follows by analytic continuation in x (see (1.10), (3.1)), since the convergence in (5.11) for f the little q-Jacobi function is uniform on compact sets for x.

The statements for $\tilde{A}_\nu^{(a,b)}$ follow from the corresponding statements for $A_\nu^{(a,b)}$ and Lemma 5.2 and (5.8). $\square$

Proof of the second statement of Theorem 2.3. Define

$$\begin{aligned} A_{\nu,\mu}(a,b) &= \tilde{A}_\mu^{(aq^\nu,b)} \circ A_\nu^{(a,b)} \\ &= S(aq^\nu, bq^\mu) \circ A_\mu^{(b,aq^\nu)} \circ S(b,aq^\nu) \circ A_\nu^{(a,b)}. \end{aligned}$$

Then, it follows from Lemma 5.3 that the intertwining property is valid. The action on a function f can be calculated and, for f compactly supported in $(0,\infty)$, we find the explicit result with the ${}_3\varphi_2$-series as kernel. We can extend the result to bounded f if we require $\nu > 0$.

The action of $A_{\nu,\mu}(a,b)$ on the little q-Jacobi function follows from Lemma 5.3. $\square$

6. Quantum group theoretic interpretation. The quantized universal enveloping algebra $U_q(\mathfrak{su}(1,1))$ has representations in $\ell^2(\mathbf{Z}_{\geq 0})$

for the discrete series representations and in $\ell^2(\mathbf{Z})$ for the principal unitary series, the complementary series and strange series representations. For the harmonic analysis, the so-called twisted primitive elements, as analogues of self-adjoint Lie algebra elements, play an important role, and in each of these representations they give rise to a three-term recurrence relation in which there is essentially one degree of freedom. In the discrete series representations the three-term recurrence relations can be solved in terms of Al-Salam and Chihara polynomials [**10, 15**] and in the other series in terms of little q-Jacobi functions [**8**, Section 6].

The transition of the generalized basis of eigenvectors of two different twisted primitive elements in a strange series representation of $U_q(\mathfrak{su}(1,1))$ is given by the dual transmutation kernel of Theorem 2.1. For the complementary series and principal unitary series we can deduce the corresponding dual transmutation kernel from Proposition 3.1 in a similar way using other specializations, cf. [**8**, Section 6, Appendix A]. For the discrete series representation we refer to [**15**] and [**11**].

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