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PARABOLIC WAVELET TRANSFORMS AND LEBESGUE SPACES OF PARABOLIC POTENTIALS

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ABSTRACT. Parabolic wavelet transforms associated with heat operators $-\mathcal{D}_x + \partial/\partial t$ and $I - \mathcal{D}_x + \partial/\partial_t$ in $\mathbf{R}^{n+1}$ are introduced. A Calderón-type reproducing formula for functions $f \in L^p(\mathbf{R}^{n+1})$ is proven. By making use of these transforms, new explicit inversion formulas for the Jones-Sampson parabolic potentials are obtained, and characterization of the corresponding anisotropic Lebesgue spaces is given.

1. Introduction and main results. Continuous wavelet transforms

$$Wf(x,a) = \frac{1}{a^n}\int_{\mathbf{R}^n} f(y)w\left(\frac{|x-y|}{a}\right)dy,$$

$x \in \mathbf{R}^n$, $a > 0$, $\int_{\mathbf{R}^n} w(|y|)\,dy = 0$, play an important role in analysis and have many applications, (see, e.g., [**7**–**9, 13, 15, 16, 22, 23, 26**] and references therein). Due to the formula

$$(1.1) \qquad \int_0^\infty Wf(x,a)\frac{da}{a^{1+\alpha}} = c(-\mathcal{D})^{\alpha/2}f(x), \quad c = c(\alpha, w),\ \alpha \in \mathbf{C},$$

which gives an integral representation of powers of the Laplacian $\mathcal{D} = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$, the wavelet transforms Wf are applicable to a variety of problems in PDE, integral geometry and function theory [**13, 15, 16, 22, 24**]. The formula (1.1) can be justified in the framework of the L^p-theory [**22**].

In the present paper, we introduce anisotropic analogues of Wf which enable one to obtain wavelet-type representations, like (1.1), of powers

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of the heat operators $-\mathcal{D}_x + \partial/\partial t$, $I - \mathcal{D}_x + \partial/\partial t$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}^1$. The following remark illuminates difficulties related to this problem. It would be natural to define the parabolic wavelet transform by

$$(1.2) \qquad A_m f(x,t;a) = \int_{\mathbf{R}^{n+1}} f(x - \sqrt{a}y, t - a\tau)\, dm(y,\tau),$$

where m is a wavelet measure. In contrast to the standard wavelet transform Wf, for which various inversion procedures are known [**26**], it is not clear how to invert (1.2) for arbitrary m. On the Fourier side (in R^{n+1}), the multiplier of A_m is $M(\sqrt{a}\xi, a\tau)$, where $M = F[m]$ is the Fourier transform of m so that the Calderón formula reads

$$\int_0^\infty M(\sqrt{a}\xi, a\tau)\frac{da}{a} = 1$$

for $(\xi,\tau) \neq 0$. This equality is satisfied if M is parabolically radial, i.e., constant on spheres for a parabolic norm, e.g., $M \equiv M(\sqrt{|\xi|^4 + \tau^2})$. Such an approach is classical, (see the papers by Calderón and Torchinsky, Fabes and Rivere, Hofmann and others), but it does not lead to representation of powers of the heat operator $-\mathcal{D}_x + \partial/\partial t$ because its symbol, $|\xi|^2 + i\tau$, is not constant on parabolic spheres.

We chose m in a different way so that $M \equiv M(|\xi|^2 + i\tau)$ and introduced parabolic wavelet transforms associated with the heat operators $-\mathcal{D}_x + \partial/\partial t$, $I - \mathcal{D}_x + \partial/\partial t$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}^1$. Such a choice of m enables one to obtain the relevant integral representations of the (1.1) type and to justify them in the framework of L^p-spaces. In particular, we obtain new explicit inversion formulas for parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ defined in the Fourier terms by

$$(1.3) \qquad F[H^\alpha f](\xi,\tau) = (|\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi,\tau),$$

$$(1.4) \qquad F[\mathcal{H}^\alpha f](\xi,\tau) = (1 + |\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi,\tau).$$

These potentials were introduced by B.F. Jones, Jr. [**14**] and C.H. Sampson [**28**] and studied in [**4**–**6, 10, 11, 17**–**20**]. We also obtain a parabolic analog of Calderón's reproducing formula for L^p-functions. As an application of our results, new characterization of parabolic Sobolev spaces is given. These spaces were introduced by Sampson [**28**], studied by Bagby [**4**] and Gopala Rao [**10, 11**] and generalized by Nogin and Rubin [**20**].

One should mention the papers [**1, 2**] by Aliev devoted to parabolic potentials with the generalized translation. The techniques developed below can be applied to this class of potentials [**3**].

Main results. Let $\mathbf{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space of points (x,t), $x=(x_1,\cdots,x_n)\in\mathbf{R}^n$, $t\in\mathbf{R}^1$. Given a finite Borel measure $m\equiv m(x,t)$ on $\mathbf{R}^{n+1}$, we define an anisotropic dilation $m_{\sqrt{a},a}$ of m by

$$\int_{\mathbf{R}^{n+1}}\omega(x,t)\,dm_{\sqrt{a},a}(x,t)=\int_{\mathbf{R}^{n+1}}\omega(\sqrt{a}x,at)\,dm(x,t), \tag{1.5}$$

$a>0$, $\omega\in C_0=C_0(\mathbf{R}^{n+1})$, the space of continuous functions vanishing at infinity. Then (1.2) can be written as a convolution:

$$A_m f(x,t;a)=(f*m_{\sqrt{a},a})(x,t). \tag{1.6}$$

In order to be consistent with heat operators and the relevant parabolic potentials, we choose m in a special way as follows: Let

$$W(x,t)=(4\pi t)^{-n/2}\exp(-|x|^2/4t),\quad x\in\mathbf{R}^n, t>0, \tag{1.7}$$

be the Gauss-Weierstrass kernel possessing the following properties [**29**]:

$$\begin{aligned}
&1)\quad \int_{\mathbf{R}^n}W(x,t)\,dx=1;\\
&2)\quad \int_{\mathbf{R}^n}W(y,t)W(x-y,\tau)\,dy=W(x,t+\tau);\\
&3)\quad F[W(\cdot,t)](\xi)=\int_{\mathbf{R}^n}e^{-ix\cdot\xi}W(x,t)\,dx=e^{-t|\xi|^2}.
\end{aligned} \tag{1.8}$$

Let μ be a finite Borel measure supported by $\mathbf{R}^1_+=[0,\infty)$ such that $\mu(\mathbf{R}^1_+)=0$ and $\mu(\{0\})=0$. We call μ a *wavelet measure* and set

$$dm(x,t)=W(x,t)\,dx\,d\mu(t), \tag{1.9}$$

$W(x,t)$ being extended to $t\le 0$ by zero. Clearly, m is finite and $m(\mathbf{R}^{n+1})=0$.

Definition 1.1. The wavelet transform (1.6) with m defined by (1.9) will be called a parabolic wavelet transform, associated with the heat operator $-\mathcal{D}_x + \partial/\partial t$ and denoted by $P_\mu f(x,t;a)$. Thus,

$$P_\mu f(x,t;a) = \int_{\mathbf{R}^n\times\mathbf{R}^1_+} f(x-\sqrt{a}y, t-a\tau)W(y,\tau)\,dy\,d\mu(\tau) \tag{1.10}$$

$$= \int_{\mathbf{R}^n\times\mathbf{R}^1_+} f(x-\sqrt{a\tau}z, t-a\tau)W(z,1)\,dz\,d\mu(\tau). \tag{1.11}$$

If $\tilde{\mu}(z) = \int_0^\infty e^{-zt}\,d\mu(t)$ is the Laplace transformation of μ, then (1.8) yields

$$M(\xi,\tau) = F[m](\xi,\tau) \equiv \int_{\mathbf{R}^{n+1}} e^{-ix\cdot\xi - it\tau}\,dm(\xi,\tau) = \tilde{\mu}(|\xi|^2 + i\tau),$$

which motivates Definition 1.1. Given two measures μ and ν on $\mathbf{R}^1_+$, we have

$$P_{\mu*\nu} f(x,t;a) = P_\mu[P_\nu f(\cdot,\cdot;a)](x,t;a). \tag{1.12}$$

Let $f \in L^p \equiv L^p(\mathbf{R}^{n+1})$. For $p=2$, the following statement contains an analogue of Calderón's formula in terms of the Laplace transformation of μ.

Theorem A. *Let μ be a wavelet measure such that the integral $c_\mu = \int_0^\infty \tilde{\mu}(\eta)\,d\eta/\eta$ converges as an improper one. Then*

$$\overset{(L^2)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} \int_\varepsilon^\rho \frac{P_\mu f(x,t;a)}{a}\,da = c_\mu f(x,t).$$

In order to state an L^p-analogue of this theorem, we give the following

Definition 1.2. A wavelet measure μ is called admissible if

$$k(t) \overset{\text{def}}{=} \frac{\mu([0,t))}{t} \in L^1(0,\infty). \tag{1.13}$$

Theorem B. *Let μ be an admissible measure, and let*

$$k_0 = \int_0^\infty k(t)\,dt, \quad \textit{see (1.13).} \tag{1.14}$$

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$\int_0^\infty P_\mu f(x,t;a)\frac{da}{a} \equiv \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho (\cdots) = k_0 f(x,t) \tag{1.15}$$

where $\lim = \overset{(L^p)}{\lim}$.

(ii) *If $f \in C_0$, then* (1.15) *holds with the limit interpreted in the* $\sup_{(x,t)\in\mathbf{R}^{n+1}}$*-norm.*

(iii) *If $f \in L^p$, $1 < p < \infty$ and $k(t)$ has a decreasing integrable majorant, then* (1.15) *holds almost everywhere on* $\mathbf{R}^{n+1}$.

Due to (1.12), Theorems A and B can be easily reformulated in terms of the two measures as it is usually done in the wavelet literature. These measures are called an *analyzing measure* and a *reconstructing measure*, respectively.

As in [**23**, page 180] and [**22**, Section 12], a measure μ is admissible if $\mu(\mathbf{R}^1_+) = 0$ and one of the following conditions is satisfied:

$$\int_0^\infty |\log t| d|\mu(t) < \infty \tag{1.16}$$

or

$$d\mu(t) = g(t)\,dt, \quad g \in H^1 \text{ (the real Hardy space on } R^1). \tag{1.17}$$

Furthermore,

$$k_0 = \begin{cases} \int_0^\infty \log(1/t)\,d\mu(t) & \text{in the case (a),}\\ (\pi/2)\int_{-\infty}^\infty Hg(t)\operatorname{sgn} t\,dt & \text{in the case (b),}\end{cases} \tag{1.18}$$

$Hg(t) = p.v.\pi^{-1}\int_0^\infty g(\tau)(t-\tau)^{-1}\,d\tau (\in L^1(0,\infty))$ being the Hilbert transform of g.

Here and on, the notation like $\int_a^b f(t)\,d\mu(t)$ designates $\int_{[a,b)} f(t)\,d\mu(t)$.

Remark 1.3. Wavelet people are prone to think of the Gaussian as a wavelet function (after differentiation). As one could notice, in our case the role of the Gaussian is completely different. Note also that the Gaussian represents an inalienable part of the kernel of the parabolic potentials (1.3) and (1.4), which can be written as integral operators

(1.19)
$$\begin{aligned} H^\alpha f(x,t) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\mathbf{R}^n\times(0,\infty)} \tau^{\frac{\alpha}{2}-1} W(y,\tau) f(x-y,t-\tau)\,dy\,d\tau \\ &= (h_\alpha * f)(x,t), \\ h_\alpha(x,t) &= \frac{1}{\Gamma(\frac{\alpha}{2})} t_+^{\alpha/2-1} W(x,t); \end{aligned}$$

(1.20)
$$\begin{aligned} \mathcal{H}^\alpha f(x,t) &= \frac{1}{\Gamma(\frac{\alpha}{2})} \int_{\mathbf{R}^n\times(0,\infty)} \tau^{\frac{\alpha}{2}-1} e^{-\tau} W(y,\tau) f(x-y,t-\tau)\,dy\,d\tau \\ &= (\tilde{h}_\alpha * f)(x,t), \\ \tilde{h}_\alpha(x,t) &= e^{-t} h_\alpha(x,t). \end{aligned}$$

It is worth noting that numerous constructions of continuous wavelet transforms can be built without any connection with the Fourier analysis by starting out from the corresponding analytic families of "fractional integrals" (this approach was developed in [**22**] and [**24**]). Parabolic potentials (1.19) and (1.20) can be included in this scheme.

Remark 1.4. We do not know whether the statement (iii) of Theorem B holds for $p = 1$ because we cannot assert the validity of the weak estimate for the relevant mixed partial maximal functions, (see the proof of Theorem B in Section 3). These maximal functions are inherent in the anisotropic case and represent a more difficult object than the usual Hardy-Littlewood maximal functions associated with classical wavelet transforms.

The following known theorem characterizes the action of parabolic potentials on L^p-functions.

Theorem 1.5 [**4, 10**]. 1. *Let* $f \in L^p$, $1 \le p < \infty$, $0 < \alpha < (n+2)/p$, $q = (n+2-\alpha p)^{-1}(n+2)p$.

(a) *The integral* $H^\alpha f(x,t)$ *converges absolutely for almost all* $(x,t) \in \mathbf{R}^{n+1}$.

(b) *For* $p > 1$, *the operator* H^α *is bounded from* L^p *into* L^q.

(c) *For* $p = 1$, H^α *is an operator of the weak* $(1,q)$ *type*

$$|\{(x,t) : |H^\alpha f(x,t)| > \gamma\}| \le \left(\frac{c\|f\|_1}{\gamma}\right)^q.$$

2. *The operator* $\mathcal{H}^\alpha$ *is bounded on* L^p *for all* $\alpha \ge 0$, $1 \le p \le \infty$.

Remark 1.6. The proof of the weak estimate (c), sketched very briefly in [**10**], seems to be wrong. A detailed proof of this result in a more general setting can be found in [**3**].

Remark 1.7. If $\alpha \ge (n+2)/p$, the integral (1.19) can be divergent for $f \in L^p$. In this case we interpret $H^\alpha f$ as a distribution defined by duality $\langle H^\alpha f, \omega\rangle = \langle f, H^\alpha_- \omega\rangle$, $H^\alpha_- \omega = U H^\alpha U \omega$, $U\omega(x,t) = \omega(-x,-t)$. Here the test function ω belongs to the space $\Phi_0 = \Phi_0(\mathbf{R}^{n+1})$ of Schwartz functions orthogonal to all polynomials [**22**, page 19], and the abbreviation $\langle f, g\rangle$ is used for $\int f\bar{g}$. Since $F[H^\alpha_- \omega](\xi,\tau) = (|\xi|^2 - i\tau)^{-\alpha/2} F[\omega](\xi,\tau)$, H^α_- is an automorphism of Φ_0 for any α.

A straightforward calculation enables us to represent (1.19) and (1.20) via the relevant wavelet transforms. Namely, for $\operatorname{Re}\alpha > 0$,

$$\begin{gathered} H^\alpha f(x,t) = c^{-1}_{\alpha,\mu} \int_0^\infty P_\mu f(x,t;a) \frac{da}{a^{1-\alpha/2}}, \\ c_{\alpha,\mu} = \Gamma(\alpha/2) \int_0^\infty \tau^{-\alpha/2}\, d\mu(\tau), \end{gathered} \tag{1.21}$$

provided that $\int_0^\infty \tau^{-\operatorname{Re}\alpha/2}\, d|\mu|(\tau) < \infty$ and $c_{\alpha,\mu} \ne 0$, cf., (1.1). Similarly,

$$\mathcal{H}^\alpha f(x,t) = c^{-1}_{\alpha,\mu} \int_0^\infty \mathcal{P}_\mu f(x,t;a) \frac{da}{a^{1-\alpha/2}} \tag{1.22}$$

where

(1.23)
$$\mathcal{P}_\mu f(x,t;a) = \int_{\mathbf{R}^n\times\mathbf{R}^1_+} f(x-\sqrt{a}y, t-a\tau)W(y,\tau)e^{-a\tau}\,dy\,d\mu(\tau)$$
$$= \int_{\mathbf{R}^n\times\mathbf{R}^1_+} f(x-\sqrt{a\tau}z, t-a\tau)W(z,1)e^{-a\tau}\,dz\,d\mu(\tau),$$

will be called a *weighted parabolic wavelet transform associated with the heat operator* $I-\mathcal{D}_x+\partial/\partial_t$.

The idea to introduce *weighted wavelet transforms* associated with inhomogeneous differential operators and generalizing the notion of weighted finite difference [**21, 22**] seems to be new.

The formula (1.12) remains true if P is substituted for $\mathcal{P}$, and analogues of Theorems A and B also hold for the weighted transform (1.23).

In view of (1.3) and (1.4), explicit inverses of H^α and $\mathcal{H}^\alpha$ can be obtained from (1.21) and (1.22) if one replaces formally α by $-\alpha$. This observation leads to the following:

Theorem C. *Let μ be a finite Borel measure on* $[0,\infty]$*, such that*

(1.24)
$$\int_0^\infty t^j\,d\mu(t)=0, \quad \forall j=0,1,\cdots[\alpha/2] \text{ (the integral part of } \alpha/2);$$

(1.25)
$$\int_0^\infty t^\beta\,d|\mu|(t)<\infty \quad \text{for some } \beta>\alpha/2.$$

Suppose that $\varphi=H^\alpha f$, $f\in L^p$, $1\le p<\infty$, $0<\alpha<(n+2)/p$ and $P_\mu\varphi$ is the wavelet transform defined by (1.11)*. Then*

$$\int_0^\infty P_\mu\varphi(x,t;a)\frac{da}{a^{1+\alpha/2}} \equiv \lim_{\varepsilon\to 0}\int_\varepsilon^\infty(\cdots) = d_{\alpha,\mu}f(x,t), \tag{1.26}$$

$$d_{\alpha,\mu} = \begin{cases} \Gamma(-\alpha/2)\int_0^\infty t^{\alpha/2}\,d\mu(t) & \text{if } \alpha/2\notin\mathbf{N}, \\ ((-1)^{1+\alpha/2}/(\alpha/2)!)\int_0^\infty t^{\alpha/2}\log t\,d\mu(t) & \text{if } \alpha/2\in\mathbf{N}. \end{cases} \tag{1.27}$$

The limit in (1.26) *is interpreted in the L^p-norm for $1 \le p < \infty$ and almost everywhere on $\mathbf{R}^{n+1}$ for $1 < p < \infty$.*

The same statement holds for all $\alpha > 0$ and $1 \le p \le \infty$ (L^∞ is identified with C_0), provided that H^α and P_μ are replaced by $\mathcal{H}^\alpha$ and $\mathcal{P}_\mu$, (see (1.23)), respectively.

Our next result concerns application of parabolic wavelet transforms to characterization of anisotropic spaces $\mathcal{L}^\alpha_{p,r}(\mathbf{R}^{n+1})$ of parabolic potentials that were introduced and studied in [**20**]. We recall that, given $\alpha > 0$, $1 < p < \infty$, $1 < r < \infty$, the space $\mathcal{L}^\alpha_{p,r} = \mathcal{L}^\alpha_{p,r}(\mathbf{R}^{n+1})$ is defined by

$$\mathcal{L}^\alpha_{p,r} = \{f : \|f\|_{\mathcal{L}^\alpha_{p,r}} \equiv \|f\|_r + \|F^{-1}(|\xi|^2 + i\tau)^{\alpha/2} F f\|_p < \infty\}, \tag{1.28}$$

where the Fourier transform F is understood in the sense of Φ_0'-distributions, (see Remark 1.7). The spaces (1.28) generalize the scale $\mathcal{L}^p_\alpha = \mathcal{H}^\alpha(L^p)$ of parabolic Bessel potentials studied in [**4, 28**] and coincide with them for $r = p$. In comparison with $\mathcal{L}^p_\alpha$, the spaces (1.28) have a number of important additional features, (see [**20**] for details).

Theorem D. *Let $\alpha > 0$, $1 < p < \infty$, $1 < r < \infty$. Then*

$$\mathcal{L}^\alpha_{p,r} = \left\{ f \in L^r : \sup_{\varepsilon > 0} \left\| \int_\varepsilon^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}}\, da \right\|_p < \infty \right\} \tag{1.29}$$

for any μ satisfying conditions of Theorem C.

Example 1.8. Let

$$\mu = \sum_{k=0}^{l} \binom{l}{k} (-1)^k \delta_k, \quad l > \alpha/2, \tag{1.30}$$

$\delta_k = \delta_k(t)$ being the unit mass at the point $t = k$. By (1.11),

$$\begin{aligned} P_\mu f(x,t;a) &= \sum_{k=0}^{l} \binom{l}{k} (-1)^k \int_{\mathbf{R}^n} f(s - \sqrt{ak}z, t - ka) W(z,1)\, dz \\ &= \int_{\mathbf{R}^n} \mathcal{D}^l_{y,\alpha} f(x,t) W(y,a)\, dy, \end{aligned}$$

where $\mathcal{D}^l_{y,a} f(x,t) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k f(x-\sqrt{k}y, t-ka)$ is an anisotropic finite difference of f. Furthermore,

$$\int_0^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}}\, da = \int_{\mathbf{R}^n \times (0,\infty)} \frac{\mathcal{D}^l_{y,a} f(x,t)}{a^{1+\alpha/2}} W(y,a)\, dy\, da.$$

Hypersingular integrals of this form were introduced in [**17**–**20**]. As in [**22**, Section 17.4], it is not difficult to show that the measure (1.30) satisfies (1.24), (1.25) and

$$d_{\alpha,\mu} = \int_0^\infty \frac{(1-e^{-t})^l}{t^{1+\alpha/2}}\, dt \neq 0.$$

In Sections 2, 3, 4 and 5 we prove Theorems A, B, C and D, respectively. Section 6 contains concluding remarks and another Theorem E characterizing the range $H^\alpha(L^p)$.

2. Proof of Theorem A. Denote

$$I_{\varepsilon,\rho}(\mu, f) = \int_\varepsilon^\rho P_\mu f(x,t;a) \frac{da}{a}, \quad 0 < \varepsilon < \rho < \infty, \tag{2.1}$$

and assume first that $f \in L^1 \cap L^2$. Then $I_{\varepsilon,\rho}(\mu, f) \in L^1 \cap L^2$ and $F[I_{\varepsilon,\rho}(\mu, f)] = \hat{K}_{\varepsilon,\rho} \hat{f}$, $\hat{f} = F[f]$, where for $|\xi| \neq 0$,

$$\hat{K}_{\varepsilon,\rho}(\xi,\tau) = \int_\varepsilon^\rho \frac{\tilde{\mu}(a(|\xi|^2 + i\tau))}{a}\, da = \int_l \frac{\tilde{\mu}(z)}{z}\, dz,$$

l being the segment in the half-plane $\mathrm{Re}\, z > 0$ connecting the point $\varepsilon(|\xi|^2 + i\tau)$ and $\rho(|\xi|^2 + i\tau)$. Denote $|\xi|^2 + i\tau = re^{i\theta}$, $\theta \in (-\pi/2, \pi/2)$. By the Cauchy theorem, $\hat{K}_{\varepsilon,\rho}(\xi,\tau) = m^{(1)}_{\varepsilon,\rho}(\xi,\tau) + m^{(2)}_{\varepsilon,\rho}(\xi,\tau)$, where

$$m^{(1)}_{\varepsilon,\rho}(\xi,\tau) = \int_{\varepsilon r}^{\rho r} \frac{\tilde{\mu}(\eta)}{\eta}\, d\eta, \quad m^{(2)}_{\varepsilon,\rho}(\xi,\tau) = i \int_0^\theta [\tilde{\mu}(\rho e^{i\phi}) - \tilde{\mu}(\varepsilon e^{i\phi})]\, d\phi.$$

Since the integral $c_\mu = \int_0^\infty \tilde{\mu}(\eta)\, d\eta/\eta$ is finite, the function $\psi(t) = \int_0^t \tilde{\mu}(\eta)\, d\eta/\eta$ is continuous on $[0,\infty]$. It follows that there is a constant $A > 0$ for which $|m^{(1)}_{\varepsilon,\rho}(\xi,\tau)| \leq A$ for all $\rho > \varepsilon > 0$ and $(\xi,\tau) \in \mathbf{R}^{n+1}$.

Furthermore, for $\operatorname{Re} z > 0$, we have $|\tilde{\mu}(z)| \leq |\mu|(\mathbf{R}^1_+)$, and therefore $|m^{(2)}_{\varepsilon,\rho}(\xi,\tau)| \leq \pi|\mu|(\mathbf{R}^1_+)$. By taking into account that $\lim_{r\to 0}\tilde{\mu}(re^{i\phi}) = \int_0^\infty d\mu(t) = 0$ and $\lim_{r\to\infty}\tilde{\mu}(re^{i\phi}) = 0$ for $\phi \in (-\pi/2, \pi/2)$, due to the Lebesgue theorem on dominated convergence we obtain

$$\|I_{\varepsilon,\rho}(\mu,f) - c_\mu f\|_2 = \|m^{(1)}_{\varepsilon,\rho} + m^{(2)}_{\varepsilon,\rho} - c_\mu)\hat{f}\|_2 \longrightarrow 0$$

as $\varepsilon \to 0$ and $\rho \to \infty$. The result for arbitrary $f \in L^2$ then follows in a standard way. □

An analogue of Theorem A for the inhomogeneous wavelet transform (1.23) has the same proof with $|\xi|^2$ replaced by $1 + |\xi|^2$.

3. Proof of Theorem B. The integral (2.1) can be written as

$$I_{\varepsilon,\rho}(\mu,f) = f * K_\varepsilon - f * K_\rho, \tag{3.1}$$

$$K_\varepsilon(x,t) = \varepsilon^{-1}W(x,t)k(t/\varepsilon), \tag{3.2}$$

$k(t)$ being the function (1.13), extended by 0 to $t < 0$. Indeed, by changing the order of integration and passing to new variables, we get

$$\begin{aligned} I_{\varepsilon,\rho}(\mu,f) &= \int_0^\infty d\mu(\tau)\int_{\varepsilon\tau}^{\rho\tau}\Big(\frac{\tau}{b}\Big)^{n/2}\frac{db}{b} \\ &\qquad \times \int_{\mathbf{R}^n} f(x-z,t-b)W\left(\frac{z\sqrt{\tau}}{\sqrt{b}},\tau\right)dz \\ &= \int_0^\infty \frac{db}{b}\int_{\mathbf{R}^n} f(x-z,t-b)W(z,b)\,dz\int_{b/\rho}^{b/\varepsilon} d\mu(\tau), \end{aligned}$$

and (3.1) follows. If μ is admissible, i.e., $k(t) \in L^1(0,\infty)$, then

$$\begin{aligned} &(f * K_\varepsilon)(x,t) - k_0 f(x,t) \\ &\qquad = \int_0^\infty k(\tau)\,d\tau\int_{\mathbf{R}^n}[f(x - z\sqrt{\varepsilon\tau}, t - \varepsilon\tau) - f(x,t)]W(z,1)\,dz, \end{aligned}$$

and therefore $f * K_\varepsilon \to k_0 f$ in the L^p-norm (or in the sup-norm for $f \in C_0$). Furthermore, if $k(t) \in L^1(0,\infty)$, then

$$\lim_{\rho\to\infty}\|f * K_\rho\|_p = 0, \quad \forall f \in L^p(\mathbf{R}^{n+1}),\ 1 < p \leq \infty, \tag{3.3}$$

(L^∞ is identified with C_0). Indeed,
(3.4)

$$\begin{aligned}\|f*K_\rho\|_p &= \Big\|\Big\|\int_0^\infty k(\tau)\,d\tau\int_{\mathbf{R}^n} f(x-y,t-\rho\tau)W(y,\rho\tau)\,dy\Big\|_{L^p_x}\Big\|_{L^p_t}\\ &\le \Big\|\int_0^\infty |k(\tau)|\Big\|\int_{\mathbf{R}^n} f(x-y,t-\rho\tau)W(y,\rho\tau)\,dy\Big\|_{L^p_x}d\tau\Big\|_{L^p_t}\\ &\le \Big\|\int_0^\infty |k(\tau)|\tilde f(t-\rho\tau)\,d\tau\Big\|_{L^p_t},\end{aligned}$$

where $\tilde f(t) = \|f(\cdot,t)\|_p \in L^p(\mathbf{R}^1)$. The expression (3.4) tends to 0 as $\rho\to\infty$, use, e.g., Theorem 1.15 from [**22**, page 3]. This completes the proof of (i) and (ii).

The validity of (iii) follows in a standard way [**29**] from the maximal estimate $\|sup_{\varepsilon>0}|f_*K_\varepsilon|\|_p \le c\|f\|_p$. The latter is a consequence of the L^p-boundedness of the "partial" Hardy-Littlewood maximal functions

$$\begin{aligned}\tilde f(x,t) &= \sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y,t)|\,dy,\\ f^*(x,t) &= \sup_{s>0}\frac{1}{2s}\int_{|t-\tau|<s}\tilde f(x,\tau)\,d\tau,\end{aligned}$$

$B(x,r)=\{y\in\mathbf{R}^n : |x-y|<r\}$ due to the following estimates:

$$\begin{aligned}|(f*K_\rho)(x,t)| &\le \int_0^\infty |k(\tau)|\Big|\int_{\mathbf{R}^n} f(x-y,t-\rho\tau)W(y,\rho\tau)\,dy\Big|\,d\tau\\ &\le \int_0^\infty |k(\tau)|\Big[\sup_{s>0}\int_{\mathbf{R}^n}|f(x-y,t-\rho\tau)|W(y,s)\,dy\Big]\,d\tau\\ &\le c\int_0^\infty |k(\tau)|\tilde f(x,t-\rho\tau)\,d\tau \le c_1 f^*(x,t),\end{aligned}$$

c and c_1 being some constants independent of f. □

The proof of the analog of Theorem B for weighted wavelet transforms (1.23) follows the same lines and is based on the equality

$$\int_\varepsilon^\rho \mathcal{P}_\mu f(x,t;a)\frac{da}{a} = f*\tilde K_\varepsilon - f*\tilde K_\rho, \tag{3.5}$$

$\tilde{K}_\varepsilon(x,t) = \varepsilon^{-1}e^{-t}W(x,t)k(t/\varepsilon)$, similarly for $\tilde{K}_\rho$, cf. (3.2). Slight additional technicalities related to the extra factor e^{-t} are left to the reader.

4. Proof of Theorem C. Denote $h_a^{\alpha/2}(x,t) = a^{\alpha/2-1}W(x,t)(I_{0+}^{\alpha/2}\mu) \times (t/a)$, where

$$(I_{0+}^{\alpha/2}\mu)(t) = \frac{1}{\Gamma(\alpha/2)}\int_0^t (t-\tau)^{\alpha/2-1}\,d\mu(\tau)$$

is the Riemann-Liouville fractional integral of μ. We first show that

$$P_\mu H^\alpha f(x,t;a) = (f * h_a^{\alpha/2})(x,t). \tag{4.1}$$

Indeed, by (1.11) and (1.19),

$$\begin{aligned}
(P_\mu H^\alpha f)(x,t) &= \int_0^\infty d\mu(\tau)\int_{\mathbf{R}^n}(H^\alpha f)(x-\sqrt{a\tau}z, t-a\tau)W(z,1)\,dz \\
&= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_0^\infty d\mu(\tau)\int_{\mathbf{R}^n} W(z,1)\,dz\int_{a\tau}^\infty (\mathbf{Z}-a\tau)^{\frac{\alpha}{2}-1}d\mathbf{Z} \\
&\quad\times\int_{\mathbf{R}^n} f(x-\xi, t-\mathbf{Z})W(\xi-\sqrt{a\tau}z, \xi-a\tau)d\xi \\
&= \frac{1}{\Gamma(\frac{\alpha}{2})}\int_{\mathbf{R}^{n+1}} f(x-\xi,t-\mathbf{Z})\,d\xi\,d\mathbf{Z} \\
&\quad\times\int_0^\infty (\mathbf{Z}-a\tau)_+^{\frac{\alpha}{2}-1} I(\xi,\mathbf{Z},\tau)\,d\mu(\tau), \\
I(\xi,\mathbf{Z},\tau) &= \int_{\mathbf{R}^n} W(z,1)W(\xi-\sqrt{a\tau}z, \mathbf{Z}-a\tau)\,dz = W(\xi,\mathbf{Z})
\end{aligned}$$

for all $\tau \geq 0$, $a > 0$. This implies (4.1). Furthermore, by (4.1),

$$T_\varepsilon^\alpha \varphi(x,t) \overset{\text{def}}{=} \int_\varepsilon^\infty \frac{(P_\mu\varphi)(x,t;a)}{a^{1+\alpha/2}}\,da = f * \psi_\varepsilon, \tag{4.2}$$

where, using 3.238 from [**12**],

$$\begin{aligned}
\psi_\varepsilon(x,t) &= \frac{W(x,t)}{\Gamma(\alpha/2)}\int_\varepsilon^\infty \frac{da}{a^{1+\alpha/2}}\int_0^\infty (t-a\tau)_+^{\alpha/2-1}\,d\mu(\tau) \\
&= \varepsilon^{-1}W(x,t)\lambda_\alpha(t/\varepsilon),
\end{aligned}$$

$\lambda_\alpha(t) = t^{-1}(I_{0+}^{\alpha/2+1}\mu)(t)$. By Lemma 1.3 from [**25**], conditions (1.24) and (1.25) imply that $\lambda_\alpha(s)$ has a decreasing integrable majorant and $\int_0^\infty \lambda_\alpha(s)\,ds = d_{\alpha,\mu}$, (see (1.27)). Hence (1.26) follows by the same argument as in the proof of Theorem B.

For $\varphi = \mathcal{H}^\alpha f$, the proof is similar and relies on the equality, cf. (3.5),

$$\int_\varepsilon^\infty \frac{\mathcal{P}_\mu \varphi(x,t;a)}{a^{1+\alpha/2}}\,da = f * \tilde{\psi}_\varepsilon, \quad \tilde{\psi}_\varepsilon(x,t) = \varepsilon^{-1}e^{-t}W(x,t)\lambda_\alpha(t/\varepsilon). \qquad \square$$

Remark 4.1. By keeping track of the proof of (4.2) one can readily see that the equality

$$T_\varepsilon^\alpha H^\alpha f = f * \psi_\varepsilon, \quad \psi_\varepsilon(x,t) = \varepsilon^{-1}W(x,t)\lambda_\alpha(t/\varepsilon), \tag{4.3}$$

holds for all $\alpha > 0$, provided, e.g., $f \in \Phi_0$. This remark will be used in the next section.

5. Proof of Theorem D. By Theorem 2.1 from [**20**], $\mathcal{L}_{p,r}^\alpha = L^r \cap H^\alpha(L^p)$. In other words, $\mathcal{L}_{p,r}^\alpha$ consists of functions $f \in L^r$, such that $f = H^\alpha g$ for some $g \in L^p$ in the Φ_0'-sense. Thus, it suffices to prove the equivalence

$$\sup_{\varepsilon>0} \left\| \int_\varepsilon^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}}\,da \right\|_p < \infty \Longleftrightarrow f \overset{(\Phi_0)}{=} H^\alpha g \tag{5.1}$$

for some $g \in L^p$. Assuming the right equality, we first show that

$$T_\varepsilon^\alpha f(x,t) = \int_\varepsilon^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}}\,da = (\psi_\varepsilon * g)(x,t) \tag{5.2}$$

cf., (4.2) and (4.3). Let u and $v \in \Phi_0$ be such that $u = H_-^\alpha v$, (see Remark 1.7). Then

$$\langle T_\varepsilon^\alpha f, u\rangle = \langle f, U\overline{T_\varepsilon^\alpha U\bar{u}}\rangle = \langle f, U\overline{T_\varepsilon^\alpha H^\alpha U\bar{v}}\rangle \overset{(4.3)}{=} \langle f, U\overline{[\psi_\varepsilon * U\bar{v}]}\rangle.$$

Since $\psi_\varepsilon \in L^1$, there is a sequence $\{\psi_{\varepsilon,l}\} \subset C_c^\infty$ that converges to ψ_ε as $l \to \infty$ in the L^1-norm. Moreover, $\psi_{\varepsilon,l} * U\bar{v} \in \Phi_0$ for all l, and

$$|\langle f, U\overline{[\psi_\varepsilon * U\bar{v}]}\rangle - \langle f, U\overline{[\psi_{\varepsilon,l} * U\bar{v}]}]\rangle| \le \|f\|_r \|v\|_{r'} \|\psi_\varepsilon - \psi_{\varepsilon,l}\|_1 \longrightarrow 0$$

$(1/r + 1/r' = 1)$, as $l \to \infty$. Hence

$$\begin{aligned}\langle T_\varepsilon^\alpha f, u\rangle &= \lim_{l\to\infty} \langle f, U\overline{[\psi_{\varepsilon,l} * U\bar{v}]}\rangle = \lim_{l\to\infty} \langle g, H_-^\alpha U\overline{[\psi_{\varepsilon,l} * U\bar{v}]}\rangle \\ &= \lim_{l\to\infty} \langle g, U\overline{H^\alpha[\psi_{\varepsilon,l} * U\bar{v}]}\rangle = \lim_{l\to\infty} \langle g, U\overline{[\psi_{\varepsilon,l} * H^\alpha U\bar{v}]}\rangle \\ &= \lim_{l\to\infty} \langle g, U\overline{[\psi_{\varepsilon,l} * U\bar{u}]}\rangle = \langle g, U\overline{[\psi_\varepsilon * U\bar{u}]} = \langle g * \psi_\varepsilon, u\rangle.\end{aligned}$$

We have prove that the functions $T_\varepsilon^\alpha f \in L^r$ and $g * \psi_\varepsilon \in L^p$ coincide as the Φ_0'-distributions. By Corollary 1.1 from [**20**], they coincide pointwise almost everywhere on $\mathbf{R}^{n+1}$, and (5.2) follows. The latter implies the left inequality in (5.1).

Conversely, if $\sup_{\varepsilon>0} \|T_\varepsilon^\alpha f\|_p < \infty$, then the set of functionals $\varphi \to \langle T_\varepsilon^\alpha f, \varphi\rangle$, $\varphi \in L^p$, $1/p + 1/p' = 1$ is bounded in $(L^p)^*$. Since the bonded set in the space which is dual to the reflexive Banach space is compact in the weak* topology, a function $g \in L^p$ and a sequence $\varepsilon_k \to 0$ exist such that $\langle T_{\varepsilon_k}^\alpha f, \varphi\rangle \to \langle g, \varphi\rangle$ as $\varepsilon_k \to 0$ for all $\varphi \in L^p$. For this g and any test function $\omega \in \Phi_0$ we have

$$\begin{aligned}\langle H^\alpha g, \omega\rangle &= \langle g, H_-^\alpha \omega\rangle = \lim_{\varepsilon_k\to 0} \langle T_{\varepsilon_k}^\alpha f, H_-^\alpha \omega\rangle \\ &= \lim_{\varepsilon_k\to 0} \langle f, U\overline{T_{\varepsilon_k}^\alpha U H_-^\alpha \bar{\omega}}\rangle = \lim_{\varepsilon_k\to 0} \langle f, U\overline{T_{\varepsilon_k}^\alpha H^\alpha U\bar{\omega}}\rangle \\ &\overset{(4.3)}{=} \lim_{\varepsilon_k\to 0} \langle f, U\overline{[\psi_{\varepsilon_k} * U\bar{\omega}]}\rangle = \lim_{\varepsilon_k\to 0} \langle f * \psi_{\varepsilon_k}, \omega\rangle = \langle f, \omega\rangle,\end{aligned}$$

i.e., $f = H^\alpha g$ in the Φ_0'-sense. This completes the proof. $\square$

6. Concluding remarks. Implementation of wavelet-type integrals $\int_0^\infty P_\mu f(x,t;a)\, da/a^{1+\alpha/2}$ (and their inhomogeneous modifications, including $\mathcal{P}_\mu f$) enables one to look with "bird's-eye view" at the method of hypersingular integrals having been used in [**17–20, 22, 27**]. The essence of the latter is represented by orthogonality relations (1.24). Without going into details, we note that the equality (4.1) can be extended to $\alpha \ge (n+2)/p$, thus demonstrating a regularizing effect of the wavelet transform $P_\mu \varphi$ when $\varphi = H^\alpha f$ is a Φ_0'-distribution. By Lemma 4.12 from [**22**], conditions (1.24) and (1.25) imply $h_a^{\alpha/2} \in L^1$ so that the righthand side of (4.1) is a usual function belonging to L^p for all $1 \le p \le \infty$.

By using the argument, which is similar to that in the proofs of Theorems C and D, one can obtain the following characterization of potentials $H^\alpha f$, $f \in L^p$.

Theorem E. *Let $\alpha > 0$, $f \in L^p$, $\varphi \in L^r$; $1 \le p$, $r < \infty$. Suppose that the wavelet measure μ satisfies the conditions of Theorem* C. *If $\varphi = H^\alpha f$ pointwise almost everywhere for $\alpha < (n+2)/p$ or $\varphi = H^\alpha f$ in the Φ_0'-sense, then* (1.26) *holds with the limit interpreted in the L^p-norm. If $p > 1$ this limit can be understood also in the a.e. sense. Conversely, if $d_{\alpha,\mu} \neq 0$, see* (1.27), *and*

$$\overset{(L^p)}{\lim_{\varepsilon \to 0}} \frac{1}{d_{\alpha,\mu}} \int_\varepsilon^\infty P_\mu \varphi(x,t;a) \frac{da}{a^{1+\alpha/2}} = f,$$

then $\varphi = H^\alpha f$ in the Φ_0'-sense, or pointwise almost everywhere for $\alpha < (n+2)/p$.

A similar statement, which does not involve distributions, also holds for inhomogeneous potentials $\mathcal{H}^\alpha f$.

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