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GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY LOCALLY UNIVALENT FUNCTIONS ON THE UNIT DISK

YONG CHANG KIM AND TOSHIYUKI SUGAWA

ABSTRACT. In this note we shall give a sharp distortion estimate for a uniformly locally univalent holomorphic function on the unit disk in terms of the norm of pre-Schwarzian derivative. As applications, we shall investigate the growth of coefficients and integral means of such a function and mention a connection with Hardy spaces. We also give norm estimates for typical classes of univalent functions.

1. Introduction. We will say that a holomorphic function f on the unit disk $\mathbf{D}$ is *uniformly locally univalent* if f is univalent on each hyperbolic disk $D(a,\rho) = \{z \in \mathbf{D}; |(z-a)/(1-\bar{a}z)| < \tanh\rho\}$ with radius ρ and center $a \in \mathbf{D}$ for a positive constant ρ. In particular, a holomorphic universal covering map of a plane domain D is uniformly locally univalent if and only if the boundary of D is uniformly perfect, see [**18**] or [**22**]. Also it is well known, cf. [**24**], that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative (or nonlinearity) $T_f = f''/f'$ of f is hyperbolically bounded, i.e., the norm

$$\|T_f\| = \sup_{z\in\mathbf{D}}(1-|z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch semi-norm of the function $\log f'$. We remark that a holomorphic function f is locally univalent at the point z if and only if $T_f = f''/f'$ is a well-defined holomorphic function near z. Roughly speaking, the quantity T_f measures the deviation of f from orientation-preserving similarities (nonconstant linear functions). In the following it is sometimes essential to consider the semi-norm

$$\|T_f\|_0 = \varlimsup_{|z|\to 1-0}(1-|z|^2)|T_f(z)| = 2\varlimsup_{|z|\to 1-0}(1-|z|)|T_f(z)|$$

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instead of $\|T_f\|$. Also, it is usually much easier to calculate $\|T_f\|_0$ than $\|T_f\|$. We note that $\|T_f\|_0 \le \|T_f\|$ always holds. A nonconstant analytic function f on the unit disk is said to be *almost uniformly locally univalent* if $\|T_f\|_0 < \infty$. For general properties of almost uniformly locally univalent functions, the reader may consult the lecture note [**25**] written by Yamashita.

In this note we will investigate the growth of various quantities for a uniformly locally univalent function in terms of the norm of pre-Schwarzian derivative. Because T_f is invariant under the post-composition by a nonconstant linear function, we may assume that a holomorphic function f on the unit disk is normalized so that $f(0) = 0$ and $f'(0) = 1$. We denote by $\mathcal{A}$ the set of such normalized holomorphic functions on the unit disk. Also we denote by $\mathcal{B}$ the set of normalized uniformly locally univalent functions: $\mathcal{B} = \{f \in \mathcal{A}; \|T_f\| < \infty\}$. The space $\mathcal{B}$ has a structure of nonseparable complex Banach spaces under the Hornich operation [**23**].

For a nonnegative real number λ, we set

$$\mathcal{B}(\lambda) = \{f \in \mathcal{A}; \|T_f\| \le 2\lambda\},$$

here the factor 2 is due to only some technical reason. The functions in $\mathcal{B}(\lambda)$ can be characterized as the following.

Proposition 1.1. *Let a nonnegative constant λ be given. A locally univalent function $f \in \mathcal{A}$ belongs to $\mathcal{B}(\lambda)$ if and only if for any pair of points z_1, z_2 in $\mathbf{D}$ the inequality*

$$|g(z_1) - g(z_2)| \le 2\lambda d_{\mathbf{D}}(z_1, z_2) \tag{1.1}$$

holds, where $g(z) = \log f'(z)$ and $d_{\mathbf{D}}(z_1, z_2) = \tanh^{-1} |(z_1 - z_2)/(1 - \bar{z}_1 z_2)|$ stands for the hyperbolic distance between z_1 and z_2 in the unit disk $\mathbf{D}$.

Proof. First of all note that we can take a holomorphic branch g of $\log f'$ for a locally univalent holomorphic function f on the unit disk. The "only if" part is shown by integrating the inequality $|g'(z)| = |T_f(z)| \le 2\lambda/(1 - |z|^2)$ along the hyperbolic geodesic joining z_1 and z_2. The "if" part directly follows from the observation:

$$\lim_{z' \to z} \frac{|g(z') - g(z)|}{d_{\mathbf{D}}(z', z)} = (1 - |z|^2)|g'(z)|. \qquad \square$$

The following theorem is significant in connection with univalent function theory.

Theorem A (Becker and Pommerenke [**3**], [**4**]). *The set $\mathcal{S}$ of normalized univalent holomorphic functions on the unit disk is contained in $\mathcal{B}(3)$ and contains $\mathcal{B}(1/2)$. The result is sharp.*

We note that the Schwarzian derivative S_f of f can be written as $S_f = (T_f)' - (T_f)^2/2$. Thus the space $\mathcal{B}$ has a close connection with (the Bers embedding of) the universal Teichmüller space $\mathcal{T}$, which is defined as the set of Schwarzian derivatives of those functions in $\mathcal{S}$ which can be quasiconformally extended to the Riemann sphere. Note that $\mathcal{T}$ is a contractible bounded domain in the complex Banach space B_2 consisting of all holomorphic functions φ in the unit disk with finite norm $\|\varphi\|_{B_2} = \sup_{z\in\mathbf{D}}(1-|z|^2)^2|\varphi(z)|$ and that $\{\varphi \in B_2; \|\varphi\|_{B_2} < 2\} \subset \mathcal{T} \subset \{\varphi \in B_2; \|\varphi\|_{B_2} < 6\}$. Thus, one expects that investigating the pre-Schwarzian derivatives is to be effective when considering the Bers boundary of the Teichmüller spaces because the quantity T_f is easier to treat than S_f in some cases. In fact, the space $\mathcal{T}_1 := \{T_f; f \in \mathcal{S}$ has a quasiconformal extension to the Riemann sphere$\}$ can be regarded as a model of the universal Teichmüller space, cf. [**1**] and [**27**]. By the relation between S_f and T_f, we have the estimate $\|S_f\|_{B_2} \le C\|T_f\| + \|T_f\|^2/2$, where C is an absolute constant. At least, we can take $C = 4$, see [**10**]. On the other hand, as is stated in [**7**], the inequality $\|T_f\| \le \|S_f\|_{B_2}$ holds for a strongly normalized function f in the Nehari class, i.e., for a function $f \in \mathcal{A}$ with $f''(0) = 0$ such that $\|S_f\|_{B_2} \le 2$ (see also [**6**]).

However, the pre-Schwarzian derivative T_f has the disadvantage that $\|T_f\|$ is not invariant under Möbius transformations in contrast with the case of S_f. Therefore, the space $\mathcal{T}_1$ is sometimes called "poor man's model" of the universal Teichmüller space. In this respect, it is often advantageous to consider the quantity

$$U_f(z) = \frac{1}{2}\,(1-|z|^2)T_f(z) - \bar{z},$$

because this satisfies the relation $U_{f\circ\omega} = U_f\circ\omega\cdot\omega'/|\omega'|$ for $\omega \in \mathrm{Aut}\,(\mathbf{D})$. The quantity $\sup_{z\in\mathbf{D}}|U_f(z)|$ is sometimes called the order of function f and was extensively investigated by Pommerenke [**16**], see also [**12**].

Note that $\|T_f\| - 2 \le 2\sup_{z\in\mathbf{D}} |U_f(z)| \le \|T_f\| + 2$. However, the quantity $U_f(z)$ is not holomorphic in z and hence is somewhat difficult to treat.

Here, as a result in connection with the Teichmüller space, we mention the following.

Corollary. *For a constant $k \in [0,1)$, let $\mathcal{S}_k$ be the subset of $\mathcal{S}$ consisting of those functions which can be extended to k-quasiconformal self-mappings of the Riemann sphere $\hat{\mathbf{C}}$. Then we have*

$$\mathcal{B}(k/2) \subset \mathcal{S}_k.$$

In particular, we have $\cup_{0<\lambda<1/2}\mathcal{B}(\lambda) \subset \mathcal{T}_1$.

This implication is easily obtained by the λ-lemma, see, for example, [**19**, p. 121]. This already appeared (implicitly) in the paper [**3**] of Becker.

Now we briefly explain the structure of this note. In Section 2, we state sharp distortion, growth and covering theorems for the class $\mathcal{B}(\lambda)$. Those are simple analogues of the results of their paper [**6**], in which Chuaqui and Osgood obtained sharp distortion, growth and covering theorems and an estimate of Hölder continuity for normalized functions in the Nehari class in terms of the Nehari norm of Schwarzian derivatives. (Further developments in this direction can be found in [**8**].) One of the special natures of our class $\mathcal{B}$ is the fact that the condition $\|T_f\| < 2$ implies boundedness of the function f. We shall investigate boundedness criteria in more detail.

As applications of those theorems, Section 3 discusses the growth of coefficients and integral means for the class $\mathcal{B}(\lambda)$.

Section 4 is devoted to explicit estimates of the norm of pre-Schwarzian derivatives for typical classes of univalent functions. To this end, we will employ the subordination method. We will be convinced that the estimate of norm of pre-Schwarzian is easier than that of Schwarzian, in general.

2. Growth estimate for the class $\mathcal{B}(\lambda)$. In the class $\mathcal{B}(\lambda)$ for $0 \le \lambda < \infty$, the function

$$F_\lambda(z) = \int_0^z \left(\frac{1+t}{1-t}\right)^\lambda dt \tag{2.1}$$

is extremal as we shall see later. We remark that $F_\lambda \in \mathcal{A}$ can be defined for any complex number λ and satisfies $T_{F_\lambda} = 2\lambda(1-z^2)^{-1}$, thus $\|T_{F_\lambda}\| = 2|\lambda|$. F_λ may provide an example of a function with small pre-Schwarzian norm which does not belong to typical classes of univalent functions when λ is sufficiently small and $\lambda \notin \mathbf{R}$. In practice, it is important to know the mapping property of F_λ for a real λ. We state a few results about the nature of F_λ.

Lemma 2.1. *For a nonnegative real number λ, the function F_λ is univalent in the unit disk if and only if $0 \le \lambda \le 1$.*

Proof. First, we compute the Schwarzian derivative S_{F_λ} of F_λ. Then we have

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^2|S_{F_\lambda}(z)| = \sup_{z\in\mathbf{D}}(1-|z|^2)^2\,\frac{2\lambda|2z-\lambda|}{|1-z^2|^2} = 2\lambda(\lambda+2).$$

In particular, if $1 < \lambda$, then $2\lambda(\lambda+2) > 6$; thus, the Nehari-Kraus theorem implies that F_λ is not univalent.

On the other hand, if $0 \le \lambda \le 1$, we have $\operatorname{Re} F_\lambda'(z) > 0$ in the unit disk; hence, the Noshiro-Warschawski theorem ensures the univalence of F_λ in this case. □

Lemma 2.2. *For each $\lambda > 0$, the function F_λ never takes the value $F_\lambda(-1)$ in the unit disk, i.e., $F_\lambda(-1) \notin F_\lambda(\mathbf{D})$.*

Remark. This result is not trivial for $\lambda > 1$ because $F_\lambda(-1)$ is an isolated boundary point of $F_\lambda(\mathbf{D})$ in that case.

Proof. Set

$$g(z) = F_\lambda(z) - F_\lambda(-1) = \int_{-1}^z \left(\frac{1+t}{1-t}\right)^\lambda dt.$$

We consider the family of circular arcs

$$\gamma_\alpha(t) = \frac{-ie^{i\alpha t}}{\sin\alpha} + i\cot\alpha, \quad -1 \le t \le 1,$$

which connect -1 and 1 in the unit disk, where α are real numbers with $0 < |\alpha| < \pi/2$. Note the relation

$$\frac{1+\gamma_\alpha(t)}{1-\gamma_\alpha(t)} = e^{-i\alpha} \frac{\sin(\frac{1+t}{2}\alpha)}{\sin(\frac{1-t}{2}\alpha)}.$$

Then we calculate

$$g(\gamma_\alpha(t)) = e^{-i\alpha\lambda} \frac{\alpha}{\sin\alpha} \int_{-1}^{t} \left(\frac{\sin(\frac{1+s}{2}\alpha)}{\sin(\frac{1-s}{2}\alpha)} \right)^{\lambda} e^{i\alpha s}\, ds.$$

Since $\cos(\alpha s) \ge \cos\alpha$ for $-1 \le s \le 1$, we have

$$\operatorname{Re}\{e^{i\alpha\lambda} g(\gamma_\alpha(t))\} \ge \frac{\alpha}{\tan\alpha} \int_{-1}^{t} \left(\frac{\sin(\frac{1+s}{2}\alpha)}{\sin(\frac{1-s}{2}\alpha)} \right)^{\lambda} ds > 0.$$

Hence, $g(\gamma_\alpha(t)) \ne 0$ for $-1 < t < 1$. Since the curve family γ_α sweeps out the unit disk, we obtain the desired conclusion. □

The following result is elementary, but we shall include the proof because of its importance for our aim.

Theorem 2.3 (Distortion theorem). *Let λ be a nonnegative real number. For an $f \in \mathcal{B}(\lambda)$, we have*

$$F_\lambda'(-|z|) = \left(\frac{1-|z|}{1+|z|} \right)^{\lambda} \le |f'(z)| \le \left(\frac{1+|z|}{1-|z|} \right)^{\lambda} = F_\lambda'(|z|) \tag{2.2}$$

and

$$|f(z)| \le F_\lambda(|z|) \tag{2.3}$$

in the unit disk. Furthermore, if f is univalent, then

$$-F_\lambda(-|z|) \le |f(z)| \le F_\lambda(|z|). \tag{2.4}$$

If equality occurs in any of the above inequalities at some point $z_0 \neq 0$, then f must be a rotation of F_λ, i.e., $f(z) = \bar{\mu} F_\lambda(\mu z)$ for a unimodular constant μ.

Proof. Applying Proposition 1.1 to the case $z_1 = z$ and $z_2 = 0$, we see

$$|\log f'(z)| \leq \lambda \log \frac{1+|z|}{1-|z|}. \tag{2.5}$$

Taking the real part of $\log f'$, we obtain (2.2). Furthermore, the integration of (2.2) yields (2.3). Inequality (2.4) can be shown by the same method as in the proof of the Koebe distortion theorem. Equality cases are obvious. □

Corollary 2.4 (Growth and covering theorem). *For $\lambda > 1$, any $f \in \mathcal{B}(\lambda)$ satisfies the growth condition*

$$f(z) = O(1-|z|)^{1-\lambda}$$

as $|z| \to 1$. On the other hand, for $\lambda < 1$, any function $f \in \mathcal{B}(\lambda)$ is bounded with the uniform bound $F_\lambda(1)$.

For all $\lambda > 0$, the image $f(\mathbf{D})$ contains the disk $\{|z| < -F_\lambda(-1)\}$ for $f \in \mathcal{B}(\lambda)$. Furthermore, $\min_{w \in \partial f(\mathbf{D})} |w| = -F_\lambda(-1)$ if and only if f is a rotation of the function F_λ.

Proof. The former part can be directly deduced by integrating inequality (2.2). We now prove the latter part. Let $f \in \mathcal{B}(\lambda)$. Choose a boundary point w_0 of $f(\mathbf{D})$ so that $|w_0| \leq |w_1|$ for all $w_1 \in \partial f(\mathbf{D})$. Let γ be the connected component of $f^{-1}([0, w_0))$ which contains the origin. Note that γ is a properly embedded analytic arc in $\mathbf{D}$ and that f is injective on γ since f is locally biholomorphic. Therefore, by (2.2), we obtain

$$\begin{aligned} |w_0| &\geq \int_{f(\gamma)} |dw| = \int_\gamma |f'(z)|\,|dz| \geq \int_\gamma \left(\frac{1-|z|}{1+|z|}\right)^\lambda |dz| \\ &\geq \int_0^1 \left(\frac{1-t}{1+t}\right)^\lambda dt = -F_\lambda(-1). \end{aligned}$$

If equality holds, then obviously f must be a rotation of F_λ. On the other hand, when $f = F_\lambda$, we can see that equality holds from Lemma 2.2. $\square$

By the same method, we have a similar conclusion to the first half in the above for a function $f \in \mathcal{A}$ with $\|T_f\|_0 \le 2\lambda$. In particular, if $\|T_f\|_0 < 2$, then f is bounded.

We note again that, for $\lambda \le 1/2$, the function $f \in \mathcal{B}(\lambda)$ must be univalent. We also note that, for $0 \le \lambda \le 1$, we have $-F_\lambda(-1) \ge -F_1(-1) = 2\log 2 - 1 = 0.38629\ldots$, therefore the result above is an improvement of the Koebe one-quarter theorem.

Remark. The function F_λ can be expressed in terms of the incomplete beta function or the Gauss hypergeometric function. The incomplete beta function $B_z(p,q)$ is defined and expressed by

$$B_z(p,q) = \int_0^z t^{p-1}(1-t)^{q-1}\,dt = \frac{z^p}{p}\,F(1-q,p;p+1;z)$$

for $z \in \mathbf{C}$ with $0 < \operatorname{Re} z < 1$, where $F(\alpha,\beta;\gamma;x)$ denotes the Gauss hypergeometric function, see [**20**].

Now, by the change of variable $s = (1+t)/2$ in (2.1), we have

$$F_\lambda(z) = 2\int_{1/2}^{(1+z)/2} s^\lambda (1-s)^{-\lambda}\,ds.$$

Thus we have the expression

$$\begin{aligned} F_\lambda(z) &= 2[B_{(1+z)/2}(1+\lambda,1-\lambda) - B_{1/2}(1+\lambda,1-\lambda)] \\ &= \frac{1}{(1+\lambda)2^\lambda}\left[(1+z)^{1+\lambda}F\left(\lambda,\lambda+1;\lambda+2;\frac{1+z}{2}\right)\right. \\ &\qquad\qquad\qquad\qquad \left. - F\left(\lambda,\lambda+1;\lambda+2;\frac{1}{2}\right)\right]. \end{aligned}$$

In particular, we have

$$-F_\lambda(-1) = \frac{1}{2^\lambda(\lambda+1)}\,F(\lambda,\lambda+1;\lambda+2;1/2),$$

which may also be rewritten in terms of the Digamma function [**20**, p. 489]:

$$-F_\lambda(-1) = \lambda\left[\psi\left(\frac{\lambda+1}{2}\right) - \psi\left(\frac{\lambda}{2}\right)\right] - 1, \qquad \psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}.$$

Similarly, we have $F_\lambda(1) = \lambda[\psi(-\lambda/2) - \psi((1-\lambda)/2)] - 1$. It may be useful to note the following elementary estimate:

$$\frac{1}{(\lambda)2^\lambda} < -F_\lambda(-1) < \frac{1}{\lambda+1}.$$

In Corollary 2.4, the case $\lambda = 1$ is critical. In this case, by Theorem 2.3, we can see that, for $f \in \mathcal{B}(1)$,

$$|f(z)| \le F_1(|z|) = 2\log\frac{1}{1-|z|} - |z|.$$

In particular, a function in $\mathcal{B}(1)$ need not be bounded (for instance, F_1). The next proposition gives a boundedness criterion for functions in $\mathcal{B}(1)$.

Proposition 2.5. *If a holomorphic function f on the unit disk satisfies that*

$$\beta(f) := \overline{\lim_{|z|\to 1-0}}\{(1-|z|^2)|T_f(z)| - 2\}\log\frac{1}{1-|z|^2} < -2, \tag{2.6}$$

then f is bounded. Here the constant -2 on the righthand side is sharp.

Proof. By assumption, there exists a $\beta < -2$ such that the lefthand side in (2.6) is less than β. Thus, for some $0 < r_0 < 1$, $(1-|z|^2)|T_f(z)| - 2 \le \beta/\log(1/(1-|z|^2))$, i.e.,

$$|T_f(z)| \le \frac{2}{1-|z|^2} + \frac{\beta}{(1-|z|^2)\log(1/(1-|z|^2))} \tag{2.7}$$

for any $z \in \mathbf{C}$ with $r_0 < |z| < 1$. Here we may choose r_0 sufficiently close to 1 so that $1 - r_0^2 < e^{-1}$. Integrating inequality (2.7), we see that, for $|z| > r_0$,

$$\begin{aligned} |\log f'(z)| &\le \log \frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta\, dt}{(1-t^2)\log \frac{1}{1-t^2}} + C_1 \\ &\le \log \frac{1+|z|}{1-|z|} + \int_{r_0}^{|z|} \frac{\beta\, dt}{2(1-t)\log \frac{1}{2(1-t)}} + C_1 \\ &= \log \frac{1-|z|}{1+|z|} + \frac{\beta}{2} \log\log \frac{1}{2(1-|z|)} + C_2, \end{aligned}$$

where C_1 and C_2 are constants depending only on f and r_0. In particular, we have

$$|f'(z)| \le e^{C_2} \frac{1+|z|}{1-|z|} \left(\log \frac{1}{2(1-|z|)} \right)^{\beta/2}.$$

Since $\beta_1/2 < -1$, the function $((1+t)/(1-t))(\log(1/(2(1-t))))^{\beta/2}$ is integrable on the interval $[r_0, 1)$. Thus f is bounded.

The sharpness follows from the example below. □

Example 2.1. Let a constant $\beta < 0$ be given. Choose a constant $c > 0$ so that $c\beta + 2 \ge 0$. Now we consider the function $f \in \mathcal{A}$ determined by

$$f'(z) = \frac{K}{1-z} \left(1 + c \log \frac{2}{1-z} \right)^{\beta},$$

where $K = (1 + c\log 2)^{-\beta}$. Then this function satisfies that $\|T_f\| = 2$. Moreover, f is bounded in the unit disk if and only if $\beta < -1$.

In fact, first observe that

$$\begin{aligned} T_f(z) &= \frac{1}{1-z} + \frac{c\beta}{(1-z)(1 + c\, \log \frac{2}{1-z})} \\ &= \frac{1}{1-z} \left[1 + \frac{\beta}{\frac{1}{c} + \log \frac{2}{1-z}} \right]. \end{aligned}$$

By the fact that $\mathrm{Re}\,(2/(1-z)) > 1$, one can conclude that $\mathrm{Re}\, w > 1/c \ge -\beta/2$ and $|\mathrm{Im}\, w| < \pi/2$, where $w = 1/c + \log(2/(1-z))$.

Noting that $|1+\beta/w|^2 = 1+\beta(2\mathrm{Re}\, w+\beta)/|w|^2 \le 1$, one can see that $|T_f(z)| \le 1/|1-z| \le 1/(1-|z|)$. In particular, it holds that $(1-|z|^2)|T_f(z)| \le 1+|z| < 2$. On the other hand, it is easy to see that $\lim_{x\to 1-0}(1-x^2)|T_f(x)| = 2$, thus $\|T_f\| = 2$.

Next we shall show that $\beta(f) = 2\beta$. Since $|1+\beta/w| = [1+\beta(2\mathrm{Re}\, w+\beta)/|w|^2]^{1/2} \sim 1+\beta(\mathrm{Re}\, w+\beta/2)/|w|^2 \sim 1+\beta/\mathrm{Re}\, w \sim 1-\beta/\log|1-z|$ as $z\to 1$ and, since the function $t(1+\beta/\log t)$ of t is monotonically increasing for sufficiently large t, we have

$$\begin{aligned}
\beta(f) &= \varlimsup_{\mathbf{D}\ni z\to 1}\{(1-|z|^2)|T_f(z)|-2\}\log\frac{1}{1-|z|^2}\\
&= \varlimsup_{\mathbf{D}\ni z\to 1}\left\{\frac{(1-|z|^2)}{|1-z|}\left(1+\frac{\beta}{\log 1/|1-z|}\right)-2\right\}\log\frac{1}{1-|z|^2}\\
&= \varlimsup_{\mathbf{D}\ni z\to 1}\left\{(1+|z|)\left(1+\frac{\beta}{\log 1/(1-|z|)}\right)-2\right\}\log\frac{1}{1-|z|^2}\\
&= \varlimsup_{x\to 1-0}\left\{-(1-x)\log\frac{1}{1-x^2}+(1+x)\beta\frac{\log\frac{1}{1-x^2}}{\log\frac{1}{1-x}}\right\} = 0+2\beta.
\end{aligned}$$

In particular, we can conclude that f is bounded if $\beta < -1$ by Proposition 2.5.

On the other hand, noting that $\int_{r_0}^1 (1/(1-x))[\log(1/(1-x))]^\beta = \infty$, in the case that $\beta \ge -1$, we can directly see $\varlimsup_{x\to 1-0} f(x) = +\infty$; thus f is unbounded.

We conclude this section with the Hölder continuity of functions in $\mathcal{B}(\lambda)$. Recall the following fundamental fact due to Hardy and Littlewood.

Theorem B (cf. [**9**]). *Let α be a constant such that $0 < \alpha \le 1$. A holomorphic function f on the unit disk is Hölder continuous of exponent α if and only if $f'(z) = O(1-|z|)^{\alpha-1}$ as $|z|\to 1$.*

Combining this with Theorem 2.3, we have

Theorem 2.6. *Let $0 \le \lambda < 1$. Then any function $f \in \mathcal{B}(\lambda)$ is Hölder continuous of exponent $1-\lambda$ on the unit disk.*

Remarks. 1. We can directly see that $|f(z_1) - f(z_2)| \le C(1-\lambda)|z_1 - z_2|^{1-\lambda}$ for any pair of points $z_1, z_2 \in \mathbf{D}$, where C is an absolute constant owing to the estimate $\int_r^s ((1+t)/(1-t))^\lambda \, dt \le 2^\lambda(1-\lambda)^{-1}((1-r)^{1-\lambda} - (1-s)^{1-\lambda}) \le 2^\lambda(1-\lambda)^{-1}(s-r)^{1-\lambda}$ for $0 < r < s < 1$.

2. Chuaqui and Osgood proved in [**6**] that a strongly normalized function f in the Nehari class is Hölder continuous of exponent $\sqrt{1-\lambda}$ where $\|S_f\|_{B_2} = 2\lambda$. Their result is better than that obtained by combining the estimate $\|T_f\| \le \|S_f\|_{B_2}$ with the above theorem.

3. Coefficient estimate for the class $\mathcal{B}(\lambda)$. Let $f(z) = z + a_2 z^2 + \cdots \in \mathcal{B}(\lambda)$. Then, by definition, $|T_f(0)| \le 2\lambda$, which implies $|a_2| \le \lambda$. Of course, this is sharp because equality holds for the function F_λ. But, a function in $\mathcal{B}(\lambda)$ essentially different from F_λ may attain this maximum. For instance, consider the function $f(z) = (e^{2\lambda z} - 1)/2\lambda$.

As for the growth of coefficients of a holomorphic function $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ in the unit disk, it is convenient to consider the integral mean of exponent $p \in \mathbf{R}$:

$$I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

For a function $f(z) = z + a_2 z^2 + \cdots$ in $\mathcal{B}(\lambda)$, by Theorem 2.3 we have $I_1(r, f') = O(1-r)^{-\lambda}$ so that $I_1(r, f) = O(1-r)^{-\lambda+1}$. Then, for $n > 1$ and $r = 1 - 1/n$, the Cauchy integral formula implies

$$\begin{aligned} |a_n| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})(re^{i\theta})^{-n} \, d\theta \right| \le r^{-n} I_1(r, f) \le M r^{-n} (1-r)^{-\lambda+1} \\ &= M\left(1 - \frac{1}{n}\right)^{-n} n^{\lambda-1} < \frac{eMn}{n-1} \, n^{\lambda-1}. \end{aligned}$$

Hence $|a_n| = O(n^{\lambda-1})$ as $n \to \infty$. Moreover, if $\lambda < 1$ and if f is univalent, then f is bounded by Corollary 2.4, so

$$\mathrm{Area}\,(f(\mathbf{D})) = \pi\left(1 + \sum_{n=2}^{\infty} n|a_n|^2\right) < \infty.$$

By this simple observation we have $a_n = o(n^{-1/2})$ as $n \to \infty$.

But we can improve the exponents in these order estimates. We now explain this.

For $\lambda > 0$, we set

$$\alpha(\lambda) = \frac{\sqrt{1+4\lambda^2}-1}{2}.$$

Noting $\alpha(\lambda) = 2\lambda^2/(\sqrt{1+4\lambda^2}+1)$, then we have

$$\frac{\lambda^2}{\lambda+1} < \alpha(\lambda) < \min\left\{\lambda^2, \frac{2\lambda^2}{2\lambda+1}\right\} \le \min\{\lambda^2, \lambda\}.$$

We also note that

$$\alpha(\lambda) = \lambda - \frac{1}{2} + \frac{1}{8\lambda} + O\left(\frac{1}{\lambda^3}\right), \quad \lambda \to \infty.$$

For this number, we have the next result.

Theorem 3.1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be in $\mathcal{B}(\lambda)$. Then, for any $\varepsilon > 0$ and a real number p, we have $I_p(r, f') = O(1-r)^{-\alpha(|p|\lambda)-\varepsilon}$, in particular, $a_n = O(n^{\alpha(\lambda)-1+\varepsilon})$.*

This immediately follows from the next result.

Theorem C [**17**, Lemma 5.3]. *Let h be a holomorphic function in the unit disk such that*

$$(1-|z|)\left|\frac{h'(z)}{h(z)}\right| \le c, \quad r_0 \le |z| < 1,$$

for constants $c > 0$ and $r_0 < 1$. Then $I_p(r, h) = O(1-r)^{-\beta}$ where $\beta = (\sqrt{1+4p^2c^2}-1)/2$ and $p \in \mathbf{R}$.

We note that this is a consequence of the differential inequality of Fuchsian type:

$$I_p''(r, h) \le \frac{p^2}{2\pi}\int_0^{2\pi} |h(z)|^p \left|\frac{h'(z)}{h(z)}\right|^2 d\theta \le \frac{p^2c^2}{(1-r)^2}\, I_p(r, h).$$

Moreover, if f is univalent, we may have a better growth estimate for the coefficients. First we remind the reader of the following result due to Littlewood, Paley, Clunie, Pommerenke and Baernstein II (see [**2**], [**19**, Theorem 8.8] and [**13**, Theorem 3.7]).

Theorem D. *Suppose that* $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$ *satisfies* $f(z) = O(1-|z|)^{-\alpha}$. *If* $0.491 < \alpha \le 2$, *then* $\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta = O(1-r)^{-\alpha}$ *and* $a_n = O(n^{\alpha-1})$. *If* $\alpha = 0$, *in other words if* f *is bounded, then* $\int_0^{2\pi} |f'(re^{i\theta})|\, d\theta = O(1-r)^{-0.491}$ *and* $a_n = O(n^{0.491-1})$.

In view of Corollary 2.4, we have the following result:

Theorem 3.2. *Let* $f(z) = z + a_2z^2 + \cdots \in \mathcal{S}$. *If* $f \in \mathcal{B}(\lambda)$ *with* $1.491 < \lambda \le 3$, *then it holds that* $a_n = O(n^{\lambda-2})$ *as* $n \to \infty$. *This order estimate is best possible.*

In order to see the sharpness, we may consider the function $f(z) = (1-z)^{1-\lambda} = 1 + a_1z + a_2z^2 + \cdots$ for $1 < \lambda$. We note that f is univalent in the unit disk if $1 < \lambda \le 3$. For this function we can see that $\|T_f\| = 2\lambda$ and $a_n = \Gamma(\lambda+n-1)/n!\Gamma(\lambda-1) \sim n^{\lambda-2}$ as $n \to \infty$ by Stirling's formula.

On the other hand, in the case that f is univalent with $\|T_f\| < 3$, the situation seems rather complicated. Given a holomorphic function $f(z) = z + a_2z^2 + \cdots$ in the unit disk, let $\gamma(f)$ denote the infimum of exponents γ such that $a_n = O(n^{\gamma-1})$ as $n \to \infty$, i.e.,

$$\gamma(f) = \varlimsup_{n\to\infty} \frac{\log n|a_n|}{\log n}.$$

Also, for a subset X of $\mathcal{A}$, we denote by $\gamma(X)$ the supremum of $\{\gamma(f); f \in X\}$. As for $\gamma(\mathcal{S}_b)$, where $\mathcal{S}_b$ denotes the class of normalized bounded univalent functions in the unit disk, it has been shown ([**5**] and [**14**]) that $0.24 < \gamma(\mathcal{S}_b) < 0.4886$ and conjectured by Carleson and Jones that $\gamma(\mathcal{S}_b) = 0.25$. We also remark that the growth of coefficients seems to involve the irregularity of boundary of image under f when f is bounded and univalent (see [**19**, Chapter 10]) and, recently, Makarov and Pommerenke observed a remarkable phenomenon of phase transition of the functional $\gamma(f)$ with respect to the Minkowski dimension of the boundary curve [**14**].

Now we turn to our case. Theorem 3.1 implies $\gamma(\mathcal{B}(\lambda)) \le \alpha(\lambda)$. The above example $(1-z)^{1-\lambda}$ (or $-\log(1-z)$ when $\lambda = 1$) shows $\lambda - 1 \le \gamma(\mathcal{B}(\lambda))$. By standard calculations, we can see that the extremal function F_λ also satisfies $\gamma(F_\lambda) = \lambda - 1$.

To construct an analytic function with curious boundary behavior, the Hadamard gap series is often used, e.g., [**19**, Section 8.6]. Here we present a simple example of such a kind to improve the above lower estimate of $\gamma(\mathcal{B}(\lambda))$.

Example 3.1 (Gap series construction). Let q be a fixed integer greater than 1. We consider the function

$$g(z) = z + z^q + z^{q^2} + z^{q^3} + \cdots$$

in the unit disk, which can be characterized by the functional equation $g(z) = z + g(z^q)$ with the initial condition $g(0) = 0$. We note that this is a Bloch function satisfying $\|g'\| \le q/(q-1)$, cf. [**19**, Section 8.6]. Let $t > 0$ be a constant. Then the function $h(z) = e^{tg(z)} = b_0 + b_1 z + b_2 z^2 + \cdots$ obeys the functional equation $h(z) = e^{tz} h(z^q)$. Thus the coefficients b_n are all positive and calculated by the relations

$$b_{kq+m} = \sum_{l=0}^{k} c_{lq+m} b_{k-l},$$

where $c_n = t^n/n!$. Letting $m = 0$, we have $b_{kq} = c_0 b_k + \cdots + c_{kq} b_0 > b_k$. In particular, we know $b_{q^k} > b_{q^{k-1}} > \cdots > b_1 = t$. Therefore, we have $\overline{\lim} \log b_n / \log n \ge 0$.

On the other hand, the function $f \in \mathcal{A}$ determined by $f' = h$ satisfies $T_f = tg'$; therefore, $\|T_f\|$ can be made arbitrarily small by letting t be sufficiently small. This shows $\gamma(\mathcal{B}(\lambda)) \ge 0$ for any $\lambda > 0$.

Summarizing these observations, we have the next result.

Theorem 3.3. *For any* $\lambda \in (0, \infty)$, *we have*

$$\max\{0, \lambda - 1\} \le \gamma(\mathcal{B}(\lambda)) \le \alpha(\lambda) = \frac{\sqrt{1 + 4\lambda^2} - 1}{2}. \tag{3.1}$$

In particular, $\gamma(\mathcal{B}(\lambda)) = O(\lambda^2)$ *as* $\lambda \to 0$.

Remarks. 1. Recently, Chuaqui, Osgood and Pommerenke [**7**] proved $\gamma(\mathcal{B}(\lambda)) \geq c\lambda^2$ for some positive constant c when λ is sufficiently small. Their construction is fairly technical and complicated, so our simple Example 3.1 still seems meaningful to be mentioned here.

2. More generally, by Theorem C, for any $f \in \mathcal{A}$ we have the estimate

$$\gamma(f) \leq \frac{1}{2}\Big(\sqrt{1+\|T_f\|_0^2} - 1\Big).$$

3. For $0 < \lambda \leq 1/2$, we note that $\alpha(\lambda) \leq \lambda^2 - 2\lambda^4/3 \leq 5/24 = 0.2083\ldots$ because $\sqrt{1+x} < 1 + x/2 - x^2/(6+4\sqrt{2}) < 1 + x/2 - x^2/12$ for $0 < x \leq 1$. We remark again that $\mathcal{B}(1/2) \subset \mathcal{S}_b$.

Now we mention a connection with integral means for univalent functions. For a univalent function $f \in \mathcal{S}$ and a real number p, we set

$$\beta_f(p) = \varlimsup_{r\to 1-0} \frac{\log \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta}{\log \frac{1}{1-r}} = \varlimsup_{r\to 1-0} \frac{\log I_p(r, f')}{\log \frac{1}{1-r}}.$$

The Brennan conjecture asserts that $\beta_f(-2) \leq 1$ for every univalent holomorphic function f, cf. [**19**, Chapter 8].

For $f \in \mathcal{B}(\lambda)$, as a corollary of Theorem 3.1, we have the next

Theorem 3.4. *For* $f \in \mathcal{B}(\lambda)$ *and* $p \in \mathbf{R}$*, the inequality*

$$\beta_f(p) \leq \alpha(|p|\lambda) = \frac{\sqrt{1+4p^2\lambda^2} - 1}{2}$$

holds. In particular, the Brennan conjecture is true for any univalent function f *with* $\|T_f\| \leq \sqrt{2}$.

A similar statement can be found in [**19**, Exercise 8.3.4].

4. Subordination principles for norm estimate of pre-Schwarzian derivative. First we state general and useful principles for estimation of the norm of T_f. A holomorphic function $f$on the unit disk is said to be *weakly subordinate* to another g if f can be written as $f = g \circ \omega$, where ω is a holomorphic self-mapping of the unit disk. Furthermore, if ω can be taken so as to satisfy $\omega(0) = 0$, the function f is said to be *subordinate* to g.

We remark that the Schwarz-Pick lemma states that any holomorphic self-mapping ω of the unit disk satisfies

$$\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \leq \frac{1}{1 - |z|^2} \tag{4.1}$$

for any point $z \in \mathbf{D}$.

We also note that if $g \in \mathcal{S}$, then f is weakly subordinate to g if and only if $f(\mathbf{D}) \subset g(\mathbf{D})$.

The following always generates a sharp result for fixed g. The idea is due to Littlewood.

Theorem 4.1 (Subordination principle I). *Let $g \in \mathcal{B}$ be given. For a holomorphic function f in the unit disk, if f' is weakly subordinate to g', then we have $\|T_f\| \leq \|T_g\|$. In particular, f is uniformly locally univalent on the unit disk.*

Proof. By assumption, a holomorphic function $\omega : \mathbf{D} \to \mathbf{D}$ exists such that $f' = g' \circ \omega$. Therefore, $T_f = T_g \circ \omega \cdot \omega'$. Thus (4.1) implies the following:

$$(1 - |z|^2)|T_f(z)| = (1 - |z|^2)|T_g(\omega)||\omega'| \leq (1 - |\omega|^2)|T_g(\omega)| \leq \|T_g\|,$$

which leads to the conclusion. $\square$

The analogous statement does not follow for the semi-norm $\| \cdot \|_0$; however, the following form can be proved.

Proposition 4.2. *There exists an absolute constant $c_0 > 0$ such that for any $g \in \mathcal{B}$ the inequality*

$$c_0 \|T_g\| \leq \sup_f \|T_f\|_0 \leq \|T_g\|$$

holds, where the supremum is taken over all holomorphic functions f on $\mathbf{D}$ for which f' are weakly subordinate to g'.

Proof. Actually, a single ω is sufficient. Take the holomorphic function f in the unit disk satisfying $f' = g' \circ \omega$, where $\omega(z) = \exp[-(1+z)/(1-z)]$ is a holomorphic universal covering map of the punctured disk $\mathbf{D} \setminus \{0\}$. The preimage of the circle $|w| = e^{-a}$ under ω is a horocircle, say C_a, tangent to $\partial\mathbf{D}$ at 1. Since $(1-|z|^2)|\omega'(z)|/(1-|\omega(z)|^2) = a/\sinh a$ along the horocircle, we know

$$\varlimsup_{C_a \ni z \to 1} (1-|z|^2)|T_f(z)| = \frac{a}{\sinh a} \max_{|w|=e^{-a}} (1-|w|^2)|T_g(w)| = 2ae^{-a}M_a,$$

where $M_a = \max_{|w|=e^{-a}} |T_g(w)|$. In particular, we have

$$2ae^{-a}M_a \le \varlimsup_{z\to 1}(1-|z|^2)|T_f(z)| \le \|T_f\|_0.$$

When $a \ge 1$, we have $(1-e^{-2a})M_a \le M_a \le M_1 \le (e/2)\|T_f\|_0$. When $a < 1$, we have $(1-e^{-2a})M_a \le (\sinh a/a)\|T_f\|_0 \le \sinh 1\|T_f\|_0 \le (e/2)\|T_f\|_0$. Therefore, we have $\|T_g\| = \sup_{a>0}(1-e^{-2a})M_a \le (e/2)\|T_f\|_0$. $\square$

As a typical application of the subordination principle, we exhibit the following.

Example 4.1 (Nunokawa [**15**]). If $f \in \mathcal{A}$ satisfies $\operatorname{Re} f' > 0$ on the unit disk, then $\|T_f\| \le 2$. The bound is sharp.

Proof. The condition $\operatorname{Re} f' > 0$ is equivalent to the assertion that f' is subordinate to the function $F_1'(z) = (1+z)/(1-z)$. Thus we have $\|T_f\| \le \|T_{F_1}\| = 2$. $\square$

This result remains true if we allow f' to be a Gelfer function where an analytic function g on the unit disk with $g(0) = 1$ is called *Gelfer* when $g(z) + g(w) \ne 0$ for all $z, w \in \mathbf{D}$. In fact, this follows directly from the estimate $(1-|z|^2)|g'(z)/g(z)| \le 2$ for a Gelfer function g, see [**26**].

The next is a variant of the subordination principle.

Theorem 4.3 (Subordination principle II). *Let $g \in \mathcal{B}$ be given. For $f \in \mathcal{A}$, if $zf'(z)/f(z)$ is subordinate to g', then we have*

$$\|T_f\| \le \sup_{z\in\mathbf{D}}(1-|z|^2)\left(\left|\frac{g'(z)-1}{z}\right| + |T_g(z)|\right) \tag{4.2}$$

$$\le \sup_{z\in\mathbf{D}}(1-|z|^2)\left|\frac{g'(z)-1}{z}\right| + \|T_g\|. \tag{4.3}$$

Proof. By assumption, a holomorphic function $\omega : \mathbf{D} \to \mathbf{D}$ exists with $\omega(0) = 0$ such that $zf'(z)/f(z) = g'(\omega(z))$. By taking the logarithmic derivative, we have the following formula.

$$T_f = \frac{f'}{f} - \frac{1}{z} + \frac{g''(\omega)}{g'(\omega)}\,\omega' = \frac{\omega}{z}\,\frac{g'(\omega)-1}{\omega} + T_g(\omega)\omega'.$$

From this, we can easily have the desired estimate. □

The following is a simple application of this principle.

Theorem 4.4. *If $f \in \mathcal{A}$ satisfies $|zf'(z)/f(z)-1| < 1$, then we have the estimate $\|T_f\| \le 2.25$. Equality holds if and only if f is a rotation of the function ze^z.*

Remark. In this case f satisfies $\operatorname{Re} zf'(z)/f(z) > 0$; thus, f is starlike, in particular, univalent in the unit disk.

Proof. We only have to apply estimate (4.2) with $g(z) = z + z^2/2$. Then we have $\|T_f\| \le \sup(2+|z|-|z|^2) = 9/4$, where the supremum is attained only by $|z| = 1/2$. Thus if $\|T_f\| = 9/4$, then $|\omega|$ must be the constant 1, whence f is a rotation of ze^z. Conversely, it is clear that the function $f(z) = ze^{\mu z}$ with $|\mu| = 1$ satisfies $\|T_f\| = 9/4$. □

Finally we consider uniformly convex functions

$$\mathrm{UCV} = \left\{ f \in \mathcal{S};\ \operatorname{Re}\left(1 + (z-\zeta)\frac{f''(z)}{f'(z)}\right) \ge 0, \quad \forall\, z,\ \forall\, \zeta \in \mathbf{D} \right\}.$$

For the geometric meaning of this class, see [**11**]. Rønning gave a simple characterization for this class.

Theorem E (Rønning [**21**]). *A function $f \in \mathcal{A}$ is uniformly convex if and only if $zT_f(z) \in W$ for any $z \in \mathbf{D}$ where W is the domain $\{w = u + iv; v^2 < 2u + 1\}$.*

We note that a conformal map $g : \mathbf{D} \to W$ with $g(0) = 0$ is given by

$$g(z) = \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 = \frac{8z}{\pi^2}\left(1 + \frac{z}{3} + \frac{z^2}{5} + \frac{z^3}{7} + \cdots\right)^2. \tag{4.4}$$

Therefore, $f \in \mathcal{A}$ is uniformly convex if and only if $zT_f(z)$ is subordinate to the function g, i.e., a holomorphic function $\omega : \mathbf{D} \to \mathbf{D}$ exists with $\omega(0) = 0$ such that $zT_f(z) = g(\omega(z))$. Since g has positive Taylor coefficients, we see that $|zT_f(z)| \le g(|\omega(z)|) \le g(|z|)$. Hence, we have

$$\|T_f(z)\| \le \sup_{0<x<1} (1 - x^2)\frac{g(x)}{x} = \sup_{0<t<\infty} h(t),$$

where

$$h(t) = \frac{8t^2}{\pi^2}\frac{\cosh t}{\sinh^2 t}$$

and $(1 + \sqrt{x})/(1 - \sqrt{x}) = e^t$. By the logarithmic differentiation, we have

$$\frac{h'(t)}{h(t)} = \frac{2\sinh 2t - t(\cosh 2t + 3)}{t \sinh 2t} = \frac{N(t)}{t\sinh 2t}.$$

Since $N''(t) = 4(\tanh 2t - t)/\cosh 2t$ has the unique zero t_0 in $(0, \infty)$, the function $N'(t) = 3(\cosh 2t - 1) - 2t \sinh 2t$ attains its maximum at t_0. Since $N'(0) = 0$ and $N'(t) \to -\infty$ as $t \to \infty$, the function $N'(t)$ has the unique zero $t_1 > t_0$ in $(0, \infty)$. For exactly the same reason, the function $N(t)$ has the unique zero $t_2 > t_1$ in $(0, \infty)$. Thus, $h(t)$ assumes its maximum at the point $t = t_2$. By a numerical calculation, we have $t_2 = 1.6061152988\ldots$, and $h(t_2) = 0.94774221287\ldots$. Therefore, we summarize as follows.

Theorem 4.5. *If $f \in \mathcal{A}$ is uniformly convex, then we have*

$$\|T_f\| \le h(t_2) = 0.94774\ldots,$$

where equality occurs only when f is a rotation of the function $F \in \mathcal{A}$ determined by $T_F(z) = g(z)/z$, where g is given by (4.4).

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REFERENCES

1. K. Astala and F.W. Gehring, *Injectivity, the BMO norm and the universal Teichmüller space*, J. Analyse Math. **46** (1986), 16–57.

2. A. Baernstein II, *Coefficients of univalent functions with restricted maximum modulus*, Complex Variables Theory Appl. **5** (1986), 225–236.

3. J. Becker, *Löwnersche Differentialgleichung and quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23–43.

4. J. Becker and Ch. Pommerenke, *Schlichtheitskriterien und Jordangebiete*, J. Reine Angew. Math. **354** (1984), 74–94.

5. L. Carleson and P.W. Jones, *On coefficient problems for univalent functions and conformal dimension*, Duke Math. J. **66** (1992), 169–206.

6. M. Chuaqui and B. Osgood, *Sharp distortion theorems associated with the Schwarzian derivative*, J. London Math. Soc. (2) **48** (1993), 289–298.

7. M. Chuaqui, B. Osgood and Ch. Pommerenke, *On the coefficients of small univalent functions*, Resultate Math. **33** (1998), 79–86.

8. M. Chuaqui and Ch. Pommerenke, *Characteristic properties of Nehari functions*, Pacific J. Math. **188** (1999), 83–94.

9. P.L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

10. P.L. Duren, H.S. Shapiro and A.L. Shields, *Singular measures and domains not of Smirnov type*, Duke Math. J. **33** (1966), 247–254.

11. A.W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), 87–92.

12. R. Harmelin and D. Minda, *Quasi-invariant domain constants*, Israel J. Math. **77** (1992), 115–127.

13. W.K. Hayman, *Multivalent functions*, 2nd ed., Cambridge University Press, London, 1994.

14. N.G. Makarov and Ch. Pommerenke, *On coefficients, boundary size and Hölder domains*, Ann. Acad. Sci. Fenn. Ser. A I Math. **22** (1997), 305–312.

15. M. Nunokawa, *On the univalency and multivalency of certain analytic functions*, Math. Z. **104** (1968), 394–404.

16. Ch. Pommerenke, *Linear-invariante Familien analytischer Funktionen* I, II, Math. Ann. **155** (1964), 108–154, ibid. **156** (1964), 226–262.

17. ———, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

18. ———, *Uniformly perfect sets and the Poincaré metric*, Ark. Mat. **32** (1979), 192–199.

19. ———, *Boundary behaviour of conformal maps*, Springer-Verlag, Berlin, New York, 1992.

20. A.P. Prudnikov, Yu. A. Brychkov and O.I. Marichev, *Integrals and series*, Vol. 3, Gordon & Breach Science Publ., London, New York, 1990.

21. F. Rønning, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), 189–196.

22. T. Sugawa, *Various domain constants related to uniform perfectness*, Complex Variables Theory Appl. **36** (1998), 311–345.

23. S. Yamashita, *Banach spaces of locally schlicht functions with the Hornich operations*, Manuscripta Math. **16** (1975), 261–275.

24. ———, *Almost locally univalent functions*, Monatsh. Math. **81** (1976), 235–240.

25. ———, *Lectures on locally schlicht functions*, Mathematics Seminar notes, Tokyo Metropolitan Univ., Tokyo, 1977.

26. ———, *Gelfer functions, integral means, bounded mean oscillation, and univalency*, Trans. Amer. Math. Soc. **321** (1990), 245–259.

27. I.V. Zhuravlev, *Model of the universal Teichmüller space*, Siberian Math. J. **27** (1986), 691–697.

Department of Mathematics, College of Education, Yeungnam University, 214-1 Dae-Dong, Gyongsan 712-749, Korea
E-mail address: kimyc@ynucc.yeungnam.ac.kr

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: sugawa@kusm.kyoto-u.ac.jp

Current address: Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan
E-mail address: sugawa@math.sci.hiroshima-u.ac.jp