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ON THE SINGULARITIES AT INFINITY OF PLANE ALGEBRAIC CURVES

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ABSTRACT. We study polynomials in two complex variables with no critical points and with at most one irregular value at infinity. We give some applications to polynomial automorphisms.

Introduction. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial of degree $d > 1$ with finite set of critical points, i.e., such that the partial derivatives $(\partial f/\partial X)$, $(\partial f/\partial Y)$ do not have common factors. Then the polynomials $f - t$, $t \in \mathbf{C}$ have no multiple factors.

Let C^t be the projective closure of the fiber $f^{-1}(t)$. If $F(X, Y, Z)$ is the homogeneous form corresponding to $f = f(X, Y)$, then C^t is given by the equation $F(X, Y, Z) - tZ^d = 0$. Let $L_\infty \subset \mathbf{P}^2(\mathbf{C})$ be the line at infinity defined by $Z = 0$, and let $C_\infty = C^t \cap L_\infty$. Then the set C_∞ is described by equations $F(X, Y, 0) = Z = 0$ in $\mathbf{P}^2(\mathbf{C})$. For every point $p \in C_\infty$, we consider the Milnor number $\mu_p^t = \mu_p(C^t)$, and we put $\mu_p^{\min} = \inf_{t \in \mathbf{C}} \mu_p^t$. The set $\Lambda(f) = \{t \in \mathbf{C} : \mu_p^t > \mu_p^{\min} \text{ for a } p \in C_\infty\}$ is finite (see [**6**]). The elements of $\Lambda(f)$ are called irregular values of f. We put according to Broughton $\lambda^t(f) = \sum_{p \in C_\infty} (\mu_p^t - \mu_p^{\min})$ and $\lambda(f) = \sum_{t \in \mathbf{C}} \lambda^t(f)$.

Equivalent definitions of irregular values are discussed in [**10**]. A polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ is called a coordinate polynomial if there is a polynomial $g : \mathbf{C}^2 \to \mathbf{C}$ such that $\mathbf{C}[X, Y] = \mathbf{C}[f, g]$. The famous Abhyankar-Moh theorem [**2**] can be stated as follows: *an affine plane curve is isomorphic to a line if and only if its minimal equation is a coordinate polynomial.* Using the Abhyankar-Moh theorem, Ephraim proved [**11**] that a polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ is a coordinate polynomial if and only if f has no critical points and $\Lambda(f) = \varnothing$.

In this note we study polynomials in two complex variables with no critical points in $\mathbf{C}^2$. Our aim is to characterize polynomials with one

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irregular value. Improving a recent result by Assi ([**3**], [**4**]), we give a description of the irregular fiber of such a polynomial (Theorem 1) and apply it to the estimation of the number of points at infinity (Theorem 2). Then we give a discriminant criterion for polynomials to have one irregular value (Theorem 3) and apply it to polynomial automorphisms (Theorem 4).

All theorems are stated in Section 1 and their proofs are given in Section 2. We end the paper by open questions concerning polynomials in two complex variables.

1. Results. If $f : \mathbf{C}^2 \to \mathbf{C}$ is a polynomial with no critical points, then the irreducible components of any fiber $f^{-1}(t)$ are smooth and pairwise disjoint. The following theorem is the main result of this note.

Theorem 1. *Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with no critical points. Fix $t_0 \in \mathbf{C}$. Then the following conditions are equivalent*:

(i) $\Lambda(f) = \{t_0\}$.

(ii) *The fiber $f^{-1}(t_0)$ is an affine reducible curve. All irreducible components of $f^{-1}(t_0)$ are rational and at least one component is isomorphic to a line. If there is only one component isomorphic to a line, then every component not isomorphic to a line has exactly two branches at infinity. If there are $l > 1$ components isomorphic to a line, then there is only one component of $f^{-1}(t_0)$ not isomorphic to a line. It has $l+1$ branches at infinity.*

Note here that an affine curve is isomorphic to a line if and only if it is rational, smooth and has exactly one branch at infinity. Then both cases described in (ii) can occur.

Examples. Let $f(X,Y) = YP(XY)$ where $P(T) \in \mathbf{C}[T]$ is a polynomial of one variable with simple, nonzero roots. Then f has no critical points and $f^{-1}(0) = \{Y = 0\} \cup \bigcup_i \{XY - t_i = 0\}$ where t_i are roots of $P(T)$. Theorem 1 shows that $\Lambda(f) = \{0\}$.

Let $f(X,Y) = Q(X)^2 Y + Q(X)$ where $Q(X) \in \mathbf{C}[X]$ is a polynomial with simple roots of degree $q > 0$. Then f has no critical points and the fiber $f^{-1}(0)$ has q irreducible components $\{X - x = 0\}$ where $Q(x) = 0$

isomorphic to $\mathbf{C}$ and one component $\{Q(X)Y+1=0\}$ which is rational with $q+1$ branches at infinity. Using Theorem 1, we get $\Lambda(f)=\{0\}$.

Any polynomial of degree ≤ 4 has at most one irregular value. The polynomial $f(X,Y)=Y((XY-1)^2+X^2Y)$ is of degree 5, has no critical points and has two irregular values: $\Lambda(f)=\{0,1\}$.

Corollary to Theorem 1 ([**3**], [**4**]). *If $f:\mathbf{C}^2\to\mathbf{C}$ has no critical points and $\Lambda(f)=\{t_0\}$, then $f=t_0+\phi(1+\phi\psi)$ in $\mathbf{C}[X,Y]$ where ϕ is a coordinate polynomial and $\psi\notin\mathbf{C}$.*

Proof. By Theorem 1, we can write $f^{-1}(t_0)=\Gamma\cup\Gamma'$, where Γ is isomorphic to a line and Γ' is not. By the Abhyankar-Moh theorem, there is a coordinate polynomial ϕ such that $\Gamma=\phi^{-1}(0)$. Since Γ and Γ' are disjoint and ϕ is a coordinate polynomial, the curve Γ' has an equation of the form $1+\phi\psi=0$, where ψ is a polynomial. One has $\psi\notin\mathbf{C}$ for Γ' is not isomorphic to a line. Since the polynomial $f-t_0$ is reduced, we get $f-t_0=\phi(1+\phi\psi)$.

If a polynomial $f:\mathbf{C}^2\to\mathbf{C}$ is a coordinate polynomial, then the curve $f^{-1}(0)$ has exactly one point at infinity. Using Theorem 1, we shall prove

Theorem 2. *Let $f:\mathbf{C}^2\to\mathbf{C}$ be a polynomial with no critical points and with at most one irregular value. If f is of degree d and the curve $f^{-1}(0)$ has c points at infinity, then $c\leq(d+1)/2$.*

The example given below shows that our estimation is exact.

Example. Let $f(X,Y)=Y\prod_{i=1}^{k}(XY-1-iY^2)$. One checks that f has no critical points. Using Theorem 1 we see that $\Lambda(f)=\{0\}$. Here $d=2k+1$ and $c=k+1=(d+1)/2$.

Let $f=f(X,Y)\in\mathbf{C}[X,Y]$ be a polynomial such that $\deg_Y f=\deg f=d>0$, and let T be a new variable. Let us consider the Y-discriminant $\Delta(X,T)=\operatorname{disc}_Y(f(X,Y)-T)$ of the polynomial $f(X,Y)-T\in\mathbf{C}[X,T][Y]$. It is easy to see that $\Delta(X,T)\neq 0$ in $\mathbf{C}[X,T]$. Moreover, $\Delta(X,t)\neq 0$ for every $t\in\mathbf{C}$ if and only if the set of critical points of f is finite.

Theorem 3. *Let $t_0 \in \mathbf{C}$. The following two conditions are equivalent*:

(i) *The polynomial f has no critical points and $\Lambda(f) \subset \{t_0\}$.*

(ii) *The set of critical points of f is finite and deg $\Delta(X, t_0) = d - 1$.*

We can use the discriminant to characterize the coordinate polynomials.

Corollary to Theorem 3. *Let us write $\Delta(X,T) = \Delta_0(T)X^N + \cdots + \Delta_N(T)$ with $\Delta_0(T) \neq 0$ in $\mathbf{C}[T]$. Then f is a coordinate polynomial if and only if $N = d - 1$ and $\Delta_0(T) = const.$*

Proof. By the Ephraim theorem, f is a coordinate polynomial if and only if f has no critical point and $\Lambda(f) \subset \{t\}$ for all $t \in \mathbf{C}$. Use Theorem 3.

Other characterizations of coordinate polynomials were given in [**8**] and [**15**].

We call a polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ a Jacobian polynomial if there is a polynomial $g : \mathbf{C}^2 \to \mathbf{C}$ such that $(\partial f/\partial X)(\partial g/\partial Y) - (\partial f/\partial Y)(\partial g/\partial X) = 1$. Every Jacobian polynomial has no critical points and every coordinate polynomial is a Jacobian polynomial. The plane Jacobian conjecture (see [**5**], [**16**]) can be stated as follows:

(JC) Every Jacobian polynomial is a coordinate polynomial.

Theorem 4. *Let $f = f(X,Y) \in \mathbf{C}[X,Y]$ be a polynomial such that* $\deg_Y f = \deg f = d$. *Put* $\Delta_f = \operatorname{disc}_Y f(X,Y)$. *Then the following two conditions are equivalent*:

(i) *f is a coordinate polynomial*

(ii) *f is a Jacobian polynomial and* $\deg \Delta_f = d - 1$.

In [**7**] the authors proved that if f has finite set of critical points, then $\deg \Delta_f \geq d - 1$. Theorem 4 shows that the plane Jacobian conjecture is equivalent to the following statement

(JC′) If f is a Jacobian polynomial such that $\deg_Y f = \deg f = d$, then $\deg \Delta_f = d - 1$.

2. Proofs. Let $\Gamma \subset \mathbf{C}^2$ be an affine algebraic curve. We put $\chi(\Gamma)$ equal to the Euler characteristic of Γ, and $r_\infty(\Gamma)$ equal to the number of branches at infinity of Γ. To prove Theorem 1, we need the following lemmas.

Lemma 1. *Let $\Gamma \subset \mathbf{C}^2$ be an affine irreducible and smooth algebraic curve. Let γ be the genus of the Riemann surface corresponding to the projective closure of Γ. Then $\chi(\Gamma) = 2 - 2\gamma - r_\infty(\Gamma)$. In particular, $\chi(\Gamma) \leq 1$ and $\chi(\Gamma) = 1$ if and only if Γ is isomorphic to a line.*

Proof. See, for example, Proposition 2.4 in [**13**].

Lemma 2. *Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with no critical points, and let $t_0 \in \mathbf{C}$. Then the following two conditions are equivalent.*

(i) $\Lambda(f) = \{t_0\}$.

(ii) $\chi(f^{-1}(t_0)) = 1$.

Proof. See Lemma 1 in [**18**].

Lemma 3. *Let $\Gamma \subset \mathbf{C}^2$ be an affine, irreducible curve of degree > 1. Suppose that there exists a pencil L of l parallel lines not meeting Γ. Then Γ has at least $l + 1$ branches at infinity.*

Proof. Let p be the point at infinity of the lines of the pencil L. Then p is a point at infinity of the curve Γ and every line of L is tangent to Γ at p. If p is the unique point at infinity of Γ, then the line at infinity is also tangent to Γ at p. Therefore through p, $l + 1$ tangents pass to Γ, consequently there are at least $l + 1$ branches of Γ centered at p. If there is another point at infinity $q \neq p$ of Γ, then there are at least l branches of Γ centered at p and at least one branch centered at q. This proves the lemma.

Now we can give

Proof of Theorem 1. (i) $\Rightarrow$ (ii). Assume $\Lambda(f) = \{t_0\}$ and write $f^{-1}(t_0) = \bigcup_{i=1}^{s} \Gamma_i$ where Γ_i are irreducible components of $f^{-1}(t_0)$. We have $\chi(f^{-1}(t_0)) = \sum_{i=1}^{s} \chi(\Gamma_i)$ for Γ_i are pairwise disjoint. Let $\gamma_i = \gamma(\Gamma_i)$ and $r_{\infty,i} = r_\infty(\Gamma_i)$. Then

$$1 = \sum_{i=1}^{s} (2 - 2\gamma_i - r_{\infty,i}) \tag{1}$$

by Lemmas 1 and 2.

All numbers $2 - 2\gamma_i - r_{\infty,i}$ are integers less than or equal to 1. Therefore there is $i_0 \in \{1, \ldots, s\}$ such that $2 - 2\gamma_{i_0} - r_{\infty,i_0} = 1$. Obviously $\gamma_{i_0} = 0$ and $r_{\infty,i_0} = 1$. Consequently, Γ_{i_0} is isomorphic to $\mathbf{C}$. If Γ_i ($i \neq i_0$) are not isomorphic to $\mathbf{C}$, then for $i \neq i_0$ we get $2 - 2\gamma_i - r_{\infty,i} = 0$, that is $\gamma_i = 0$ and $r_{\infty,i} = 2$.

Suppose that $\Gamma_1, \ldots, \Gamma_l$ are isomorphic to $\mathbf{C}$ ($l > 1$) and $\Gamma_{l+1}, \ldots, \Gamma_s$ are not. Then by (1) we get $\sum_{i=l+1}^{s}(2 - 2\gamma_i - r_{\infty,i}) = 1 - l$. By applying a polynomial automorphism we may assume that $\Gamma_1, \ldots, \Gamma_l$ are affine lines. Therefore, by Lemma 3, we get $r_{\infty,i} \geq l + 1$ for $i \geq l + 1$. We see easily that $s = l + 1$, $\gamma_{l+1} = 0$ and $r_{\infty,l+1} = l + 1$.

(ii) $\Rightarrow$ (i). Assume that condition (ii) holds and write $f^{-1}(t_0) = \bigcup_{i=1}^{s} \Gamma_i$ where Γ_i are irreducible components of $f^{-1}(t_0)$. Then $\chi(f^{-1}(t_0)) = \sum_{i=1}^{s} \chi(\Gamma_i) = 1$ by Lemma 2 and $\Lambda(f) = \{t_0\}$ by Lemma 1.

To prove Theorem 2, we need

Lemma 4. *Let $\Gamma \subset \mathbf{C}^2$ be an affine algebraic curve of degree d with c points at infinity. Then $c \leq r_\infty(\Gamma) \leq d$. If $c = d$, then the projective closure of Γ has no singularities at infinity.*

Proof. Obvious.

Proof of Theorem 2. Since f has an irregular value, then $d \geq 3$. Let $f^{-1}(t_0) = \bigcup_{i=1}^{s} \Gamma_i$ be the decomposition of $f^{-1}(t_0)$ into irreducible components. Let us consider two cases.

Case 1. Exactly one irreducible component of $f^{-1}(t_0)$ is isomorphic to a line. Suppose that Γ_1 is isomorphic to a line and $\Gamma_2, \dots, \Gamma_s$ are not. Let Γ_∞ be the set of points at infinity of an affine curve $\Gamma \subset \mathbf{C}^2$. The curves Γ_1, Γ_i, $i = 2, \dots, s$, are disjoint, so $(\Gamma_1)_\infty \cap (\Gamma_i)_\infty \neq \varnothing$. Since Γ_1 is isomorphic to $\mathbf{C}$, we have $(\Gamma_1)_\infty = \{p_1\}$ and we may write $(\Gamma_i)_\infty = \{p_1, p_i\}$ for $i = 2, \dots, s$, for Γ_i has at most two points at infinity. Now we get $(f^{-1}(t_0))_\infty = (\Gamma_1)_\infty \cup \dots \cup (\Gamma_s)_\infty \subset \{p_1, \dots, p_s\}$. Therefore $\#(f^{-1}(t_0))_\infty \leq s$. On the other hand, $d = \deg f^{-1}(t_0) = \sum_{i=1}^{s} \deg \Gamma_i \geq 1 + 2(s-1) = 2s - 1$. Finally, we get $s \leq (d+1)/2$.

Case 2. At least two irreducible components of $f^{-1}(t_0)$ are not isomorphic to $\mathbf{C}$. By Theorem 1, we may assume that $\Gamma_1, \dots, \Gamma_{s-1}$ are isomorphic to $\mathbf{C}$ and Γ_s has s branches at infinity. Note that $\deg \Gamma_s > s > 2$; the inequality $\deg \Gamma_s \geq s$ follows from Lemma 4. To check that $\deg \Gamma_s > s$ suppose $\deg \Gamma_s = s$. Then Γ_s would have no singular points by Lemma 4, which leads to a contradiction because a rational (projective) curve of degree > 2 always has singular points.

We may write $(\Gamma_i)_\infty = \{p_1\}$ for $i = 1, \dots, s-1$ and $(\Gamma_s)_\infty = \{p_1, p_2, \dots, p_s\}$ for $r_\infty(\Gamma_s) = s$. Therefore, $\#(f^{-1}(t_0))_\infty \leq s$. On the other hand, $d = \deg f^{-1}(t_0) = \sum_{i=1}^{s} \deg \Gamma_i \geq s - 1 + \deg \Gamma_s \geq (s-1) + (s+1) = 2s$. Hence, $s \leq d/2$.

Thus we have proved in both cases $\#(f^{-1}(t_0))_\infty \leq$ number of irreducible components of $f^{-1}(t_0) \leq (d+1)/2$.

Proof of Theorem 3. Keep the notation from the Introduction. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with a finite set of critical points. According to Krasinski [**14**, Theorem 6.4], we have

$$\deg \Delta(X, t_0) = d(d-2) + c - \sum_{p \in C_\infty} \mu_p^{t_0}. \tag{2}$$

On the other hand, by Cassou-Noguès' formula [**9**, Proposition 1.2], we can write

$$d(d-3) + c + 1 = \sum_{p \in C_\infty} \mu_p^{\min} + \mu(f) + \lambda(f) \tag{3}$$

where $\mu(f)$ is the total Milnor number of f (see [**6**], [**9**]). Combining (2) and (3), we get

$$\deg \Delta(X, t_0) = \mu(f) + \lambda(f) - \lambda^{t_0}(f) + d - 1. \tag{4}$$

Now Theorem 3 follows easily from (4) since f has no critical points if and only if $\mu(f) = 0$ and the condition $\Lambda(f) \subset \{t_0\}$ is equivalent to the equality $\lambda(f) = \lambda^{t_0}(f)$.

Proof of Theorem 4. Implication (i) $\Rightarrow$ (ii) follows from the corollary to Theorem 3. To check (ii) $\Rightarrow$ (i) assume that f is a Jacobian polynomial such that $\deg \Delta_f = d - 1$. Then by Theorem 3, we get $\Lambda(f) \subset \{0\}$. By the Ephraim theorem it suffices to check that $\Lambda(f) = \varnothing$. If we had $\Lambda(f) \neq \varnothing$, i.e., $\Lambda(f) = \{0\}$, then by a result of Assi [**4**, Corollary 4.8], f would not be a Jacobian polynomial.

3. Questions. We end this note by a few questions concerning polynomials in two complex variables. We indicate partial answers we got by using elementary tools (resultants and discriminants, Puiseux series, Newton diagrams).

Question 1. How many points at infinity can a polynomial have without critical points of degree $d > 3$?

Using Theorem 2 and a result of Le Van Than and Oka [**17**, Theorem 2.4], we checked that the curve $f^{-1}(0)$ has at most $d-2$ points at infinity provided f is of degree $d > 3$ with no critical points. We do not know if our estimate is exact.

Question 2. Suppose that a polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ of degree $d > 5$ has a finite number of critical points. What is the optimal bound on the number $\#\Lambda(f)$ of irregular values at infinity?

Our result is $\#\Lambda(f) \leq \max(1, d-3)$. Is it exact if $d > 5$?

Question 3. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial with a finite number of critical points. What is the optimal bound on $\lambda(f)$?

We checked that $\lambda(f) \leq d^2 - 3d$ for $d > 3$. If $d = 4$, then our estimation is optimal (take $f(X, Y) = X^4 - X^2Y^2 + 2XY - 1$, then $\lambda(f) = 4$).

If $f : \mathbf{C}^2 \to \mathbf{C}$ is a polynomial with a finite number of critical points, then $|\operatorname{grad} f(z)| \geq \operatorname{const} |z|^{-\lambda(f)-1}$ for large $|z|$. We can complete the last question by the following

Question 4. Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial of degree $d > 4$ with a finite number of critical points. What is the optimal bound on the exponent θ in the inequality $|\mathrm{grad}\, f(z)| \geq \mathrm{const}\, |z|^\theta$ for large $|z|$?

Note that explicit formulae for the Lojasiewicz exponent (the best θ in the inequality above) are known ([**12**], [**8**]).

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REFERENCES

1. S.S. Abhyankar, *Expansion techniques in algebraic geometry*, Tata Inst. Fund. Res. Stud. Math., Tata Institute of Fundamental Research, Bombay, 1977.

2. S.S. Abhyankar and T. Moh, *Embeddings of the line in the plane*, J. Reine Angew. Math. **276** (1975), 148–166.

3. A. Assi, *Familles des courbes planes ayant une seule valeur irregulière*, C.R. Acad. Sci. Paris Ser. I Math. **322** (1996), 1203–1207.

4. ———, *Meromorphic plane curves*, Math. Z. **230** (1999), 165–183.

5. H. Bass, E. Connell and D. Wright, *The Jacobian conjecture*: *Reduction of degree and formal expansion of the inverse*, Bull. Amer. Math. Soc. **7** (1982), 287–330.

6. S.A. Broughton, *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math. **92** (1988), 217–241.

7. J. Chadzyński and T. Krasiński, *Properness and the Jacobian conjecture in* $\mathbf{C}^2$, Bull. Soc. Sci. Lett. Łódź. **XIV** (1992), 13–19.

8. ———, *On the Lojasiewicz exponent at infinity for polynomial mappings of* $\mathbf{C}^2$ *into* $\mathbf{C}^2$ *and components of polynomial automorphism of* $\mathbf{C}^2$, Ann. Polon. Math. **58** (1992), 291–302.

9. P. Cassou-Noguès, *Sur la généralisation d'un théorème de Kouchnirenko*, Compositio Math. **103** (1996), 95–121.

10. A.H. Durfee, *Five definitions of critical point at infinity*, *Singularities*: *The Brieskorn anniversary volume*, Progr. Math. **162** (1998), 345–360.

11. R. Ephraim, *Special polars and curves with one place at infinity*, Proc. Sympos. Pure Math. **40** (1983), 353–359.

12. H.V. Ha, *Nombres de Lojasiewicz at singularités à l'infini des polynômes de deux variables complexes*, C.R. Acad. Sci. Paris Ser. I **311** (1990), 429–432.

13. Z. Jelonek, *The automorphism groups of Zariski open affine subsets of the affine plane*, Ann. Pol. Math. **LX.2** (1994), 163–171.

14. T. Krasiński, *The level sets of polynomials in two variables and the Jacobian conjecture*, Acta Univ. Łódź. Folia Math. (published dissertation) (1991). (Polish)

15. ———, *Matrix identities connected with the Jacobian conjecture*, Univ. Iagel. Acta Math. Fasc. **XXXII** (1995), 175–185.

16. Le Dung Trang and C. Weber, *A geometrical approach to the Jacobian conjecture for* $n = 2$, Kodai Math. J. **17** (1994), 374–381.

17. Le Van Thanh and M. Oka, *Note on estimation of the numbers of the critical values at infinity*, Kodai Math. J. **17** (1994), 409–419.

18. M.G. Zaidenberg and V.Ya. Lin, *An irreducible simply connected algebraic curve in* $\mathbf{C}^2$ *is equivalent to a quasihomogeneous curve*, Soviet Math. Dokl. **28** (1983), 200–203.

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