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THE STABLE SET OF ASSOCIATED PRIMES OF THE IDEAL OF A GRAPH

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ABSTRACT. Let G be a graph and let I be the edge ideal of G. We give a constructive method for determining primes associated to the powers of I. Brodmann showed that the sets of associated primes stabilize for large powers of I. Our construction will yield this stable set and an upper bound on where the stable set will occur.

1. Introduction. In this paper we will study the sets of prime ideals that are associated to the powers of the edge ideal of a graph. In [**1**], Brodmann showed that when R is a Noetherian ring and I is an ideal of R, the sets $\operatorname{Ass}(R/I^n)$ stabilize for large n. That is, there exists a positive integer N such that $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^N)$ for all $n \geq N$. Although the sets $\operatorname{Ass}(R/I^n)$ have been studied extensively (see [**5**] for instance), little is known about where the stability occurs or about which primes are in the stable set. If the ideal is generated by a regular sequence, then it is shown in [**3**, 2.1] that $\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I)$ for all n. If the ring R is Gorenstein and if I is a strongly Cohen-Macaulay perfect ideal generated by a d-sequence, then in [**6**, Theorem 2.2] the stable set is described and an upper bound on where it occurs is given. However, if the generators of the ideal do not form a d-sequence, very little is known. Even for special classes of ideals such as monomial ideals or ideals defining simplicial complexes, the stable set is unknown.

In this paper we will work with a class of monomial ideals, the edge ideals of graphs. These are ideals whose generators are square-free monomials of degree two. We will give a construction that produces the primes that are in the stable set of the ideal of a graph and that gives an upper bound for where the stability occurs.

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First, we recall some standard definitions. Let I be an ideal of a ring R. A prime ideal P of R is a *minimal prime* of I if $I \subset P$ but there does not exist a prime $Q \neq P$ of R such that $I \subset Q \subset P$. The set of all minimal primes of I is written $\operatorname{Min}(R/I)$. A prime ideal P of R is an *associated prime* of I if there exists an element c in R such that $P = (I : c)$ where $(I : c) = \{r \in R \mid rc \in I\}$. The notation for the set of all associated primes of an ideal I is $\operatorname{Ass}(R/I)$. Thus

$$\operatorname{Ass}(R/I^n) = \{P \subset R \mid P \text{ is prime and } P = (I^n : c) \text{ for some } c \in R\}.$$

In general, $\operatorname{Min}(R/I) \subseteq \operatorname{Ass}(R/I^n)$ for all positive integers n. In the case where equality holds for all n, the ideal I is said to be *normally torsion-free*.

A *primary decomposition* of an ideal is a way to write the ideal as an intersection of primary ideals. This is analogous to the factorization of an integer into a product of prime integers. An ideal I can be written as an intersection of primary ideals

$$I = q_1 \cap \cdots \cap q_t \cap Q_1 \cap \cdots \cap Q_s$$

where $\sqrt{q_i} \in \operatorname{Min}(R/I)$ and $\sqrt{Q_j}$ are *embedded* primes, that is, associated primes which contain one of the minimal primes. See [**4**, Section 6] for more details regarding the associated primes of an ideal. When I is a monomial ideal of a polynomial ring $R = k[x_1, \ldots, x_d]$, the associated primes will be monomial primes, which are primes generated by a subset of the variables. Moreover, there is a well-known algorithm for computing a primary decomposition of a monomial ideal, see, for example, [**2**, Chapter 3].

Formally, a *graph* G is a set of vertices $V = \{v_1, \ldots, v_n\}$ together with a set of edges $E \subseteq \{v_i v_j \mid v_i, v_j \in V\}$. Two vertices, v_i and v_j, of a graph are *adjacent* if $v_i v_j$ is in E, in other words, if they are connected by an edge of the graph. A *path* is a set of distinct vertices $\{v_{i_1}, \ldots, v_{i_s}\}$ of G together with edges $v_{i_j} v_{i_{j+1}}$ for $1 \leq j \leq s-1$. A *cycle* of length s is a path together with an edge $v_{i_s} v_{i_1}$. For general terminology and notation regarding graphs, see for instance [**8**].

To form the edge ideal of a graph G, let k be a field, let d be the number of vertices of G, and let R be the polynomial ring in d variables over $k, R = k[x_1, \ldots, x_d]$. Define $I = (\{x_i x_j \mid v_i v_j \in E\})$ to be the ideal whose generators are the edges of G. Then I is the *edge ideal* of G. A *minimal vertex cover* of a graph G is a subset U of the vertices such that every edge of G has at least one of its two endpoints in U and no proper subset of U has this property. Note that $P = (x_{i_1}, \dots, x_{i_s})$ is a minimal prime of the edge ideal I of G if and only if $\{v_{i_1}, \dots, v_{i_s}\}$ is a minimal vertex cover of G. Since the minimal primes of I and thus of I^n can be found from the minimal vertex covers, the focus of this work will be to find the embedded primes.

2. Preliminaries. In [**7**, Theorem 5.9] it is shown that the graph G is bipartite if and only if I is normally torsion-free, that is, if and only if there are no embedded primes of I^n for all n. Thus to find embedded primes we will restrict our consideration to graphs that contain at least one odd cycle. The following lemma (see also [**7**, Corollary 5.7]) will allow us to restrict our attention to connected graphs.

Lemma 2.1. *Suppose G is a disconnected graph which is the disjoint union of subgraphs G_1 and G_2. Let $I = (I_1, I_2)$ be the edge ideal of G, where I_1 and I_2 are the edge ideals of G_1 and G_2, respectively. Then $P \in \operatorname{Ass}(R/I^n)$ if and only if $P = (P_1, P_2)$ where $P_1 \in \operatorname{Ass}(R/I_1^{n_1})$ and $P_2 \in \operatorname{Ass}(R/I_2^{n_2})$ for some positive integers n_1 and n_2 such that $n_1 + n_2 = n + 1$.*

Proof. Let $R = k[x_1, \dots, x_l, y_1, \dots, y_m]$, where $x_1, \dots, x_l$ are the vertices of G_1 and $y_1, \dots, y_m$ are the vertices of G_2. Then $P_1 \subseteq (x_1, \dots, x_l) = (\underline{x})$, $P_2 \subseteq (y_1, \dots, y_m) = (\underline{y})$ and $x_i y_j$ is not an edge for every i, j.

We first prove the converse. If $P_i \in \operatorname{Ass}(R/I_i^{n_i})$, then $P_i = (I_i^{n_i} : c_i)$ for $i = 1, 2$ where $c_1 \in I_1^{n_1 - 1}$ is a monomial in $k[\underline{x}]$ and $c_2 \in I_2^{n_2 - 1}$ is a monomial in $k[\underline{y}]$. Let u be a generator of $P = (P_1, P_2)$. Then either $u \in P_1$ and $uc_1c_2 \in I_1^{n_1} I_2^{n_2 - 1} \subset I^n$ or $u \in P_2$ and $uc_1c_2 \in I_1^{n_1 - 1} I_2^{n_2} \subset I^n$. Thus $P \subseteq (I^n : c_1c_2)$.

Notice that $c_1c_2 \in I^{n-1} \setminus I^n$ since the graphs are disjoint. Assume $v \in k[\underline{x}, \underline{y}]$ is a monomial such that $vc_1c_2 \in I^n$. Write $v = v_1v_2$ where $v_1 \in k[\underline{x}]$ and $v_2 \in k[\underline{y}]$ are monomials. If $v_1c_1 \in I_1^{n_1 - 1} \setminus I_1^{n_1}$ and $v_2c_2 \in I_2^{n_2 - 1} \setminus I_2^{n_2}$, then $(v_1c_1)(v_2c_2) \notin I^n$ since the graphs are disjoint. But this is a contradiction, so either $v_1 \in P_1$ or $v_2 \in P_2$, and thus $v \in P$.

Now suppose $P \in \operatorname{Ass}(R/I^n)$. Then $P = (P_1, P_2)$, where $P_1 = (P \cap k[\underline{x}])R$ and $P_2 = (P \cap k[\underline{y}])R$. Now $P = (I^n : c)$ for some monomial $c \in I^{n-1} \setminus I^n$. Write $c = c_1c_2$ where $c_1 \in k[\underline{x}]$ and $c_2 \in k[\underline{y}]$ are monomials. Then $c_1 \in I_1^k$ and $c_2 \in I_2^s$ for some $0 \leq k,\ s \leq n-1$ with $k + s = n - 1$. Suppose x is a generator of P_1. Then $xc \in I^n$ forces $xc_1 \in I_1^{k+1}$ so $P_1 \subseteq (I_1^{k+1} : c_1)$. Suppose $u \in (I_1^{k+1} : c_1)$ is a monomial. Then $u = u_1u_2$, where $u_1 \in k[\underline{x}]$ and $u_2 \in k[\underline{y}]$ are monomials. Since the graphs are disjoint, $u_1 \in (I_1^{k+1} : c_1)$. But then $u_1c = u_1c_1c_2 \in I_1^{k+1}I_2^s \subseteq I^n$ so $u_1 \in P$. Since $u_1 \in P \cap k[\underline{x}]$, $u_1 \in P_1$ and $P_1 = (I_1^{k+1} : c_1)$.

A similar argument shows that $P_2 \in \operatorname{Ass}(R/I_2^{s+1})$. Let $n_1 = k + 1$ and $n_2 = s + 1$. Then $n_1 + n_2 = k + s + 2 = n + 1$. $\square$

Corollary 2.2. *Suppose G is a graph with connected components $G_1, \dots, G_s$ and suppose $I = (I_1, \dots, I_s)$ is the edge ideal of G. Then $P \in \operatorname{Ass}(R/I^n)$ if and only if $P = (P_1, \dots, P_s)$ where $P_i \in \operatorname{Ass}(R/I_i^{n_i})$ and $n - 1 = (n_1 - 1) + (n_2 - 1) + \cdots + (n_s - 1)$.*

Corollary 2.3. *Suppose $I = (I_1, I_2)$, where I_1 is a monomial prime ideal and I_2 is the edge ideal of a graph which has no variables in common with I_1. Then $P \in \operatorname{Ass}(R/I^n)$ if and only if $P = (I_1, P_2)$ where $P_2 \in \operatorname{Ass}(R/I_2^{n_2})$ for some $n_2 \leq n$.*

Proof. Since $\operatorname{Ass}(R/I_1^{n_1}) = \{I_1\}$ for all n_1 and $I_1 = \{I_1^{n_1} : c_1)$ if and only if $c_1 \in I_1^{n_1} \setminus I_1^{n_1 - 1}$, the above proof holds. Notice that if $n_1 = 1$, c_1 can be chosen to be any unit in the ring. $\square$

Corollary 2.3 will prove useful when localization techniques are employed. Before stating the next lemma, we introduce some terminology and notations.

Definition 2.4. Let G be a graph with vertex set $V = \{v_1, \dots, v_n\}$ and let A be a subset of V. The neighbor set of A is the set

$$N(A) = \{v \in V \mid v \text{ is adjacent to some vertex in } A\}.$$

Definition 2.5. Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and let A be a subset of V. The induced subgraph $\langle A \rangle$ is the maximal subgraph of G with vertex set A. In other words, two vertices of A are adjacent in $\langle A \rangle$ if and only if they are adjacent in G. Define $I(A)$ to be the edge ideal of $\langle A \rangle$.

One case in which we will use the above definition is the case where $A = G \setminus x_i$, by which we mean the set of all vertices of G except for the vertex corresponding to x_i.

Lemma 2.6. *Suppose G is a graph, I is its edge ideal and $P = (I^n : c)$ is an associated prime of I^n for some $n \in \mathbf{N}$. Suppose some vertex $x \in G$ does not divide c. Let $I' = I(G \setminus x)$. If $x \in P$, let P' be the ideal generated by all generators of P except for x. Otherwise, let $P' = P$. Then $P' = ((I')^n : c)$.*

Proof. Let $x_a \in P'$. Then $x_a \in P$, so $x_a c \in I^n$. Since x does not divide c and $x_a \neq x$, x does not divide $x_a c$. Thus $x_a c \in (I')^n$ and $P' \subseteq ((I')^n : c)$.

Let $v \notin P'$ be a monomial. Assume $vc \in (I')^n$. Then $v \in P$ so $v = x^t a$ for some positive integer t and some monomial $a \notin P$. Then $ac \notin (I')^n$ and since P' is a vertex cover of $\langle G \setminus x \rangle$, no edge of $\langle G \setminus x \rangle$ can divide v. Thus there exists y such that $xy \in I'$ and y divides c. This is a contradiction since $xy \notin I'$. So it must be the case that $vc \notin (I')^n$. Thus $P' = ((I')^n : c)$. $\square$

3. Building associated primes. Let G be a graph with d vertices, let $R = k[x_1, \ldots, x_d]$ be a polynomial ring in d variables, let $\mathfrak{m} = (x_1, \ldots, x_d)$ be the unique homogeneous maximal ideal, and let $I \subset R$ be the edge ideal of G. In this section we will describe a process that can be used to produce prime ideals in $\operatorname{Ass}(I^n)$ for any n. For the remainder of the paper, a graph G is assumed to be connected and not bipartite unless otherwise indicated.

We will first treat the case where G is a cycle, in which case we completely determine $\operatorname{Ass}(R/I^n)$ for all n.

Lemma 3.1. *Suppose G is a cycle of length $2k+1$ and I is the edge ideal of G. Then $\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I)$ if $n \le k$ and $\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I) \cup \{\mathfrak{m}\}$ if $n \ge k+1$.*

Proof. Since $\operatorname{Min}(R/I) \subseteq \operatorname{Ass}(R/I^n)$ for all n, we first show that $\mathfrak{m} \in \operatorname{Ass}(R/I^n)$ if and only if $n \ge k+1$. Let $c_{k+1} = x_1 x_2 \cdots x_{2k+1}$ be the product of the variables. Since I is generated by monomials of degree two and c_{k+1} has degree $2k+1$, $c_{k+1} \notin I^{k+1}$. Thus $(I^{k+1} : c_{k+1}) \neq R$. Now $x_i c_{k+1} = (x_i x_{i+1})(x_{i+2} x_{i+3}) \cdots (x_{i-1} x_i)$ is a product of $k+1$ edges of G, so $x_i c_{k+1} \in I^{k+1}$ for all i. Thus $\mathfrak{m} \subseteq (I^{k+1} : c_{k+1})$ which forces equality since the ideals in question are homogeneous. So $\mathfrak{m} \in \operatorname{Ass}(R/I^{k+1})$.

For $n \ge k+1$, let $c_n = c_{k+1}(x_1 x_2)^{n-k-1}$. Then it is easy to check that $\mathfrak{m} = (I^n : c_n)$ and $\mathfrak{m} \in \operatorname{Ass}(R/I^n)$.

Now suppose $n < k+1$ and $\mathfrak{m} = (I^n : c)$ for some $c \notin I^n$. Then $c \notin I^k$, so there must be some vertex x_i that does not divide c. Let $I' = I(G \setminus x_i)$ be the ideal generated by all edges of G except for those containing x_i. Notice that I' is normally torsion-free since $\langle G \setminus x_i \rangle$ is bipartite. Let $P' = (x_1, \ldots, \hat{x}_i, \ldots, x_d)$. By Lemma 2.6, P' is an associated prime of $(I')^n$, a contradiction since P' does not correspond to a minimal vertex cover of $\langle G \setminus x_i \rangle$. So $\mathfrak{m}$ is not an associated prime of I^n for any $n < k+1$.

Now suppose $P \neq \mathfrak{m}$ is a prime ideal. Since $P \neq \mathfrak{m}$, there is some variable not in P. Without loss of generality, assume $x_{2k+1} \notin P$. Localize at $Q = (x_1, \ldots, x_{2k})$. Then $I_Q = (I_1, I_2)$ where I_1 is a prime ideal generated by the vertices adjacent to v_{2k+1} and I_2 is the ideal of a graph which consists of the vertices $v_2, \ldots, v_{2k-1}$ connected in a path. Then by Corollary 2.3, P is an associated prime of I^n for some n if and only if $PR_Q = (I_1, P_2)$ where P_2 is a prime corresponding to a minimal vertex cover of I_2. However, if $PR_Q = (I_1, P_2)$, then P corresponds to a minimal vertex cover of G. Thus P is an associated prime of I^n if and only if P corresponds to a minimal vertex cover of G. $\square$

We now describe the process by which we form embedded associated primes of I^n for more general graphs. Fix an odd cycle C of length $2k+1$ contained in a graph G. Define R_{k+1} to be the set of vertices of

C and define B_{k+1} to be the set of vertices in $N(C) \setminus C$. Let

$$d_{k+1} = \prod_{x_i \in R_{k+1}} x_i$$

be the product of the $2k+1$ vertices in the cycle. If V is any minimal subset of vertices for which $R_{k+1} \cup B_{k+1} \cup V$ is a vertex cover of G, then we will see in Theorem 3.3 that $P = (R_{k+1}, B_{k+1}, V)$ is an associated prime of I^n for all $n \geq k+1$.

We will now recursively build sets R_n. As the powers of I increase, the associated primes will be built by working outward from the sets R_n. Suppose x_i is in R_s for some $s \geq k+1$ and suppose $x_i x_j$ is an edge of G. Then, by definition, x_j is in either R_s or B_s. If $x_j \in R_s$, let $R_{s+1} = R_s$ and let $B_{s+1} = B_s$. If $x_j \in B_s$, let $R_{s+1} = R_s \cup \{x_j\}$ and let $B_{s+1} = (B_s \cup N(x_j)) \setminus R_{s+1}$. In either case, let $d_{s+1} = d_s(x_i x_j)$. Notice that at each stage there may be many choices for x_i. Thus there will be a collection of possible sets R_s, each with corresponding B_s and d_s. Notice also that each choice of a cycle contained in G will produce different collections of sets.

For each $n \geq k+1$, let $R(C)_n$ be the set of all R_n produced from C by the above process. Notice that $R(C)_n \subseteq R(C)_{n+1}$, although the corresponding d_n and d_{n+1} will differ by an edge.

Lemma 3.2. *Suppose I is the edge ideal of a graph containing a cycle C of length $2k+1$. Let $n \geq k+1$. Then the following are true for each $R_n \in R(C)_n$ and corresponding B_n and d_n:*

1. *Every vertex that divides d_n is in R_n and thus is adjacent only to vertices in $R_n \cup B_n$.*

2. *d_n has degree $2n-1$, so $d_n \notin I^n$, but $d_n \in I^{n-1}$.*

3. *For each vertex x in R_n, $d_n/x \in I^{n-1}$ and for each vertex $x \in R_n \cup B_n$, $xd_n \in I^n$.*

Proof. Parts 1 and 2 are clear from the definitions.

For part 3 let $x \in R_n$ and notice that when $n = k+1$, d_{k+1}/x is the product of $2k$ adjacent vertices. If $n > k+1$, then for some d_{n-1} we have $d_n = d_{n-1} x_i x_j$ where $x_i \in R_{n-1}$ and $x_i x_j$ is an edge of G. Since $x \in R_n$, then either $x \in R_{n-1}$ or $x = x_j$. In the first case

$d_{n-1}/x \in I^{n-2}$ by induction, and in the second case $d_{n-1} \in I^{n-1}$, so $d_n/x = d_{n-1}x_i \in I^{n-1}$.

Now suppose $x \in R_n \cup B_n$. Then there is an $x_b \in R_n$ that is adjacent to x. Then x_b divides d_n and $d_n/x_b \in I^{n-1}$, so $xd_n = xx_b(d_n/x_b) \in I^n$. $\square$

Theorem 3.3. *Let C be any fixed odd cycle of a graph G and let R_n be in $R(C)_n$ with B_n and d_n corresponding to R_n. If $P = (R_n, B_n, V)$, where V is any minimal set of additional vertices needed to make P a vertex cover of G, then $P \in \operatorname{Ass}(R/I^t)$ for $t \geq n$.*

Proof. Let $\{y_1, \dots, y_t\}$ be the set of vertices in $G \setminus P$. Define

$$c = d_n \prod_{i=1}^{t} y_i.$$

We claim that $P = (I^n : c)$. Let $x \in P$. If $x \in R_n \cup B_n$, then by Lemma 3.2, $xd_n \in I^n$ so $xc \in I^n$ as well. If $x \in V$, then x is adjacent to y_j for some j, else V was not minimal. Then $d_n(xy_j)$ divides xc and since $d_n \in I^{n-1}$, $xc \in I^n$. Thus $P \subseteq (I^n : c)$.

Now suppose $v \notin P$ is a monomial. Then v is the product of a subset of $\{y_1, \dots, y_t\}$. Since P is a vertex cover, y_iy_j is not an edge for all y_j. Also, if x divides d_n, then x is not adjacent to y_j for all i by Lemma 3.2, so $vc \notin I^n$. Thus $P = (I^n : c)$ is an associated prime of I^n.

Now, let x_ix_j be any edge of G with $x_i, x_j \in R_n$. If $t \geq n$, let $c_t = c(x_ix_j)^{t-n}$. Then it is easy to show that $P = (I^t : c_t)$, so $P \in \operatorname{Ass}(R/I^t)$ for all $t \geq n$. $\square$

In the above theorem, one should notice that for each R_n there might be several choices for V. Each choice will produce an associated prime.

Corollary 3.4. *Let I be the edge ideal of a graph G and let $\mathfrak{m}$ be the unique homogeneous maximal ideal of R. Then $\mathfrak{m} \in \operatorname{Ass}(R/I^n)$ for $n \gg 0$.*

Proof. It suffices to notice that $R_n \cup B_n = \mathfrak{m}$ for $n \gg 0$ since G is connected and contains an odd cycle. $\square$

If a graph contains a single cycle, then we shall see in Theorem 5.6 that Theorem 3.3 is actually a description of all of the embedded associated primes. If G has more than one cycle, the picture is more complicated. We will see in Definition 3.5 how to combine the sets produced from each cycle to form the remaining associated primes.

Suppose G is a graph containing a cycle C of length $2k+1$ for some $k \in \mathbf{N}$. Let $S_n(C)$ be the set of all possible $A = R_n \cup B_n$ where $R_n \in R(C)_n$. Let $d(A)$ be the monomial d_n corresponding to R_n. Let $S_n = \cup S_n(C)$ where the union is over all odd cycles C of length at most n. For each cycle C of a graph G, $S_n(C) \subseteq S_{n+1}(C)$ and so $S_n \subseteq S_{n+1}$. Notice that the primes in Theorem 3.3 are of the form (A, V) where $A \in S_n$.

Definition 3.5. Suppose G is a graph, let $n \in \mathbf{N}$ and let $s \geq 2$. Define U_n to be the set of all possible $A = A_1 \cup \cdots \cup A_s$, such that $A_i \in S_{n_i}$ where $n_1, \ldots, n_s < n$ are such that $n-1 = (n_1 - 1) + \cdots + (n_s - 1)$ and in addition if x divides $d(A_i)$ for some i, then $x \notin A_j$ for all $i \neq j$. Define $d(A) = d(A_1)d(A_2)\cdots d(A_s)$.

Lemma 3.6. *Let $A \in U_n$ and $d = d(A)$ for some $n \in \mathbf{N}$. Then the following properties hold*:

1. *Every vertex that divides d is adjacent only to vertices in A.*
2. *$d \notin I^n$, but $d \in I^{n-1}$.*
3. *For each vertex x in A, $xd \in I^n$.*

Proof. Since $A \in U_n$, $A = (A_1, \ldots, A_s)$ where $s \geq 2$, $A_i \in S_{n_i}$ for each i and $n_1, \ldots, n_s < n$ are such that $n-1 = (n_1-1)+\cdots+(n_s-1)$. Let $d(A_i) = d_i$ and $d = d(A) = d_1 \cdots d_s$.

Part one follows from Lemma 3.2 since $A_i \subseteq A$ and any vertex that divides d must divide d_i for some i. Part three also follows from Lemma 3.2. For part two, $d_i \in I^{n_i - 1}$ for each $i \in \{1, 2, \ldots, s\}$, so $d = d_1 \cdots d_s \in I^{n-1}$. If $d \in I^n$, there exists some edge $x_i x_j$ that divides d but does not divide d_a for any a. Then x_i divides d_a and x_j divides d_b for some $a \neq b$. By Lemma 3.2, x_i is adjacent only to vertices in A_a, so $x_j \in A_a$. But x_j divides d_j, so by Definition 3.5 $x_j \notin A_a$, a contradiction. Thus $d \notin I^n$. $\square$

We now use the sets U_n to build additional associated primes of I^n.

Theorem 3.7. *Let G be a graph and let I be the edge ideal of G. Let $P = (A, V)$ where $A \in U_n$ for some $n \in \mathbf{N}$ and where V is a minimal set of vertices such that P is a vertex cover of G. Then P is an associated prime of I^t for all $t \geq n$.*

Proof. The proof follows that of Theorem 3.3 using Lemma 3.6 in place of Lemma 3.2. □

4. The stable set. In this section we will show that the primes described in Theorems 3.3 and 3.7 are the only embedded primes that appear in the stable set of associated primes, that is, they are the only embedded primes in $\mathrm{Ass}\,(R/I^n)$ for $n \gg 0$. For ease of notation we define P_n to be the set of all primes $P = (A, V)$ where either $A \in U_n$ or $A \in S_n(C)$ for some odd cycle C and where V is a minimal set of vertices needed for P to be a vertex cover of G. Notice that $P_n \subseteq P_{n+1}$ for all n.

Theorem 4.1. *If $P \in \mathrm{Ass}\,(R/I^s)$ for some s, then either $P \in \mathrm{Min}\,(R/I)$ or $P \in P_n$ for some $n \geq s$.*

Proof. We will prove the theorem by induction on the number of vertices of the graph. The theorem holds for a three-cycle by Lemma 3.1.

Suppose $P \in \mathrm{Ass}\,(R/I^s)$ for some $s > 0$. By Corollary 3.4, $\mathfrak{m} \in P_n$ for $n \gg 0$. Thus we may assume that $P \neq \mathfrak{m}$. Then there is some vertex x of G which is not contained in P. Localize at the prime ideal Q generated by all of the vertices except for x. Then $I_Q = (I_1, I_2)$ where $I_1 = (y_1, \ldots, y_t)$ is the prime ideal generated by all vertices y_i adjacent to x, and I_2 is the edge ideal of the graph $G' = G \setminus \{x, y_1, \ldots, y_t\}$. By Corollary 2.3, $PR_Q = (I_1, P_2)$ where P_2 is an associated prime of $I(G')$. If G' is bipartite, then P_2 is a minimal vertex cover of G' and P is a minimal vertex cover of G and thus a minimal prime of I.

Suppose G' is not bipartite. If G' is connected, then $P_2 \in P_n(G')$ for some n by induction. So $P_2 = (A, V)$ where V is a minimal set such that P_2 is a vertex cover of G' and either $A \in S_n(C)$ for some cycle C of G' or A is the union of such sets. Now every cycle of G' is also a cycle of G so $A \in S_n(C)$, where C is now viewed as a cycle of G or $A \in U_n$. As before, I_1 is a minimal set such that $(A, V, I_1) = P$ is a vertex cover of G, so $P \in P_n$.

Suppose G' is not connected. Then by Corollary 2.2, $P_2 = (P_{2_1}, P_{2_2}, \ldots, P_{2_s})$ where $P_{2_i} \in \operatorname{Ass}(R/I_i^{n_i})$ where I_i is the edge ideal of the connected component G_i of G' and $(n_1-1)+(n_2-1)+\cdots+(n_s-1) = n-1$. Then by induction each P_{2_i} is either a minimal vertex cover of G_i, in which case define $A_i = \varnothing$ and $V_i = P_{2_i}$, or P_{2_i} is in $P_{l_i}(G_i)$ for some $l_i \geq n_i$, in which case $P_{2_i} = (A_i, V_i)$ where A_i and V_i are as in the connected case. So $P = (A_1, \ldots, A_s, V_1, \ldots, V_s, I_1)$. Let $A = A_1 \cup \cdots \cup A_s$ and let $V = (V_1 \cup \cdots \cup V_s \cup I_1)$. Then by Definition 3.5, $A \in U_n$ and V is minimal such that (A, V) is a vertex cover of G. So $P \in P_n$. $\square$

Notice that we now have an upper bound on where the stable set will occur. By working outward from the smallest odd cycle of G, we can produce the longest possible chain of primes in the stable set. The hypothesis $d > 2k+1$ in the proposition below guarantees that the graph is not a cycle. If G is a cycle, Lemma 3.1 applies.

Proposition 4.2. *Suppose G is a connected, nonbipartite graph with d vertices and suppose the smallest odd cycle which is a subgraph of G has length $2k+1$. Assume $d > 2k+1$. Then $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^{d-k-1})$ for all $n \geq d-k-1$.*

Proof. Let C be any odd cycle in G. If C has length $2s+1$, then $R_{s+1} \in R(C)_{s+1}$ will contain $2s+1$ vertices. If $d = 2s+1$, then (R_{s+1}) will be the maximal ideal. Otherwise, by adding one vertex at each step in the construction process, one can construct a set $R_{(s+1)+(d-(2s+1))-1} = R_{d-s-1}$ which contains all but one vertex of G. Since G is connected, this vertex must be in B_{d-s-1}, and so $P = (R_{d-s-1}, B_{d-s-1})$ is the maximal ideal and thus the process must stop. So $S_t(C) = S_{d-s-1}(C)$ for all $t \geq d-s-1$ when $d > 2s+1$ and $S_t(C) = S_{d-s}(C)$ for all $t \geq d-s$ when $d = 2s+1$. Notice that $s \geq k$, so $d-s-1 \leq d-k-1$, and if $d = 2s+1$ then $s > k$, so $d-s \leq d-k-1$

as well.

Suppose $P \in \operatorname{Ass}(R/I^n)$ for some n. Then either $P \in \operatorname{Min}(R/I) \subset \operatorname{Ass}(R/I^{d-k-1})$, or by Theorem 4.1, $P = (A, V)$ where $A \in S_l$ or $A \in U_l$ for some l. Choose l to be the least integer such that $P \in P_l$. If $A \in S_l$ then as above $l \leq d-k-1$. If $A \in U_l$, then $A = A_1 \cup \cdots \cup A_q$ where $A_i \in S_{l_i}(C_i)$ where C_i is a cycle of length $2_{k_i}+1$ and $q \geq 2$. By definition, $A_i = R_i \cup B_i$ for some set R_i. Since l is the least integer such that $P \in P_l$, we can assume that $R_i \cap R_j = \varnothing$. Then $l_i = k_i + 1 + m_i$ where

$$\sum_{i=1}^{q} m_i \leq d - \sum_{i=1}^{q} (2k_i + 1).$$

Then

$$l-1 = \sum_{i=1}^{q}(l_i-1) = \sum_{i=1}^{q}(k_i+m_i) \leq \sum_{i=1}^{q} k_i + d - 2\sum_{i=1}^{q} k_i - q = d - \sum_{i=1}^{q} k_i - q,$$

and so $l-1 \leq d - \sum k_i - q \leq d-k-2$ and $l \leq d-k-1$. $\square$

Notice that in general the above bound is not strict. For instance, suppose G is the suspension of a cycle of length $2k+1$; that is, G consists of a cycle with vertices $x_1, \ldots, x_{2k+1}$ together with vertices $y_1, \ldots, y_{2k+1}$ such that $x_i y_i$ is an edge for each i and each y_i is a vertex of degree one (a cycle with "whiskers"). Then $\operatorname{Ass}(R/I^{k+1}) = \operatorname{Ass}(R/I^n)$ for all $n \geq k+1$. However, for a graph which is an odd cycle together with a single path leading off of one vertex of the cycle (a "kite"), the above bound is strict.

If a connected graph has vertices of degree one, we can strengthen the bound given above.

Corollary 4.3. *Suppose G is a connected, nonbipartite graph with d vertices and suppose the smallest odd cycle which is a subgraph of G has length $2k+1$. If G has s vertices of degree one, then $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^{d-k-s})$ for all $n \geq d-k-s$.*

5. Graphs with only one cycle. In Theorem 4.1 it was shown that if P is an embedded associated prime of I^s for some s, then $P \in P_n$ for some $n \geq s$. We would like to say that $P \in P_s$, however, if the graph contains several odd cycles, this need not be the case. A careful examination of the proof of Theorem 4.1 shows that if for all graphs G, $\mathfrak{m} \in P_n$ for the least positive integer n such that $\mathfrak{m} \in \mathrm{Ass}\,(R/I^n)$, then $P_n \cup \mathrm{Min}\,(R/I) = \mathrm{Ass}\,(R/I^n)$ for all n. Unfortunately this need not be true. However, if a graph G contains a unique cycle, then the above holds as will be seen in Theorem 5.6.

Before proving Theorem 5.6 we will need a few technical lemmas.

Lemma 5.1. *Suppose G is a graph, I is its edge ideal and $n \in \mathbf{N}$. Let $c \in I^n$ be a monomial. Suppose x_1 is a vertex of degree one in G and x_2 is the vertex adjacent to x_1. If $(x_1x_2)^a$ divides c, then $c/(x_1x_2)^a \in I^{n-a}$.*

Proof. Suppose x_1x_2 divides c. Since $c \in I^n$, we can write $c = e_1e_2\cdots e_n \cdot b$ where e_i is an edge of G and b is some monomial. If x_1 divides e_i for some i, then $e_i = x_1x_2$. Similarly, if x_1 and x_2 both divide b then we are done. If x_1 divides b and x_2 divides e_i for some i, then $e_i = x_2y$ for some y adjacent to x_2. Rewrite c as $c = e_1\cdots e_i'\cdots e_n \cdot b'$ where $e_i' = x_1x_2$ and $b' = (by)/x_1$. The remainder of the proof follows by induction. □

Lemma 5.2. *Let G be a graph, let I be the edge ideal of G, and let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Suppose x is the vertex of G of degree one. Let n be the least positive integer such that $\mathfrak{m} = (I^n : c)$ for some monomial c. Then x does not divide c.*

Proof. Let x_2 be the unique vertex adjacent to x. Let a be the largest integer such that x^a divides c. Now $c \notin I^n$ and $xc \in I^n$, so $(xx_2)^a x_2$ divides c.

Suppose $a \neq 0$ and let $d = c/(xx_2)^a$. Let $x_i \in \mathfrak{m}$. Then $x_ic \in I^n$ and $(xx_2)^a$ divides x_ic. By Lemma 5.1, $x_id = x_ic/(xx_2)^a \in I^{n-a}$, so $\mathfrak{m} \subseteq (I^{n-a} : d)$. But $(I^{n-a} : d) \neq R$ since $d \notin I^{n-a}$, so $\mathfrak{m} = (I^{n-a} : d)$. Thus $\mathfrak{m}$ is an associated prime of I^{n-a}. This is a contradiction if $a \neq 0$, so x does not divide c. □

Lemma 5.3. *Let G be a graph with a unique cycle C. Suppose C*

has length $2k+1$ and $A \in S_n(C)$ for some $n \geq k+2$. Let $G' = \langle A \rangle$. Let $A' \in S_{n-1}(C)$ and $x_j \in A'$ be such that $A = A' \cup N(x_j)$. Let $G'' = \langle A' \rangle$. If $x \in A \setminus A'$, then x has degree one in G'.

Proof. Let $x \in A \setminus A'$. Then $x \in N(x_j)$. Assume that $x \in N(y)$ for some vertex y in G' with $y \neq x_j$. If $y \in N(x_j)$, then x_j, x and y form a cycle in G', a contradiction. If $y \notin N(x_j)$, then $y \in A'$. Since G'' is connected, there is a path between y and x_j that lies entirely in G''. Since $x_j x$, $xy \in G' \setminus G''$, there is a cycle other than C in G', a contradiction. So x must have degree one in G'. □

Lemma 5.4. *Let G be a graph containing a cycle C of length $2k+1$ and no other cycles. Let I be the edge ideal of G. Suppose $P = (A, V) \in P_{k+1}$. Let $G' = \langle A \rangle$ and $I' = I(A)$. Then $P' = (A)$ is not an associated prime of $(I')^n$ for all $n < k+1$.*

Proof. Suppose $n \leq k$. If $P = ((I')^n : c)$ for some monomial c, then there is a vertex $x \in C$ that does not divide c. Let $G'' = \langle G'' \setminus x \rangle$. Note that G'' is bipartite. Let $I'' = I(G'')$, and let $P'' = (A \setminus x)$ be the prime ideal generated by all the vertices of A except for x. Then I'' is normally torsion-free. By Lemma 2.6, P'' is an associated prime of $(I'')^n$, a contradiction since P'' is not a minimal vertex cover of G''. So P' is not an associated prime of $(I')^n$ for any $n < k+1$. □

We are now ready to prove the main proposition of this section.

Proposition 5.5. *Let G be a graph containing an odd cycle C of length $2k+1$ and no other cycles. Let I be the edge ideal of G. Let N be the least positive integer such that $\mathfrak{m} \in P_N$. Then $\mathfrak{m} \notin \operatorname{Ass}(R/I^n)$ for all $n \leq N$.*

Proof. If $\mathfrak{m} \in S_{k+1}(C)$, by Lemma 5.4, $\mathfrak{m}$ is not an associated prime of I^n for any $n < N$. So suppose $N > k+1$. Since $\mathfrak{m} \in S_N(C)$, there exist $A \in S_{N-1}(C)$ and x_i, x_j such that $x_i, x_j \in A$, x_i and x_j are adjacent and $\mathfrak{m} = A \cup N(x_j)$. Let $G' = \langle A \rangle$, let $I' = I(A)$ and let $P = (A)$. Notice that P is the maximal ideal of the ring corresponding to the graph G'. Notice that $\mathfrak{m} \neq P$ and that $N-1$ is the least positive integer such that $A \in S_{N-1}(C)$.

Assume that there exists $n < N$ such that $\mathfrak{m}$ is an associated prime of I^n. Also assume that n is the least such integer. Then $\mathfrak{m} = (I^n : c)$ for some monomial $c \notin I^n$. Let $x \in \mathfrak{m} \setminus A$. By Lemma 5.3, x has degree one in G. By Lemma 5.2, x does not divide c. Therefore, by repeated use of Lemma 2.6, $P = ((I')^n : c)$. By induction $n \geq N-1$, so $n = N-1$. Then n is the least positive integer such that P is an associated prime of $(I')^n$.

Notice that $x_j \notin C$ since otherwise $N(x_j) \subseteq A$. Assume x_j has degree greater than one in G'. Then x_j is adjacent to some vertex in G' besides x_i, call it y. Since G' is connected, there is a path from the cycle to y. Similarly, there is a path from the cycle to x_i. These two paths, C, $x_i x$ and xy form another cycle, a contradiction. So x_j must have degree one in G'. By Lemma 5.2, since $P = ((I')^n : c)$ and n is minimal as seen above, x_j does not divide c.

If $x \in \mathfrak{m} \setminus A$, then $xc \in I^n$, so there exists z such that $xz \in I$ and z divides c. As above, x has degree one in G, so $z = x_j$, a contradiction since x_j does not divide c. So $\mathfrak{m}$ is not an associated prime of I^n for any $n < N$. □

Notice that by combining Theorem 4.1 and Proposition 5.5, we have shown that if a graph G has a unique cycle, then $\operatorname{Ass}(R/I^n)$ is precisely described for all n. If the cycle is even, then $\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I)$ for all n, and if the cycle is odd, then every embedded associated prime is produced by the construction in Section 3.

Theorem 5.6. *If a graph G contains a unique cycle C, then* $\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I) \cup P_n$.

Proof. First notice that if the cycle C has even length, then $P_n = \varnothing$ and the result holds. So we may assume C has odd length. Suppose $P \in \operatorname{Ass}(R/I^n)$ but $P \notin \operatorname{Min}(R/I)$. If $P = \mathfrak{m}$ is the homogeneous maximal ideal, then $\mathfrak{m} \in P_k$ for $k \gg 0$ by Corollary 3.4. Let s be the least positive integer such that $\mathfrak{m} \in P_s$. Then by Proposition 5.5 $s \leq n$. Since the sets P_i form an ascending chain, $\mathfrak{m} \in P_n$. If $P \neq \mathfrak{m}$, the proof now follows inductively from a careful examination of the proof of Theorem 4.1. □

Corollary 5.7. *If a graph G contains a unique cycle, then the sets* $\mathrm{Ass}\,(R/I^n)$ *form an ascending chain.*

6. Graphs with multiple cycles. We saw in Theorem 4.1 that our construction does produce all of the embedded associated primes of a graph. However, if the graph contains more than one cycle, the inequality in Theorem 4.1 can be strict. For the remainder of the paper we will see how to modify the definitions of U_n and P_n so that some of the embedded associated primes will now be in P_n for smaller values of n.

Definition 6.1. Suppose G is a graph containing a cycle C of length $2k+1$ for some $k \in \mathbf{N}$. Let $\tilde{S}_n(C)$ be the set of all possible $A = (R_n, B_n)$, $R_n \in R(C)_n$, such that $\langle A \rangle$ contains exactly one cycle. Let $d(A)$ be the monomial d_n corresponding to $R_n = R(A)$.

Notice that for each cycle C of a graph G, $\tilde{S}_n(C) \subseteq S_n(C)$. Also $\tilde{S}_n(C)$ could be empty. We will use the sets $\tilde{S}_n(C)$ to build associated primes.

Definition 6.2. Let G be a graph and let I be its edge ideal. If the smallest odd cycle C of G has length $2k+1$ for some $k \in \mathbf{N}$, define $T_{k+1} = \tilde{S}_{k+1}(C)$. For each $n > k+1$, define T_n to be the collection of all sets A such that either

1. $A \in \tilde{S}_n(C)$ for some cycle C of G, in which case $d(A)$ and $R(A)$ are as in Definition 6.1, or

2. $A = A_1 \cup A_2 \cup \cdots \cup A_s$ where $A_i \in T_{n_i}$ for some $n_i < n$, $R(A) = \cup R(A_i)$ and for each $i \in \{1, \ldots, s-1\}$, there is an edge $x_i y_i$ with $x_i \in R(A_i)$ and $y_i \in R(A_{i+1})$. In addition,

(a) if $s = 2t+1$ for some $t \in \mathbf{N}$ and there is an edge $x_s y_s$ with $x_s \in R(A_s)$ and $y_s \in R(A_1)$, then $n = n_1 + \cdots + n_s - t$. In this case $d(A) = d(A_1)d(A_2)\cdots d(A_s)$.

(b) else $n = n_1 + \cdots + n_s$ and $d(A) = (x_1 d(A_1))(x_2 d(A_2)) \cdots (x_{s-1} d(A_{s-1}))d(A_s)$.

We now prove the analogue of Lemmas 3.2 and 3.6.

Lemma 6.3. *Let G be a graph and let I be its edge ideal. Let $n \in \mathbf{N}$, $A \in T_n$ and $d = d(A)$. Then the following properties hold:*

1. *Every vertex that divides d is adjacent only to vertices in A.*
2. *d has degree $2n-1$, so $d \notin I^n$, but $d \in I^{n-1}$.*
3. *For each vertex x in A, $xd \in I^n$ and if $x \in R(A)$, then x divides d and $d/x \in I^{n-1}$.*

Proof. By Lemma 3.2 the three properties hold for $A \in T_{k+1}$ where k is such that the smallest odd cycle has length $2k+1$. Let $N \in \mathbf{N}$ and suppose that the three properties hold for all $n < N$. Let $A \in T_N$. If $A \in \tilde{S}_N(C)$ for some cycle C, the three properties hold by Lemma 3.2. So assume $A = A_1 \cup \cdots \cup A_s$ for some $s \geq 2$, $n_1, \ldots, n_s < N$ and ideals $A_1 \in T_{n_1}, \ldots, A_s \in T_{n_s}$ as in Definition 6.2. Let x_a be a vertex that divides d. Then x_a divides $d(A_i) = d_i$ for some i, or x_a divides $x_i d_i$ where $x_i \in R(A_i)$; so by induction all vertices adjacent to x_a are in $A_i \subseteq A$.

To see parts 2 and 3, first consider the case where $s = 2t+1$ for some $t \in \mathbf{N}$ and there is an edge $x_s y_s$ with $x_s \in R(A_s)$ and $y_s \in R(A_1)$. Then $d = d_1 \cdots d_s$ and $N = n_1 + \cdots + n_s - t$. Each d_i has degree $2n_i - 1$, so d has degree $2(n_1 + \cdots + n_s) - s = 2N - 1$. Thus $d \notin I^N$. For each $i \in \{1, 2, \ldots, s\}$, y_i divides d_{i+1} since y_i is in $R(A_{i+1})$ and $n_{i+1} < N$. Since $x_i \in R(A_i)$ and y_i is adjacent to x_i, $y_i \in A_i$. By induction, $d_{i+1}/y_i \in I^{n_{i+1}}$ and $d_i y_i \in I^{n_i}$. Then $d_i d_{i+1} = (d_i y_i)(d_{i+1}/y_i) \in I^{n_i + n_{i+1} - 1}$. Since $d = (d_1 d_2) \cdots (d_{2t-1} d_{2t})(d_s)$, $d \in I^{(n_1+n_2-1)+\cdots+(n_{2t-1}+n_{2t}-1)+(n_s-1)} \subseteq I^{N-1}$.

Now let x be a vertex in $R(A)$. Then $x \in R(A_j)$ for some j, so by induction x divides d_j. Then x divides d and $d/x = (d_j/x)(d_{j+1}d_{j+2}) \cdots (d_{j-2}d_{j-1})$. Since $d_j/x \in I^{n_j - 1}$ by hypothesis, $d/x \in I^{N-1}$.

Now consider the case where s is even or there is no edge $x_s y_s$ with $x_s \in R(A_s)$ and $y_s \in R(A_1)$. Then $N = n_1 + \cdots + n_s$ and $d = (d_1 x_1) \cdots (d_{s-1} x_{s-1})(d_s)$. Therefore, d has degree $2n_1 + \cdots + 2_{n_{s-1}} + (2n_s - 1) = 2N - 1$. Thus $d \notin I^N$. For each i, $x_i \in A_i$, so $d_i x_i \in I^{n_i}$. Since $d_s \in I^{n_s - 1}$, $d \in I^{n-1}$.

Let x be a vertex in $R(A)$. Then $x \in R(A_j)$ for some j. By the inductive hypothesis, $d_j/x \in I^{n_j - 1}$. Then $d/x = (d_1 x_1) \cdots (d_{j-1} x_{j-1})(d_j/x)$

$(d_{j+1}x_j)\cdots(d_s x_{s-1})$. Then $d/x \in I^{N-1}$.

So for either case $d/x \in I^{N-1}$ for each vertex x in $R(A)$. Let x be a vertex in A. Since $\langle A\rangle$ is connected, x must be adjacent to some vertex y in $R(A)$. Then $d/y \in I^{N-1}$ and $xy \in I$, so $dx = (xy)(d/y) \in I^N$. □

We now use the sets T_n to give a modified definition of U_n, from which we will again build the embedded associated primes.

Definition 6.4. Suppose G is a graph, let $n \in \mathbf{N}$ and let $s \geq 2$. Define $\tilde{U}_n$ to be the set of all possible $A = A_1 \cup \cdots \cup A_s$ such that $A_i \in T_{n_i}$ where $n_1, \ldots, n_s < n$ are such that $n-1 = (n_1-1)+\cdots+(n_s-1)$ and in addition, if x divides $d(A_i)$ for some i, then $x \notin A_j$ for all $i \neq j$. Define $d(A) = d(A_1)d(A_2)\cdots d(A_s)$.

Lemma 6.5. *Let $A \in \tilde{U}_n$ and $d = d(A)$ for some $n \in \mathbf{N}$. Then the following properties hold*:

1. *Every vertex that divides d is adjacent only to vertices in A.*
2. *$d \notin I^n$, but $d \in I^{n-1}$.*
3. *For each vertex x_a in A, $x_a d \in I^n$.*

Proof. The proof follows from Lemma 6.3 and the proof of Lemma 3.6 with minor modifications. □

Theorem 6.6. *Let G be a graph and let I be the edge ideal of G. Let $n \in \mathbf{N}$. Let $P = (A, V)$ where $A \in \tilde{U}_n$ and V is a minimal set of vertices such that P is a vertex cover of G. Then P is an associated prime of I^m for all $m \geq n$.*

Proof. The proof is similar to that of Theorem 3.7, using Lemma 6.5 in place of Lemma 3.6. □

Define $\tilde{P}_n$ to be the set of all primes $P = (A, V)$ where $A \in \tilde{U}_n$. Then the above theorem shows that $\tilde{P}_n \subseteq \operatorname{Ass}(R/I^n)$. Since $P_n \subseteq \tilde{P}_n$ the analog of Theorem 4.1 also holds. The embedded primes may appear in $\tilde{P}_n$ for smaller values of n. Thus using $\tilde{U}_n$ we produce some of the embedded primes at an earlier stage in the construction.

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