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A SYSTEMATIC GENERALIZATION PROCEDURE FOR FIXED-POINT THEOREMS

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ABSTRACT. The progress of mathematics often follows a standard path: the discovery of a new theorem, followed by a systematic exploration of that theorem. Two standard ways of exploring theorems are by weakening the hypotheses and strengthening the conclusion. This paper discusses another way to explore theorems through the sharing of hypotheses, and develops this idea in the context of three classical fixed-point theorems: the Banach Contraction Principle, the Tarski Fixed-Point Theorem for complete lattices and the Brouwer Fixed-Point Theorem for solid n-spheres.

Introduction. Mathematical research often progresses in accordance with the anthropologist Steven Jay Gould's theory of 'punctuated equilibrium,' in which dramatic breakthroughs are interspersed with long periods of quiet but gradual advance.

The Banach Contraction Principle provides a good example. When first discovered, this theorem represented a dramatic breakthrough. We state a preliminary version here as a reference point.

Banach Contraction Principle. *Let T be a map of a complete metric space X into itself. Assume there exists a constant $M \in (0,1)$ such that $d(Tx,Ty) \leq Md(x,y)$. Then T has a fixed point.*

Once the dramatic breakthrough has been made, other mathematicians will use the basic idea to discover additional theorems. Two classic ways to go about this are: (1) to strengthen the conclusion and (2) to weaken the hypothesis. In the case of the Banach Contraction Principle, a well-known strengthening of the conclusion is that the fixed point is unique. There are many examples of weakening the hypothesis; one such is Browder's result [**2**] that the conclusion holds if one assumes that u is an upper semi-continuous function on the positive reals such that $u(t) < t$ and $d(Tx,Ty) < u(d(x,y))$.

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Weakening the hypothesis and strengthening the conclusion are two time-honored and highly productive ways of developing new mathematical results. The purpose of this paper is to examine another way of developing new mathematical results. This technique, which we shall call 'sharing a hypothesis,' is probably not new, but an investigation of the AMS's Silver Platter database failed to reveal (to the author, anyway) a specific enunciation of the idea under this or related terms.

Suppose that we wish to examine potential generalizations of the Banach Contraction Principle. An obvious question to ask is the following: assume that T merely satisfies the hypothesis that for each x, $y \in X$, $d(T^n x, T^n y) \leq M d(x,y)$ for some integer $n = n(x,y)$. Must T have a fixed point?

A simple counterexample can be found on $[0, \infty)$ with the usual metric. Define $Tx = \sqrt{(x^2+1)}$. It is straightforward to show that $T^n x = \sqrt{(x^2+n)}$. Suppose $x < y$. Then

$$\begin{aligned} d(T^n x, T^n y)/d(x,y) &= [\sqrt{(x^2+n)} - \sqrt{(y^2+n)}]/(x-y) \\ &= (x+y)/[\sqrt{(x^2+n)} + \sqrt{(y^2+n)}] \end{aligned}$$

by rationalizing the numerator. Choosing n large will make this latter expression arbitrarily small. However, it is clear that T has neither fixed nor periodic points. This example is due to Prof. Joseph Bennish.

An alternative way of viewing the hypotheses of both the Banach Contraction Principle and the generalization suggested above is to consider a class $F = \{T_a : a \in A\}$ of continuous maps of X into itself. The contraction hypothesis is shared by the maps in F as follows. For each $x, y \in X$, there exists an $a \in A$, which may depend on x and y, such that $d(T_a x, T_a y) \leq M d(x,y)$. In the standard Banach Contraction Principle, the class F consists of a single map. In the generalization suggested above, which turned out to be false, the class F consists of the powers of T.

This paper will investigate 'shared-hypothesis' versions of three well-known fixed point theorems: the Banach Contraction Principle, the Tarski Fixed-Point theorem for complete lattices, and the Brouwer Fixed-Point Theorem for the solid n-sphere. We present the latter two theorems for reference purposes; the usual statement of the Brouwer Theorem has been slightly altered for reasons that will later become apparent.

Tarski Fixed-Point Theorem. *Let T be an isotone map on a complete lattice. Then T has a fixed point.*

Brouwer Fixed-Point Theorem. *Let S^n be the solid unit n-sphere in R^n. Let $T : R^n \to R^n$ be a continuous map which takes S^n into S^n. Then T has a fixed point.*

Shared-hypothesis versions can be proposed for the Tarski and Brouwer Theorems as well, because the key hypothesis in each theorem involves a restriction either on individual points in the domain of the map (in the Brouwer Theorem, $x \in S^n \Longrightarrow Tx \in S^n$), or on pairs of points (the Banach Contraction Principle has already been discussed, and, in the Tarski Theorem, the hypothesis is $x \leq y \Longrightarrow Tx \leq Ty$).

There are many potentially interesting *shared-hypothesis* versions of the three fixed-point theorems. We list some of the more interesting ways below.

Hypothesis 1 (Single map, finite powers). *Given a single map T, there is an integer N such that, for each point $x \in X$, or pair $x, y \in X$, there exists $n = n(x)$, or $n(x, y)$, with $n \leq N$ such that the hypothesis is true for $T^n x$ (or the pair $T^n x$, $T^n y$).*

In the remaining hypotheses, we shall simply state the situation for the single point (rather than the pairs of points) case.

Hypothesis 2 (Finitely many maps). *Given finitely many maps $T_1, \dots, T_N$, for each $x \in X$ there exists a $k \in \{1, \dots, N\}$ such that the hypothesis is true for $T_k x$.*

Hypothesis 3 (Single map, arbitrary powers). *Given a single map T, for each point $x \in X$ there exists an n such that the hypothesis is true for $T^n x$.*

Hypothesis 4 (Single map, all but finitely many powers). *Given a single map T, for each point $x \in X$ the hypothesis is true for all but finitely many $T^n x$.*

Hypothesis 5 (At least one of countably many maps). *Given a family $\{T_k : k = 1, 2, \dots\}$, for each point x there exists a k such that the hypothesis is true for $T_k x$.*

Hypothesis 6 (All but finitely many maps). *Given a family $\{T_k : k = 1, 2, \dots\}$, for each point x the hypothesis is true for all but finitely many $T_k x$.*

As far as the Banach Contraction Principle is concerned, the example given at the start of this paper shows that Hypotheses 3–6 do not yield theorems involving fixed or periodic points.

Before proceeding, we note that in theorems with several hypotheses there can be a variety of ways to share the hypotheses. Also, it is sometimes necessary to rephrase the statement of a theorem in order to facilitate this process. For instance, the usual statement of the Brouwer Theorem is that any continuous map of S^n into itself has a fixed point. The hypotheses involve both continuity and S^n and, even though both could be shared, it seems more reasonable to retain continuity for all the maps. As a result, a natural 'shared hypothesis' problem would be to assume that $T_1, \dots, T_N$ are continuous maps of R^n into itself such that, for each $x \in S^n$, there is an integer k such that $T_k x \in S^n$.

The Banach Contraction Principle, Tarski Theorem and Brouwer Theorem are extremely well-known and heavily-investigated theorems, so it should not be surprising that they are quite robust with respect to the sharing of the hypothesis. As a result, a sizable portion of this paper consists of counterexamples. Nonetheless, there are, perhaps somewhat surprisingly, positive results to be obtained from pursuing this approach, particularly if we expand our investigations to include maps possessing a natural relationship to the maps which share the hypothesis.

As a final preliminary note, we add that the idea of 'hypothesis-sharing' is not new and exists in diverse areas of mathematics. This paper is concerned with 'hypothesis-sharing' for fixed-point theorems, but it is certainly likely that an investigation of 'hypothesis-sharing' in other areas could well prove fruitful.

1. Fixed-point theorems in abstract sets. We first investigate 'shared-hypothesis' fixed-point theorems in abstract sets as a precursor to a study of the Banach Contraction Principle. The following elementary example is illustrative.

Example 1. Let $X = \{1, 2, 3\}$ with the usual metric, and let $M = 1/2$. Define maps T_1, T_2, and T_3 as follows:

$$\begin{aligned} T_1(1) = T_1(2) = 3 \qquad & T_1(3) = 1 \\ T_2(1) = T_2(3) = 2 \qquad & T_2(2) = 3 \\ T_3(2) = T_3(3) = 1 \qquad & T_3(1) = 2. \end{aligned}$$

For every pair $x \neq y$ there is a k such that $T_k x = T_k y$, and so $d(T_k x, T_k y) = 0 < Md(x, y)$, but no T_k has a fixed point.

Nonetheless, in Example 1, 1 is a fixed point of $T_3\, T_2\, T_1$ (and of many other products as well). This raises the following related question: suppose that the hypothesis is shared by a collection of maps—must some member of the semi-group generated by the maps in the collection have a fixed point? This paper will investigate this question in several different settings. Throughout this paper, G will denote the semi-group generated by the maps sharing the appropriate hypothesis.

The motivation for the results in this section actually comes from considering the Banach Contraction Principle in certain discrete metric spaces. Suppose that X is the integers, $T_1, \dots, T_N$ maps of X into X, and that there is an $M \in (0, 1)$ such that, for each $x, y \in X$, there is a $k \in \{1, \dots, N\}$ with $d(T_k x, T_k y) \leq Md(x, y)$. Since $|x - y| \geq 1$, iterating the process of choosing the appropriate T_k, in accordance with the above condition, eventually produces a $T \in G$ such that $Tx = Ty$.

This argument shows that, in a discrete metric space in which any two points are separated by a minimum distance δ, Hypothesis 3 ensures the existence of a unique fixed point. Operating on both x and Tx with an appropriate power of T reduces the distance between them by a factor of M. Continue this process until, for some integer k, $d(T^k x, T^k Tx) < \delta \Longrightarrow T^k x = T^k Tx$, and so $T^k x$ is a fixed point of T. This fixed point is unique, for if $Tx = x$ and $Ty = y$, then $T^n x = x$ for all k and similarly for y. If x and y are distinct, Hypothesis 3 guarantees the existence of an integer k such that T^k reduces the distance between x and y, resulting in the usual contradiction.

If X is a discrete metric space in which any two points are separated by a minimum distance δ, suppose that $\{T_n : n = 1, 2, \dots\}$ is a commuting family of maps of X into X satisfying Hypothesis 5. Modifying the above argument, start with the points x and $T_k x$, and construct a $T \in G$ such that $Tx = TT_k x$. Therefore, Tx is a fixed point of T_k, and we see that each T_k has a fixed point.

Theorem 1. *Let X be a set and let $T_1, \dots, T_N$ map X into X. Assume that, for each $x \in X$ and $j \in \{1, \dots, N\}$, there exists an integer $k \in \{1, \dots, N\}$ and positive integers p and n such that $T_k^p T_j^n x = T_k x$. Then some T in G has a fixed point.*

Proof. Let $x \in X$. In order to avoid needlessly complicated subscripts and superscripts, let W denote the semi-group of words formed by concatenating the letters $1, \dots, N$. Define a relation $\sim$ on W as follows. If $v, w \in W$, we say $v \sim w$ if

$$T_{i_1} \cdots T_{i_n}\ x = T_{j_1} \cdots T_{j_p} x$$

where $v = i_1 \dots i_n$ and $w = j_1 \dots j_p$. Note that

(i) $\sim$ is an equivalence relation on W

(ii) $v \sim w$ and $u \in W \Longrightarrow uv \sim uw$

(iii) $uv \sim w$ and $v \sim v' \Longrightarrow uv' \sim w$.

Assume that there exist words $s, t \in W$ such that $st \sim t$ and s is not the empty word. If $s = i_1 \dots i_n$ and $t = j_1 \dots j_p$, we see that $T_{j_1} \dots T_{j_p} x$ is a fixed point of $T_{i_1} \dots T_{i_n}$. Therefore, to demonstrate the existence of fixed points for operators in G, it suffices to show that nontrivial solutions to $st \sim t$ exist.

Let $k_1 = 1$. The hypothesis of the theorem enables us to find a $k_2 \in \{1, \dots, N\}$ and positive integers n_1 and p_2 such that $k_2^{p_2} k_1^{n_1} \sim k_2$. If $k_2 = k_1$, then $(k_1^{p_2+n_1-1})k_1 \sim k_1$, and some $T \in G$ has a fixed point. Assume therefore that $k_2 \neq k_1$.

Continue inductively, and assume that integers $k_1, \dots, k_r$ belonging to $\{1, \dots, N\}$ and positive integers $n_1, \dots, n_{r-1}$ and $p_2, \dots, p_r$ have been chosen such that $k_1, \dots, k_r$ are all distinct and $k_j^{p_j} k_{j-1}^{n_{j-1}} \sim k_j$ for $2 \leq j \leq r$. The hypothesis of the theorem insures that we can find $k_{r+1} \in \{1, \dots, N\}$ and positive integers p_{r+1} and n_r such that $k_{r+1}^{p_{r+1}} k_r^{n_r} \sim k_{r+1}$.

We now assert that, if $k_{r+1} = k_q$ for some integer q with $1 \le q \le r$, then some $T \in G$ must have a fixed point. For then

$$\begin{aligned} k_q &\sim k_q^{p_{r+1}} k_r^{n_r - 1} k_r \\ &\sim k_q^{p_{r+1}} k_r^{n_r - 1} k_r^{p_r} k_{r-1}^{n_{r-1}} \\ &\sim k_q^{p_{r+1}} k_r^{n_r + p_r - 1} k_{r-1}^{n_{r-1} - 1} k_{r-1} \\ &\sim \cdots \sim \\ &\sim k_q^{p_{r+1}} k_r^{n_r + p_r - 1} \cdots k_{q+1}^{n_{q+1} + p_{q+1} - 1} k_q^{n_q - 1} k_q. \end{aligned}$$

So $k_q \sim v k_q$ and some $T \in G$ has a fixed point. We therefore proceed on the assumption that $k_1, \dots, k_{r+1}$ are all distinct. This assumption only remains valid until $r = N$, at which point it is no longer possible to choose a different k_{r+1}. This completes the proof. $\square$

A consequence of Theorem 1 is that, under the assumptions of Hypothesis 2, if X is a discrete metric space of finite diameter D in which any two points are separated by at least δ, then some $T \in G$ has a fixed point. Choose an integer p such that $DM^p < \delta$. If $T_1, \dots, T_N$ are the given maps, then given any two points x and y in X, there is a map $T = T_{k_1} \cdots T_{k_n}$ such that $Tx = Ty$ and $n \le p$. If we consider the subsemi-group H consisting of all those members of G which are composites of at most p of the $T_1, \dots, T_N$ (obviously, repetitions are allowed), then using the members of H as the maps in Theorem 1 yields the desired result for H and hence for G.

Consider the defining equality of Theorem 1: $T_k^p T_j^n x = T_k x$. This can be regarded as an equality of the form $T_k^p y = T_k^q x$ where the pair (x, y) belongs to some subset S of $X \times X$ and $q = 1$. Two immediate questions arise—is it possible to 'thin out' the subset S, thereby reducing the 'number' of equalities that must be satisfied in order to guarantee a fixed point, and can one guarantee fixed points if the equalities satisfied use a value of q other than 1?

The answer to the second question is regrettably negative, as allowing more flexibility in the choice of q would obviously generate more fixed-point theorems. Let X be the integers, and define $Tx = Sx = x + 1$.

Then $TSx = T^2x$ and $STx = S^2x$ for all x, but clearly no operator in the semi-group generated by S and T has a fixed point.

The answer to the first question is more encouraging, as the constructive exhaustion argument of Theorem 1 can easily be modified to handle other situations. One such result is given in the following theorem.

Theorem 2. *Let X be a set, and let $T_1, \dots, T_N$ map X into X. Let H be the subsemi-group of G consisting of operators which are concatenations of at least n members of $\{T_1, \dots, T_N\}$, repetitions being allowed. Suppose that, for any $x \in X$ and $S, T \in H$, $k \in \{1, \dots, N\}$ exists such that $T_kTx = T_kSx$. Then some $T \in G$ has a fixed point.*

Proof. We use the word and $\sim$ notation of the proof of Theorem 1. Let $k_1 = 1$. The hypothesis guarantees the existence of an integer $k_2 \in \{1, \dots, N\}$ such that $k_2k_1^{n+1} \sim k_2k_1^n$. If $k_2 = k_1$, then $k_1k_1^{n+1} \sim k_1^{n+1}$; as in Theorem 1 this would guarantee the existence of a fixed point for some $T \in g$. We therefore assume $k_2 \neq k_1$.

Assume inductively that $k_jk_{j-1}\dots k_2k_1^n \sim k_jk_1^n$ for $2 \leq j \leq p$ and that $k_1, \dots, k_p$ are distinct. The hypothesis guarantees the existence of an integer $k_{p+1} \in \{1, \dots, N\}$ such that $k_{p+1}k_p \cdots k_2k_1^n \sim k_{p+1}k_1^n$. If $k_{p+1} = k_1$, then $k_1^{n+1} \sim k_1k_p \cdots k_2k_1^n$. Since $k_2k_1^{n+1} \sim k_2k_1^n$, by properties (i) and (iii) of $\sim$ cited in Theorem 1, we see that $k_1^{n+1} \sim k_1k_p \cdots k_2k_1^{n+1}$. Again, this would guarantee the existence of a fixed point for some $T \in G$, so we assume $k_{p+1} \neq k_1$.

If $k_{p+1} = k_q$ for some integer q with $2 \leq q \leq p$, then $k_qk_p \cdots k_2k_1^n \sim k_qk_1^n \sim k_q \cdots k_2k_1^n$, the last equivalence coming from the inductive hypothesis. Consequently,

$$k_qk_p \cdots k_{q+1}(k_q \cdots k_2k_1^n) \sim k_q \cdots k_2k_1^n$$

again ensuring the existence of a fixed point for some $T \in G$. We therefore assume that $k_1, \dots, k_{p+1}$ are distinct. As in Theorem 1, this situation can only occur if $p \leq N$, and the theorem is proved. $\square$

In Theorem 2, the equality $T_kTx = T_kSx$ can be interpreted as using the maps T_k to generate the needed equalities for pairs of the form

(Tx, Sx). As might be expected, if we use the (infinitely many) maps from G to generate the equalities rather than the (finitely many) maps $T_1, \dots, T_N$, fixed points are no longer guaranteed, much as the example given at the start of the paper ($Tx = \sqrt{(x^2+1)}$ on $[0, \infty)$) shows that the Banach Contraction Principle is no longer valid when we use the infinitely many maps T^n rather than the single map T.

Example 2. Let X be the positive integers. Define $Tx = x+1$, and define $Sx = x+1$ if $x = 2^n$ for some integer n, and $Sx = x+2$ for all other x. Let G be the semi-group generated by S and T. Obviously, no map from G has a fixed point. However, given any two integers $x > y$, it is possible to find an operator $U \in G$ such that $Ux = Uy$. U will have the form $S\,T^i\,S\,T^j \cdots S\,T^k$ and is constructed as follows: choose p so large that $2^{p+1} - 2^p > x - y$ and $2^{p+1} > x$. Let $k = 2^{p+1} - x$. Then $ST^k x - ST^k y = (x-y) - 1$. Applying ST^k to both x and y therefore decreases the distance between the points x and y by 1. Continuing this process results in a U of the form described above such that $Ux = Uy$.

2. The Banach Contraction Principle. We have already observed that if X is a discrete metric space of finite diameter and separation, then Hypothesis 2 applied to the Banach Contraction Principle will yield maps in G with fixed points. The following theorem describes a situation in which a shared hypothesis results in fixed points for some maps in G.

Theorem 3. *Let X be a complete metric space, $M \in (0,1)$, and assume that $T_1, \dots, T_N$ are commuting maps of X into X such that there is a constant $M_k > 0$ for which $d(T_k x, T_k y) \le M_k d(x,y)$. Assume that, for each $x, y \in X$, there is an integer $k \in \{1, \dots, N\}$ such that $d(T_k x, T_k y) \le M d(x,y)$. If the product $\prod_{k=1}^N M_k < 1/M$, then some $T \in G$ has a unique fixed point.*

Proof. Let $T = T_1 T_2 \dots T_N$, and let $r = M \prod_{k=1}^N M_k < 1$. We can assume without loss of generality that each $M_k \ge 1$; otherwise, T_k is a strict contraction and the result is trivially true by the Banach Contraction Principle. Let $x, y \in X$ and choose k such that $d(T_k x, T_k y) \le$

$Md(x,y)$. By commutativity, $T = T_1 \cdots T_{k-1}T_{k+1} \cdots T_N T_k$, and so

$$\begin{aligned} d(Tx,Ty) &\leq M_1 d(T_2 \cdots T_{k-1}T_{k+1} \cdots T_N T_k x, \\ &\qquad \cdot T_2 \cdots T_{k-1}T_{k+1} \cdots T_N T_k y) \\ &\leq \cdots \leq M_1 \cdots M_{k-1}M_{k+1} \cdots M_N d(T_k x, T_k y) \\ &\leq M_1 \cdots M_{k-1}M_{k+1} \cdots M_N M d(x,y) \\ &\leq r d(x,y), \end{aligned}$$

since we have assumed each $M_k \geq 1$. Therefore T is a strict contraction, and the Banach Contraction Principle applies. $\square$

Note that in Theorem 3 the hypothesis $d(T_k x, T_k y) \leq M_k d(x,y)$ is only used for those points $x, y \in T_j X$ and so each T_k only needs to satisfy $d(T_k x, T_k y) \leq M_k d(x,y)$ on each $T_j X$.

Observe that the example given at the beginning of this paper, $Tx = \sqrt{(x^2+1)}$ on $[0,\infty)$ is a contraction such that for each $x, y \in [0,\infty)$, there exists an N such that $n \geq N \Longrightarrow d(T^n x, T^n y) \leq Md(x,y)$. This example obviously limits the extent to which Theorem 3 can be improved.

Theorem 3 involves a restrictive Lipschitz condition on the maps. If only one map is involved, as is the case with Hypothesis 1, a stronger result can be obtained.

Definition 1. A map T of a metric space into itself is said to be strongly continuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} d(x_k, y_k) < \delta \Longrightarrow \sum_{k=1}^{n} d(Tx_k, Ty_k) < \varepsilon.$$

The concept of strong continuity is motivated by absolute continuity on the real line. Maps satisfying Lipschitz conditions are easily seen to be strongly continuous. Note that the composite of strongly continuous maps is strongly continuous.

Theorem 4. *Let X be a complete metric space, $M \in (0,1)$, and let $T : X \to X$ be strongly continuous. Assume there is an integer N such* *that, if $x, y \in X$, $d(T^k x, T^k y) \leq M d(x,y)$ for some integer $k \leq N$. Then T has a unique fixed point.*

Proof. As in the standard proof of the Banach Contraction Principle, we show that, for any $x \in X$, $\{T^n x : n = 1, 2, \dots\}$ forms a Cauchy sequence. Once this is done, the proof is essentially complete, as the limit y of that sequence will be a fixed point. If z is any other fixed point, choose $n \leq N$ such that $d(T^n y, T^n z) \leq M d(y,z)$. The usual argument shows that $y = z$.

Let $n_1 = 1$. We are assured of the existence of an integer n_2 such that $d(T^{n_2} x, T^{n_2 - 1} x) \leq M d(Tx, x)$ where $n_2 - n_1 \leq N$. Now choose an integer n_3 such that $d(T^{n_3} x, T^{n_3 - 1} x) \leq M d(T^{n_2} x, T^{n_2 - 1} x) \leq M^2 d(Tx, x)$ and $n_3 - n_2 \leq N$. Continuing inductively, we construct a sequence of integers $\{n_k : k = 1, 2, \dots\}$ such that $n_{k+1} - n_k \leq N$ and $d(T^{n_k} x, T^{n_k - 1} x) \leq M^{k-1} d(Tx, x)$.

Since $T, T^2, \dots, T^N$ are strongly continuous, choose $\delta > 0$ such that $\sum_{k=1}^n d(x_k, y_k) < \delta \rightarrow \sum_{k=1}^n d(T^p x_k, T^p y_k) < 1$ for $p \leq N$. Choose J so large that $\sum_{k=J}^\infty M^{k-1} d(Tx, x) < \delta$. Observe that, if $n_p < k < n_{p+1}$, then $d(T^k x, T^{k-1} x) = d(T^q T^{n_p} x, T^q T^{n_p - 1} x)$, where $1 \leq q \leq N$. Then, for $j > J$,

$$\sum_{k=1}^{j} d(T^k x, T^{k-1} x) = \sum_{k=1}^{J-1} d(T^k x, T^{k-1} x) + \sum_{k=J}^{j} d(T^k x, T^{k-1} x).$$

We take a closer look at the second term on the right.

$$\sum_{k=J}^{j} d(T^k x, T^{k-1} x) = \sum_{q=0}^{N} \sum_{\substack{p \\ J \leq n_p \leq j}} d(T^q T^{n_p} x, T^q T^{n_p - 1} x).$$

For $q = 0$,

$$\sum_{\substack{p \\ J \leq n_p \leq j}} d(T^{n_p} x, T^{n_p - 1} x) < \delta.$$

For $q = 1, 2, \dots, N$,

$$\sum_{\substack{p \\ J \leq n_p \leq j}} d(T^q T^{n_p} x, T^q T^{n_p - 1} x) < 1.$$

Therefore,

$$\sum_{k=1}^{j} d(T^k x, T^{k-1}x) < \sum_{k=1}^{J-1} d(T^k x, T^{k-1}x) + \delta + N.$$

Consequently, the partial sums of the series $\sum_{k=1}^{\infty} d(T^k x, T^{k-1}x)$ form a bounded monotonic sequence, and thus the series converges, completing the proof. $\square$

3. Fixed-point theorems in partially-ordered sets. A major result in the theory of partially-ordered sets is the Tarski Fixed-Point Theorem [**1**, p. 115], which states that any isotone function on a complete lattice has a fixed point. The following example shows that the Hypothesis 1 version of the Tarski Theorem fails.

Example 3. Let I denote the unit interval $[0,1]$, which is a complete lattice under the usual operations of upper and lower bound. Let Q denote the rationals. Let $Q_1 = [0,1/3) \cap Q$, $I_1 = [0,1/3)\backslash Q$, $Q_2 = [1/3,2/3) \cap Q$, $I_2 = [1/3,2/3)\backslash Q$, $Q_3 = [2/3,1] \cap Q$, $I_3 = [2/3,1]\backslash Q$. I is clearly the disjoint union of all these sets. Define $T : I \to I$ as follows

$$\begin{aligned}
Tx &= (x+5)/4\pi, && x \in Q_1\\
Tx &= (x+16)/6\pi, && x \in Q_2\\
Tx &= (x+1)/2\pi, && x \in Q_3\\
Tx &= (3x+4)/6, && x \in I_1\\
Tx &= (3x-1)/6, && x \in I_2\\
Tx &= (3x+1)/6, && x \in I_3.
\end{aligned}$$

Note that $T(I)$ consists solely of irrational numbers and that T is linear on each of the six disjoint subsets whose union is I. Furthermore,

$$\begin{array}{lll}
T : Q_1 \longrightarrow (1/3,1/2) & T : Q_2 \longrightarrow (5/6,1) & T : Q_3 \longrightarrow (1/6,1/3)\\
T : I_1 \longrightarrow (2/3,5/6) & T : I_2 \longrightarrow (0,1/6) & T : I_3 \longrightarrow (1/2,2/3).
\end{array}$$

We now show that, if $x < y$, then either $Tx < Ty$, $T^2x < T^2y$, or $T^3x < T^3y$. Since T is monotone increasing on each of the six disjoint subsets, if x and y both belong to the same one of these six disjoint subsets, then $Tx < Ty$. The other 18 possible cases in which $x < y$ are tabulated below.

$$
\begin{aligned}
x \in Q_1, \quad y \in Q_2 &\Longrightarrow T^2x < T^2y \\
x \in I_1, \quad y \in I_2 &\Longrightarrow T^3x < T^3y \\
x \in Q_1, \quad y \in Q_3 &\Longrightarrow T^2x < T^2y \\
x \in I_1, \quad y \in I_3 &\Longrightarrow T^3x < T^3y \\
x \in Q_2, \quad y \in Q_3 &\Longrightarrow T^2x < T^2y \\
x \in I_2, \quad y \in I_3 &\Longrightarrow T^3x < T^3y \\
x \in I_1, \quad y \in Q_1 &\Longrightarrow T^3x < T^3y \\
x \in Q_1, \quad y \in I_1 &\Longrightarrow Tx < Ty \\
x \in I_2, \quad y \in Q_2 &\Longrightarrow Tx < Ty \\
x \in Q_2, \quad y \in I_2 &\Longrightarrow T^2x < T^2y \\
x \in I_3, \quad y \in Q_3 &\Longrightarrow T^2x < T^2y \\
x \in Q_3, \quad y \in I_3 &\Longrightarrow Tx < Ty \\
x \in I_1, \quad y \in Q_2 &\Longrightarrow Tx < Ty \\
x \in Q_1, \quad y \in I_2 &\Longrightarrow T^2x < T^2y \\
x \in I_1, \quad y \in Q_3 &\Longrightarrow Tx < Ty \\
x \in Q_1, \quad y \in I_3 &\Longrightarrow Tx < Ty \\
x \in I_2, \quad y \in Q_3 &\Longrightarrow Tx < Ty \\
x \in Q_2, \quad y \in I_3 &\Longrightarrow T^3x < T^3y.
\end{aligned}
$$

We now show that T has neither fixed nor periodic points. Since Tx is always irrational, clearly no rational number can be a fixed or periodic point of T. Restricted to I_1, I_2 or I_3, T^n is a composite of linear maps with rational coefficients. On any one of these three subsets, T^n must have the form $T^nx = ax + b$, where both a and b are rational, and both a and b lie in $(0,1)$. Therefore, if T has an irrational number x as a fixed or periodic point, $x = T^nx = ax+b$. Solving for x, $x = b/(1-a)$, which is a rational number. Therefore, T has no fixed or periodic points.

Nonetheless, it is possible to prove a weaker version of the Tarski Theorem for finitely many functions.

Theorem 5. *Let L be a complete lattice, and let $T_1, \dots, T_N$ map L into L. Suppose that* (i) *the T_k commute,* (ii) *each T_i is isotone on each T_jL and* (iii) *for each $x, y \in L$ with $x \le y$, there is an integer $k \in \{1, \dots, N\}$ such that $T_k x \le T_k y$. Then some $T \in G$ has a fixed point.*

Proof. Let $T = T_1 T_2 \cdots T_N$. If $x, y \in L$, then there is an integer $k \in \{1, \dots, N\}$ such that $T_k x \le T_k y$. By commutativity, $T = T_1 \cdots T_{k-1} T_{k+1} \cdots T_N T_k$. Since $T_k x \le T_k y$, by hypothesis (ii), $T_N T_k x \le T_N T_k y$. Repeated applications of hypothesis (ii) to $T = T_1 \cdots T_{k-1} T_{k+1} \cdots T_N T_k$ shows that T is isotone on L. The Tarski Fixed-Point Theorem shows that $T \in G$ has a fixed point. $\square$

Note that, in Theorem 5, hypothesis (ii) does not imply that $T_i T_j$ is isotone on L (which would enable us to apply the Tarski Theorem to the isotone map $T_i T_j$ on the complete lattice L). Also, $T_j L$ is in general not a complete lattice (as in Example 3) which would enable us to apply the Tarski Theorem to the isotone map T_i on the complete lattice L. The maps in Example 3 satisfy every hypothesis of Theorem 5 except commutativity.

Theorem 5 is provable because the hypotheses enable a composite of the T_k to satisfy the Tarski Theorem. We also saw this idea in conjunction with the Banach Contraction Principle.

Using hypotheses and methods similar to those of Theorems 1 and 2 enable us to prove a shared hypothesis fixed-point theorem for isotone maps on arbitrary partially-ordered sets.

Theorem 6. *Let X be a partially-ordered set, and let $T_1, \dots, T_N$ be isotone maps of X into X. Suppose that, for each $x, y \in X$, there is an integer $k \in \{1, \dots, N\}$ such that $T_k x \le T_k y$. Then maps $S, T \in G$ and $u, v \in X$ exist such that $Tu \le u$ and $v \le Sv$. If each T_k is one-to-one, then some $T \in G$ has a fixed point.*

Proof. Let $x \in X$ and $k_1 = 1$. Choose $k_2 \in \{1, \dots, N\}$ such that $T_{k_2} x \le T_{k_2}(T_{k_1} x)$. If $k_2 = k_1$, then $T_{k_1} x \le T_{k_1}(T_{k_1} x)$, and so S and v would exist such that $v \le Sv$. Therefore, assume $k_2 \ne k_1$.

We proceed by induction. Assume that distinct integers $k_1, \dots, k_p$ have been chosen such that

$$T_{k_p}x \leq T_{k_p}T_{k_{p-1}}x \leq \cdots \leq T_{k_p}T_{k_{p-1}} \cdots T_{k_1}x.$$

Choose $k_{p+1} \in \{1, \dots, N\}$ such that $T_{k_{p+1}}x \leq T_{k_{p+1}}T_{k_p}x$. The isotonicity of $T_{k_{p+1}}$ ensures that

$$T_{k_{p+1}}x \leq T_{k_{p+1}}T_{k_p}x \leq \cdots \leq T_{k_{p+1}}T_{k_p} \cdots T_{k_1}x.$$

If $k_{p+1} = k_q$ for $1 \leq q \leq p$, then $T_{k_q}T_{k_p} \cdots T_{k_{q+1}}T_{k_q}x \geq T_{k_q}x$, and so S and v would exist such that $v \leq Sv$. Assuming that $k_1, \dots, k_{p+1}$ are distinct is only possible if $p \leq N$. So $S \in G$ and $v \in X$ exist such that $v \leq Sv$. The existence of $T \in G$ and $u \in X$ follow from reversing all the inequalities in the above proof.

Now assume that each T_k is one-to-one, and that $v \in X$ and $S \in G$ satisfy $v \leq Sv$. Choose $k \in \{1, \dots, N\}$ such that $T_k(Sv) \leq T_kv$. Since T_k is isotone, $T_kv \leq T_k(Sv) \Longrightarrow T_k(Sv) = T_kv$. Since T_k is one-to-one, $Sv = v$. $\square$

Example 2, with the usual order on the integers, shows that the conclusion of Theorem 6 does not hold under the hypothesis that, for each $x, y \in X$, there is a map $U \in G$ such that $Ux \leq Uy$. Notice that each map in G is one-to-one.

The basic idea of Example 3, linear maps with rational coefficients on the irrationals, can be modified to show that the Hypothesis 6 version of the Tarski Theorem also fails.

Example 4. Let $I = [0, 1]$, and let Q denote the rationals. For $n = 1, 2, \dots$, define

$$\begin{array}{llll} a_n = 1-5/(\pi(n+1)), & b_n = 1/(\pi(n+1)), & T_n = a_nx+b_n, & x \in I \cap Q \\ \alpha_n = 1-3/(2(n+1)), & \beta_n = 1/(n+1), & T_n = \alpha_nx+\beta_n, & x \in I \backslash Q. \end{array}$$

The choice of coefficients insures that $T_n : I \to I\backslash Q$ and that T_n is increasing on $I \cap Q$ and $I\backslash Q$. Note also that

$$\begin{array}{ll} 0 < b_n < \beta_n & \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} b_n = 0 \\ 0 < a_n < \alpha_n < 1 & \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} a_n = 1. \end{array}$$

We now show that $\{T_n : n = 1, 2, \dots\}$ satisfy Hypothesis 6, i.e., given $x < y$, there is an integer $N = N(x, y)$ such that $n \geq N \Longrightarrow T_n x < T_n y$. If both x and y are rational or if both x and y are irrational, then $T_n x < T_n y$ for all n. Also, since $a_n < \alpha_n$ and $b_n < \beta_n$, if $x < y$ and $x \in I \cap Q$, $y \in I \backslash Q$, then

$$T_n x = a_n x + b_n < a_n y + b_n < \alpha_n y + \beta_n = T_n y.$$

The only remaining case is $x < y$, $x \in I \backslash Q$, $y \in I \cap Q$. Choose N_1, N_2 and N_3 such that

$$\begin{aligned} n \geq N_1 &\Longrightarrow \beta_n - b_n < (y - x)/3 \\ n \geq N_2 &\Longrightarrow \alpha_n - a_n < (y - x)/3y \\ n \geq N_3 &\Longrightarrow \alpha_n > 3/4. \end{aligned}$$

Then, if $n \geq \max(N_1, N_2, N_3)$,

$$\beta_n - b_n + (\alpha_n - a_n)y < 2(y - x)/3 < 3(y - x)/4 < \alpha_n(y - x).$$

Therefore, $\beta_n - b_n - a_n y < -\alpha_n x$, and so

$$T_n x = \alpha_n x + \beta_n < a_n y + b_n = T_n y.$$

The same argument as used at the end of Example 3 shows that no composite of the $\{T_n : n = 1, 2, \dots\}$ has any fixed points, as the image under any composite of a rational is irrational, and any composite of the $\{T_n : n = 1, 2, \dots\}$ restricted to the irrationals is a linear map with rational coefficients, which can only have rational fixed points.

However, it is possible to obtain a fixed-point theorem when Hypothesis 6 is in force. We need to introduce several definitions, which are lattice analogs of the standard concepts of inferior and superior limits.

Definition 2. Let $\{a_n : n = 1, 2, \dots\}$ be a sequence of elements in a complete lattice. Then

$$\begin{aligned} \liminf_{n\to\infty} a_n &= \bigvee_{N=1}^{\infty} \bigwedge_{n=N}^{\infty} a_n \\ \limsup_{n\to\infty} a_n &= \bigwedge_{N=1}^{\infty} \bigvee_{n=N}^{\infty} a_n. \end{aligned}$$

We state without proof the following elementary properties of $\liminf$ and $\limsup$.

(i) If $a_n \leq b_n$ for all but finitely many n,then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n \tag{i.a}$$

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n \tag{i.b}$$

(ii)

$$\liminf_{n\to\infty} a_{n+1} = \liminf_{n\to\infty} a_n \qquad \limsup_{n\to\infty} a_{n+1} = \limsup_{n\to\infty} a_n$$

Theorem 7. *Let L be a complete lattice. Assume that $\{T_n : n = 1, 2, \dots\}$ are maps of L into L such that, for every $x \leq y$ there exists an N such that $n \geq N \Longrightarrow T_n x \leq T_n y$. Define $Sx = \liminf_{n\to\infty} T_n x$. Then S has a fixed point. If there is a $T : L \to L$ such that, for every $x \leq y$, there exists an N such that $n \geq N \Longrightarrow T^n x \leq T^n y$, define $Qx = \liminf_{n\to\infty} T^n x$.*

Then, not only does Q have a fixed point, but so does TQ. Analogous results hold when $\liminf$ *is replaced by* $\limsup$.

Proof. Suppose that $x \leq y$. Property (i.a) above shows that $Sx \leq Sy$. By the Tarski Theorem, S has a fixed point.

By property (ii), for any $x \in L$,

$$\begin{aligned} QTx &= \liminf_{n\to\infty} T^n Tx = \liminf_{n\to\infty} T^{n+1} x \\ &= \liminf_{n\to\infty} T^n x \\ &= Qx. \end{aligned}$$

If u is a fixed point of Q, which exists by the first portion of the proof, then $QTu = Qu = u$, and therefore $TQTu = Tu$, and so Tu is a fixed point of TQ. $\square$

4. The Brouwer Fixed-Point Theorem. The Schauder-Tychonoff theorem [**3**, p. 456], which generalizes the Brouwer Fixed-Point Theorem, states that every continuous map of a compact convex subset of a locally convex linear topological space has a fixed point. We define a class of results which we call Schauder-Tychonoff Theorems.

Definition 3. Let A be a subset of a set X, and let F be a collection of maps of X into itself which is closed under composition. A Schauder-Tychonoff Theorem of class (X, A, F) states that any $T \in F$ which maps A into A must have a fixed point in A.

If a Schauder-Tychonoff Theorem of class (X, A, F) exists, it is possible to obtain a shared hypothesis fixed-point theorem under conditions similar to those present in Theorems 3 and 6.

Theorem 8. *Assume that a Schauder-Tychonoff Theorem of class* (X, A, F) *exists. Let* $T_1, \dots, T_N$ *belong to* F*, and assume* (i) *the* T_k *commute,* (ii) $T_j x \in A \Longrightarrow T_i T_j x \in A$*, and* (iii) *for each* $x \in A$*, there is an integer* $k \in \{1, \dots, N\}$ *such that* $T_k x \in A$*. Then some* $T \in G$ *has a fixed point in* A*.*

Proof. Once again, let $T = T_1 T_2 \cdots T_N$. Let $x \in A$, and choose k such that $T_k x \in A$. Then $Tx = T_1 \cdots T_{k-1} T_{k+1} \cdots T_N T_k x$. But $T_k x \in A \Longrightarrow T_N T_k x \in A \Longrightarrow \cdots \Longrightarrow Tx \in A$. By the Schauder-Tychonoff theorem for (X, A, F), T has a fixed point. $\square$

Theorems 3, 5 and 8 have common assumptions—commuting maps, the shared hypothesis and some restriction concerning the products of the maps. It is not clear whether this represents a possible limit to which shared hypothesis fixed-point theorems can be extended or is simply an artifact of a method of proof that is effective in all three cases.

We turn our attention back to the Brouwer Theorem. The first result is that the Hypothesis 2 version of the Brouwer Theorem fails even in dimension 1.

Example 5. Let I denote the unit interval $[0, 1]$. Define $Tx = x + 1/5$, and let $Sx = x - \pi/5$. If $x \in [0, 4/5]$, then $Tx \in [0, 1]$, and if $x \in (4/5, 1]$, then $Sx \in [0, 1]$. Since S and T commute, for any nonnegative integers n and k, $T^n S^k x = x + (n - k\pi)/5$. Therefore, the only solutions to $T^n S^k x = x$ occur when $n - k\pi = 0$, and so the irrationality of π shows that this can only occur if $n = k = 0$. Therefore, not only do S and T have no fixed points, but no member of G has a fixed point.

Shared hypothesis versions of the Brouwer Theorem can, however, be proved in dimension 1. As a matter of fact, Hypothesis 3 is enough to guarantee a fixed point for a continuous $T : R^1 \to R^1$. Suppose that, for each $x \in [0,1]$, there is an integer n such that $T^n x \in [0,1]$, and assume that T has no fixed point. If $T(0) > 0$, then the function $f(x) = Tx - x$ satisfies $f(0) > 0$. Moreover, $f(x) > 0$ for all x, otherwise the Intermediate Value Theorem shows that $f(x) = 0$ for some x, and x is therefore a fixed point of T.

Consequently, $Tx > x$ for all x. Therefore, $T(1) > 1$ and $T^2(1) = T(T(1)) > T(1) > 1$. Continuing inductively, $T^n(1) > 1$ for all n, contradicting the existence of an integer n for which $T^n(1) \in [0,1]$. Similarly, the assumption that $T(0) < 0$ leads to the conclusion that $Tx < x$ for all x, and hence that $T^n(0) < 0$ for all n. Therefore, T must have a fixed point.

A similar, but shorter, argument shows that Hypothesis 6 yields fixed points in dimension 1. Suppose that $T_n : R^1 \to R^1$, and to each $x \in [0,1]$, there is an integer N such that $n \geq N \Longrightarrow T_n x \in [0,1]$. We can therefore find an integer k such that both $T_k(0) \in [0,1]$ and $T_k(1) \in [0,1]$. If 0 or 1 is a fixed point of T_k we are done, so assume $T_k(0) > 0$ and $T_k(1) < 1$ and apply the Intermediate Value Theorem to $f(x) = T_k x - x$ as before.

However, the above result fails in higher dimensions, even if Hypothesis 6 is satisfied.

Example 6. Define $T_n : R^2 \to R^2$ on vertical strips $S_1 = \{(x,y) : x < 1 - 1/n\}$, $S_2 = \{(x,y) : 1 - 1/n \leq x < 1 - 1/2n\}$, $S_3 = \{(x,y) : 1 - 1/2n \leq x < 1\})$, and $S_4 = \{(x,y) : 1 \leq x\}$. The value of $T_n(x, y)$ is defined as follows

$$
\begin{aligned}
&T_n(x,y) = (1,0), \quad (x,y) \in S_1\\
&T_n(x,y) = (1, 2n(x-1)+2), \quad (x,y) \in S_2\\
&T_n(x,y) = (2n(1-x), 2n(2n+1)^2(x-1)(x-(1-1/2n))y+1),\\
&\qquad\qquad\qquad (x,y) \in S_3\\
&T_n(x,y) = (0,1), \quad (x,y) \in S_4.
\end{aligned}
$$

To verify that this is indeed a counterexample, we must check that each T_n is continuous, that if $\|(x, y)\| \le 1$, there exists an N such that $n \ge N \Longrightarrow \|T_n(x, y)\| \le 1$, and finally that no T_n has a fixed point.

Obviously, T_n is continuous in the interior of each of the four vertical strips, and it is easy to verify that T_n is constant on each of the vertical boundaries and that the values of these constants guarantee continuity along the boundaries.

Note that T_n maps both S_1 and S_4 into the unit circle. If $\|(x, y)\| \le 1$ and $x < 1$, then $(x, y) \in S_1$ for all but finitely many n, and if $x = 1$, then $(1, 0) \in S_4$ for all n.

Finally, to see that T_n has no fixed points, observe that because the x coordinate of the point $T_n(x, y)$ is 1 in S_1 and S_2, and because the x coordinate of the point $T_n(x, y)$ is 0 in S_4, the only fixed points must be in S_3. In order for $T_n(x, y) = (x, y)$, we must have $2n(1 - x) = x$ and so $x = 2n/(1 + 2n)$. However, the value of the y coordinate of the point $T_n(2n/(1+2n), y)$ can be seen to be $y+1$, and so T_n has no fixed points.

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