ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 2, Summer 2000

POLYNOMIALS ON SCHREIER'S SPACE

MANUEL GONZÁLEZ AND JOAQUÍN M. GUTIÉRREZ

ABSTRACT. We introduce a weakened version of the Dunford-Pettis property and give examples of Banach spaces with this property. In particular, we show that every closed subspace of Schreier's space S enjoys it. As an application we characterize the weak polynomial convergence of sequences, show that every closed subspace of S has the polynomial Dunford-Pettis property of Biström et al. and give other polynomial properties of S.

A subset $A = \{n_1 < \cdots < n_k\}$ of the natural numbers $\mathbf{N}$ is said to be *admissible* if $k \leq n_1$. Schreier's space S [**22**], [**4**] is the completion of the space c_{00} of all scalar sequences of finite support with respect to the norm:

$$\|x\|_S := \sup\left\{\sum_{j\in A} |x_j| : A \subset \mathbf{N} \text{ is admissible}\right\}, \quad \text{for } x = (x_j)_{j=1}^{\infty}.$$

Some basic properties of S may be seen in [**6**]. Schreier's space has been used to provide counterexamples in Banach space theory [**2**], [**6**], [**7**], [**20**], [**21**].

In this paper we introduce a weakened version of the Dunford-Pettis property and give examples of Banach spaces with this property. In particular, we show that every closed subspace of S enjoys it. It is well known that a reflexive Banach space with the Dunford-Pettis property must be finite dimensional. The same is true for a Banach space with the Banach-Saks property and the weak Dunford-Pettis property. As an application we investigate polynomial properties of S, characterizing the sequences which converge in the weak polynomial topology that we

Received by the editors on February 28 1998, and in revised form on January 14, 1999.
2000 AMS *Mathematics Subject Classification.* Primary 46B20.
Key words and phrases. Weak Dunford-Pettis property, polynomial Dunford-Pettis property, Schreier's space, weak polynomial convergence, Banach-Saks set, Banach-Saks property.
The first author supported in part by DGICYT grant PB 97-0349 (Spain).
The second author supported in part by DGICYT grant PB 96-0607 (Spain).

shall call the $\mathcal{P}$-topology. As far as we know, this is the first time that $\mathcal{P}$-convergent sequences are characterized for a space where $\mathcal{P}$-convergence does not coincide with either norm of weak convergence of sequences. From this we obtain that every closed subspace of S has the polynomial Dunford-Pettis property [**3**].

We also show that the relatively compact sets for the $\mathcal{P}$-topology coincide with the Banach-Saks sets, that the absolutely convex closed hull of a Banach-Saks set in S is a Banach-Saks set, and that the tensor product of two Banach-Saks sets is a Banach-Saks set in the projective tensor product $S \otimes_\pi S$. It is unknown if the Banach-Saks sets in an arbitrary Banach space are stable under convex hulls. An example of a Banach space so that the relatively $\mathcal{P}$-compact sets are not stable under convex hulls was given in [**5**]. Moreover, given two $\mathcal{P}$-null (i.e., $\mathcal{P}$-convergent to zero) sequences $(x_n), (y_n) \subset S$, we prove that $\{x_n \otimes y_n\}$ is a Banach-Saks set in $S \otimes_\pi S$. The polynomial Dunford-Pettis property of S implies that the sequence $(x_n \otimes y_n)$ is $\mathcal{P}$-null in $S \otimes_\pi S$, and that $(x_n + y_n)$ is $\mathcal{P}$-null in S. These properties have interesting consequences in infinite dimensional holomorphy, as shown in [**15**, Remark 4.7].

We shall use the facts that the unit vector basis of S is unconditional, and that every closed subspace of S contains an isomorphic copy of c_0 (so S contains no copy of l_1).

Throughout the paper, E will denote a Banach space and E^* its dual. The space of all scalar valued k-homogeneous (continuous) polynomials on E is represented by $\mathcal{P}(^k E)$. General references for polynomials on Banach spaces are [**11**], [**19**]. Given a subset $A \subset \mathbf{N}$, $\operatorname{card} A$ stands for the cardinality of A.

A sequence $(x_n) \subset E$ is $\mathcal{P}$-*convergent* to x if $P(x_n) \to P(x)$ for every $P \in \mathcal{P}(^k E)$ and all $k \in \mathbf{N}$. A set $A \subset E$ is *relatively* $\mathcal{P}$*-compact* if every sequence in A has a $\mathcal{P}$-convergent subsequence.

A subset $A \subset E$ is a *Banach-Saks set* if every sequence in A has a subsequence whose arithmetic means converge in norm. A sequence $(x_n) \subset E$ *converges uniformly weakly* to x in E [**18**, Definition 2.1] if, for each $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbf{N}$ such that $\operatorname{card}\{n \in \mathbf{N} : |\phi(x_n - x)| \geq \varepsilon\} \leq N(\varepsilon)$ for every $\phi \in E^*$ with $\|\phi\| \leq 1$. A subset $A \subset E$ is a Banach-Saks set if and only if every sequence in A has a subsequence which is uniformly weakly convergent in E [**18**, Theorem 2.9].

Recall that a Banach space E has the *Dunford-Pettis property* (DPP for short) if, for all weakly null sequences $(x_n) \subset E$ and $(\phi_n) \subset E^*$, we have $\phi_n(x_n) \to 0$. We say that E has the *polynomial Dunford-Pettis property* if, for every $\mathcal{P}$-null sequence $(x_n) \subset E$ and every weakly null sequence $(\phi_n) \subset E^*$, we have $\phi_n(x_n) \to 0$. The DPP implies the polynomial DPP. E is said to be a Λ-*space* if $\mathcal{P}$-null sequences and norm null sequences coincide in E. Spaces with the Schur property are trivially Λ-spaces. All super-reflexive spaces are Λ-spaces [**16**]. It is proved in [**13**, Corollary 3.6] that every Banach space with nontrivial type is a Λ-space.

1. The weak Dunford-Pettis property. We say that a Banach space E has the *weak Dunford-Pettis property* (wDPP for short) if, given a uniformly weakly null sequence $(x_n) \subset E$ and a weakly null sequence $(\phi_n) \subset E^*$, we have $\lim \phi_n(x_n) = 0$.

The space l_2 fails the wDPP since its unit vector basis is uniformly weakly null. Clearly, if E has the DPP, then E has the wDPP.

Denote by T the dual of the original Tsirelson space T^* [**4**]. Then the uniformly weakly convergent sequences in T are norm convergent. Indeed, suppose (x_n) is uniformly weakly convergent to $x \in T$ and $\|x_n - x\| \geq \delta > 0$. Passing to a subsequence, we may assume that the sequence $(x_n - x)$ is basic and equivalent to a subsequence of the unit vector basis (t_n) of T [**4**]. If $A \subset \mathbf{N}$ is admissible, by the definition of the norm of T, we have

$$\left\| \sum_{i \in A} t_i \right\| \geq \frac{1}{2} \operatorname{card} A$$

and so, (t_n) has no uniformly weakly null subsequence, which yields a contradiction.

Therefore, T enjoys the wDPP, but T^* does not since the unit vector basis of T^* is a Banach-Saks set. We conclude that the wDPP of a Banach space neither implies nor is implied by the wDPP of its dual.

The following simple remark will be useful.

Proposition 1.1. *A Banach space E has the* wDPP *if and only if whenever $(x_n) \subset E$ is uniformly weakly null and $(\phi_n) \subset E^*$ is weak Cauchy, we have* $\lim \phi_n(x_n) = 0$.

Proof. For the nontrivial part, if $\phi_n(x_n) \geq \delta > 0$, we can find $k_1 < \cdots < k_n < \cdots$ such that $|\phi_n(x_{k_n})| < \delta/2$. Then,

$$\delta \leq \phi_{k_n}(x_{k_n}) \leq |(\phi_{k_n} - \phi_n)(x_{k_n})| + |\phi_n(x_{k_n})|$$

and the righthand side is less than δ for n large enough, since the sequence $(\phi_{k_n} - \phi_n)$ is weakly null. $\square$

Denoting by $\mathcal{WC}o(E,F)$ the space of all weakly compact (linear) operators from E into the Banach space F, and by $\mathcal{C}_w(E,F)$ the space of all operators taking uniformly weakly null sequences in E into norm null sequences in F, we have

Proposition 1.2. *The Banach space E satisfies the* wDPP *if and only if, for all Banach spaces F, we have $\mathcal{WC}o(E,F) \subseteq \mathcal{C}_w(E,F)$.*

Proof. Suppose E has the wDPP and $(x_n) \subset E$ is uniformly weakly null. Take $L \in \mathcal{WC}o(E,F)$ with adjoint L^*. Choose (ϕ_n) in the unit ball of F^* such that $\phi_n(Lx_n) = \|Lx_n\|$. There is a subsequence (ϕ_{n_k}) such that $(L^*\phi_{n_k})$ is weakly convergent. Hence, $\phi_n(Lx_n) = (L^*\phi_n)x_n \to 0$. Conversely, if E fails the wDPP, we can find (x_n) uniformly weakly null in E and (ϕ_n) weakly null in E^* such that $\phi_n(x_n) \geq \delta > 0$. We define an operator $L : E \to c_0$ by $Lx := (\phi_n(x))$. Then L is weakly compact but $\|Lx_n\| \geq |\phi_n(x_n)| \geq \delta > 0$ for all n. $\square$

The following easy fact characterizes the reflexive Banach spaces with the wDPP.

Proposition 1.3. *Let E be a reflexive Banach space. Then E has the* wDPP *if and only if every uniformly weakly null sequence in E is norm null.*

Proof. Suppose that there is a uniformly weakly null sequence $(x_n) \subset E$ with $\|x_n\| = 1$. We can assume that (x_n) is basic and the sequence of coefficient functionals (ϕ_n) is weakly null in E^*. Since $\phi_n(x_n) = 1$, we conclude that E does not have the wDPP. The converse is clear. $\square$

Recall that a Banach space E has the *Banach-Saks property* if every bounded subset in E is a Banach-Saks set. We then have

Corollary 1.1. *If E has the Banach-Saks property and the* wDPP*, then E is finite dimensional.*

A space E has the *weak Banach-Saks property* if every weakly null sequence in E contains a subsequence whose arithmetic means converge. Equivalently [**18**], every weakly null sequence has a subsequence which converges to zero uniformly weakly in E. The space $L^1[0,1]$ has the weak Banach-Saks property. The following result is clear.

Proposition 1.4. *Assume E has the weak Banach-Saks property. Then E has the* DPP *if and only if E has the* wDPP*.*

We say that E has the *hereditary weak Dunford-Pettis property* if every closed subspace of E has the wDPP.

Proposition 1.5. *A Banach space E has the hereditary* wDPP *if and only if every normalized uniformly weakly null sequence in E contains a subsequence equivalent to the c_0-basis.*

Proof. Suppose that the uniformly weakly null sequence $(x_n) \subset E$, $\|x_n\| = 1$, has no subsequence equivalent to the c_0-basis. We can assume that (x_n) is basic. Let $(\phi_n) \subset [x_n]^*$ be the sequence of coefficient functionals where $[x_n]$ denotes the closed linear span of the set $\{x_n\}$ in E. After taking a subsequence, we can assume that either (ϕ_n) is equivalent to the l_1-basis or (ϕ_n) is weak Cauchy [**10**]. In the first case we define an operator $L : [x_n] \to c_0$ by $L(x) := (\phi_n(x))$. Clearly L is injective and has dense range. The adjoint $L^* : l_1 \to [x_n]^*$ takes the unit vector basis of l_1 into the sequence (ϕ_n) and therefore has closed range. Hence, L is a surjective isomorphism, which contradicts our assumption. So (ϕ_n) must be weak Cauchy. Since $\phi_n(x_n) = 1$, the subspace $[x_n]$ fails to have the wDPP.

For the converse, it is enough to show that E has the wDPP. Suppose it does not. Then we can find a uniformly weakly null sequence

$(x_n) \subset E$ and a weakly null sequence $(\phi_n) \subset E^*$ such that $\phi_n(x_n) \geq 1$ for all n. Passing to a subsequence, we can assume that (x_n) is equivalent to the c_0-basis. Since the dual of c_0 has the Schur property, the restriction of (ϕ_n) to the subspace $[x_k]$ is norm null, and we get a contradiction. $\square$

Remark 1.1. This simple proof also shows that a Banach space E has the hereditary DPP if and only if every normalized weakly null sequence in E has a subsequence equivalent to the c_0-basis [**8**, Proposition 2]. From this we get that every infinite-dimensional Banach space without a copy of either c_0 or l_1 contains a subspace without the DPP [**10**, p. 254]. The original proofs of these two results were based on a characterization of c_0's unit vector basis that Elton [**12**] obtained by using Ramsey's theorem.

Our aim now is to show that Schreier's space enjoys the hereditary wDPP.

Proposition 1.6. *If (x_n) is a uniformly weakly null sequence in S, then $\|x_n\|_\infty \to 0$.*

Proof. Let $x_n = (x_n^i)_{i=1}^\infty$. Since a set of ± 1's on an admissible set is a norm-one functional on S, given $\varepsilon > 0$, there is $N(\varepsilon) \in \mathbf{N}$ such that

$$\operatorname{card}\left\{n \in \mathbf{N} : \sum_{i \in A} |x_n^i| \geq \varepsilon\right\} \leq N(\varepsilon)$$

for each admissible A. Suppose our statement fails; then we can find $\delta > 0$ and two increasing sequences of indices $(n_k), (l_k)$ such that

$$|x_{n_k}^{l_k}| \geq \delta \quad \text{for all } k.$$

The set $A_m := \{l_{m+1}, \dots, l_{2m}\}$ is admissible for each $m \in \mathbf{N}$ and

$$\operatorname{card}\left\{n \in \mathbf{N} : \sum_{i \in A_m} |x_n^i| \geq \delta\right\} \geq m,$$

a contradiction which finishes the proof. $\square$

The converse is not true. Indeed, take $x_n := (e_1 + \cdots + e_n)/n$. The set $A_k := \{2^{k-1}, \ldots, 2^k - 1\}$ is admissible for each $k \in \mathbf{N}$. Denoting by (e_i^*) the unit vector basis of S^*, the functional

$$\phi_k := \sum_{i=2^{k-1}}^{2^k-1} e_i^* \in S^*$$

has norm one. Choosing n so that $2^{k-2} + 2^{k-1} \leq n \leq 2^k - 1$, we have

$$\phi_k(x_n) \geq \frac{2^{k-2}}{n} > \frac{2^{k-2}}{2^k} = \frac{1}{4}.$$

Therefore, $\|x_n\|_\infty \to 0$, but (x_n) does not converge to zero uniformly weakly. The proof of the following result is essentially contained in [**7**]. We give it for completeness.

Proposition 1.7. *Let (x_n) be a normalized sequence in S such that $\|x_n\|_\infty \to 0$. Then (x_n) contains a subsequence equivalent to the c_0-basis.*

Proof. Let us denote by $\operatorname{supp}(x)$ the support of x. Passing to a subsequence and perturbing it with a null sequence, we can assume that $\max\operatorname{supp}(x_n) < \min\operatorname{supp}(x_{n+1})$, and

$$\|x_n\|_\infty \leq \frac{1}{2^n \max\operatorname{supp}(x_{n-1})}. \tag{1.1}$$

Given $x_{n_1}, \ldots, x_{n_m}$ and an admissible set A, we take k_0 to be the minimum value of k such that $A \cap \operatorname{supp}(x_{n_k}) \neq \varnothing$. In particular, this implies that $\operatorname{card} A \leq \max\operatorname{supp}(x_{n_{k_0}})$. Denoting $x_n(i) := x_n^i$, we have

$$\begin{aligned}
\sum_{i\in A}\left|\left(\sum_{k=1}^m x_{n_k}\right)(i)\right| &= \sum_{i\in A}\left|\left(\sum_{k=k_0}^m x_{n_k}\right)(i)\right| \\
&= \sum_{k=k_0}^m \sum_{i\in A\cap\operatorname{supp}(x_{n_k})} |x_{n_k}(i)| \\
&\leq \|x_{n_{k_0}}\| + \sum_{k=k_0+1}^m \|x_{n_k}\|_\infty \cdot \operatorname{card} A \\
&\leq \|x_{n_{k_0}}\| + \sum_{k=k_0+1}^m 2^{-n_k} \leq 2,
\end{aligned}$$

where we have used (1.1). Thus we have proved that

$$\Big\| \sum_{k=1}^{m} x_{n_k} \Big\| \leq 2$$

and hence the series $\sum x_n$ is weakly unconditionally Cauchy. Therefore, (x_n) has a subsequence equivalent to the c_0-basis [**10**]. $\square$

Combining the last two results with Proposition 1.5 yields

Theorem 1.1. *Schreier's space S has the hereditary* wDPP.

We now show that the dual S^* of Schreier's space fails the wDPP. The next result follows the lines of [**17**].

Proposition 1.8. *Let (ϕ_n) be a normalized block basis of the unit basis of S^* such that $\|\phi_n\|_\infty \to 0$. Then (ϕ_n) contains a subsequence equivalent to the l_1-basis.*

Proof. Let (x_n) be a sequence in S such that $\|x_n\| < 2$, $\operatorname{supp}(x_n) = \operatorname{supp}(\phi_n)$ and $\phi_n(x_n) = 1$ for every n.

First we select n_1 such that $\min\operatorname{supp}(\phi_{n_1}) > 2^2$ and $\|\phi_{n_1}\|_\infty < 2^{-4}$. Since $\|x_{n_1}\| < 2$, the set

$$A_1 = \{i \in \mathbf{N} : |x_{n_1}(i)| \geq 2^{-1}\}$$

has fewer than 2^2 elements. We define $y_{n_1}(i) = 0$ if $i \in A_1$ and $y_{n_1}(i) = x_{n_1}(i)$ otherwise, and obtain $y_{n_1} \in S$ such that $\|y_{n_1}\| < 2$, $\|y_{n_1}\|_\infty < 2^{-1}$ and

$$|\phi_{n_1}(y_{n_1})| \geq \phi_{n_1}(x_{n_1}) - |\phi_{n_1}(y_{n_1} - x_{n_1})| > 1 - 2(2^2)2^{-4} = 2^{-1}.$$

Next we select $n_2 > n_1$ such that $\min\operatorname{supp}(\phi_{n_2}) > 2^3$ and $\|\phi_{n_2}\|_\infty < 2^{-5}$. Since $\|x_{n_2}\| < 2$, the set

$$A_2 = \{i \in \mathbf{N} : |x_{n_2}(i)| \geq 2^{-2}\}$$

has fewer than 2^3 elements. We define $y_{n_2}(i) = 0$ if $i \in A_2$ and $y_{n_2}(i) = x_{n_1}(i)$ otherwise, and obtain $y_{n_2} \in S$ such that $\|y_{n_2}\| < 2$, $\|y_{n_2}\|_\infty < 2^{-2}$ and

$$|\phi_{n_2}(y_{n_2})| \geq \phi_{n_2}(x_{n_2}) - |\phi_{n_2}(y_{n_2} - x_{n_2})| > 1 - 2(2^3)2^{-5} = 2^{-1}.$$

In this way we get a subsequence (ϕ_{n_j}) and a sequence $(y_{n_j}) \subset S$ such that $|\phi_{n_j}(y_{n_j})| > 2^{-1}$, $\|y_{n_j}\| < 2$ and $\|y_{n_j}\|_\infty < 2^{-j}$. Passing to a subsequence we can assume by Proposition 1.7 that (y_{n_j}) is equivalent to the c_0-basis, from which it easily follows that (ϕ_{n_j}) is equivalent to the l_1-basis. □

Proposition 1.9. *The dual S^* of Schreier's space S has the weak Banach-Saks property.*

Proof. Let (ϕ_n) be a normalized weakly null sequence in S^*. Passing to a subsequence we can assume that (ϕ_n) is equivalent to a block basis of the unit basis. We have that $((\phi_1 + \cdots + \phi_n)/n)$ is a weakly null sequence and $\|(\phi_1 + \cdots + \phi_n)/n\|_\infty \to 0$. If $\|(\phi_1 + \cdots + \phi_n)/n\|$ does not converge to zero, passing to a subsequence, it follows from Proposition 1.8 that $((\phi_1 + \cdots + \phi_n)/n)$ contains a subsequence equivalent to the l_1-basis, a contradiction. □

Corollary 1.2. *The dual S^* of Schreier's space does not have the* wDPP.

2. Applications to polynomials. In this section we describe the $\mathcal{P}$-convergence of sequences in S, thereby obtaining some polynomial properties of this space, and characterize the Banach-Saks sets in it.

We shall use the fact that S may be algebraically embedded in l_2 and that the natural inclusion $j : S \to l_2$ is continuous. To see this, take $x := (x_i) \in S$, $\|x\|_S = 1$ and call $y := (y_i)$ the sequence $(|x_i|)$, reordered in a nonincreasing way. Then $\|y\|_2 = \|x\|_2$ and $\|y\|_S \leq 1$. This implies $y_{2k-1} \leq k^{-1}$ for each k. Therefore,

$$\|y\|_2^2 = \sum_{i=1}^{\infty} y_i^2 \leq 1 + 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{3},$$

from which $\|j\| \le \pi/\sqrt{3}$.

As a consequence, $P(x) := \|x\|_2^2$ defines a 2-homogeneous polynomial on S.

Proposition 2.1. *Let (x_n) be a sequence in S. The following assertions are equivalent*

(a) (x_n) *is $\mathcal{P}$-null*;

(b) (x_n) *is bounded in S and* $\|x_n\|_2 \to 0$;

(c) (x_n) *is bounded in S and* $\|x_n\|_\infty \to 0$.

Proof. (a) $\Rightarrow$ (b) since $P(x) := \|x\|_2^2$ is a polynomial on S.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). It is enough to show that (x_n) has a $\mathcal{P}$-null subsequence. If $\inf\|x_n\| > 0$, then there is a subsequence of (x_n) equivalent to the c_0-basis (Proposition 1.7) and so $\mathcal{P}$-null, since the c_0-basis is $\mathcal{P}$-null. If $\inf\|x_n\| = 0$, then there is a norm null subsequence, which is $\mathcal{P}$-null a fortiori. □

A Banach space has the *hereditary polynomial* DPP if every closed subspace has the polynomial DPP.

Theorem 2.1. *The space S has the hereditary polynomial* DPP.

Proof. By Propositions 2.1 and 1.7 every normalized $\mathcal{P}$-null sequence in S contains a subsequence equivalent to the c_0-basis. Obvious modifications in the "if" part of the proof of Proposition 1.4 yield the result. □

It is shown in [**3**] that, given two $\mathcal{P}$-null sequences $(x_n), (y_n)$ in a space with the polynomial DPP, the sequence $(x_n + y_n)$ is $\mathcal{P}$-null. A Banach space where this is not true was recently found by Castillo et al. [**5**].

Proposition 2.2. *Let A be a subset of S. The following assertions are equivalent*:

(a) *A is a Banach-Saks set*;

(b) *A is relatively $\mathcal{P}$-compact*;

(c) *A is relatively weakly compact in S and relatively compact as a subset of l_∞.*

Proof. (a) $\Rightarrow$ (b). Let A be a Banach-Saks set. Given a sequence $(x_n) \subset A$, passing to a subsequence, we may assume that (x_n) converges to some x uniformly weakly in S. Then $(x_n - x)$ has a subsequence which is either norm null or equivalent to the c_0-basis. In both cases (x_n) is $\mathcal{P}$-convergent to x.

(b) $\Rightarrow$ (c). If A is relatively $\mathcal{P}$-compact, it is relatively weakly compact. Moreover, given a sequence $(x_n) \subset A$, we can assume that $(x_n - x)$ is $\mathcal{P}$-null for some x. By Proposition 2.1, $\|x_n - x\|_\infty \to 0$ and so A is relatively compact as a subset of l_∞.

(c) $\Rightarrow$ (a). Choose a sequence $(x_n) \subset A$. We may assume that (x_n) is weakly convergent to some x and $\|x_n - x\|_\infty \to 0$. Passing to a subsequence, we have either $\|x_n - x\| \to 0$ or, by Proposition 1.7, $(x_n - x)$ is equivalent to the c_0-basis and is therefore uniformly weakly null. $\square$

Corollary 2.1. *If A is a Banach-Saks set in S, then the absolutely convex closed hull of A is a Banach-Saks set.*

The following two properties were introduced in [**1**] and studied by various authors (see, e.g., [**3**], [**9**]).

(a) A Banach space E has property (P) if, given two bounded sequences $(u_n), (v_n)$ in E such that $P(u_n) - P(v_n) \to 0$ for every $P \in \mathcal{P}(^kE)$ and all k, it follows that the sequence $(u_n - v_n)$ is $\mathcal{P}$-null. Every superreflexive space and every space with the DPP have property (P). A Banach space failing to have property (P) has been found by Castillo et al. [**5**].

(b) A Banach space E has property (RP) if, given two bounded sequences $(u_n), (v_n)$ in E such that the sequence $(u_n - v_n)$ is $\mathcal{P}$-null, it follows that $P(u_n) - P(v_n) \to 0$ for every $P \in \mathcal{P}(^kE)$ and all k. Every Λ-space and every predual of a Banach space with the Schur property have property (RP). The spaces $L_1[0,1]$, $C[0,1]$ and $L_\infty[0,1]$ fail to have property (RP) [**1**].

We now show that S has property (P) and fails property (RP).

Proposition 2.3. *The space S fails property* (RP).

Proof. Consider the vectors

$$v_n := e_n; \qquad u_n := e_n + 2^{1-n}(e_{2^{n-1}} + \cdots + e_{2^n-1}).$$

Then $\|u_n - v_n\|_\infty \to 0$ and so $(u_n - v_n)$ is $\mathcal{P}$-null in S. Define

$$P(x) := \sum_{n=1}^{\infty} x_n^2 \bigg(\sum_{k=2^{n-1}}^{2^n-1} x_k \bigg), \quad \text{for } x = (x_n) \in S.$$

Since

$$|P(x)| \le \|x\|_S \cdot \|x\|_2^2 \le \frac{\pi^2}{3} \cdot \|x\|_S^3,$$

we get that $P \in \mathcal{P}(^3S)$. We have $P(v_n) = 0$ and $P(u_n) = 1$ for all $n > 1$. $\square$

In the above proof, we need a polynomial of degree greater than or equal to three. Indeed, if $P \in \mathcal{P}(^2S)$ and $(u_n), (v_n) \subset S$ are bounded with $(u_n - v_n)$ $\mathcal{P}$-null, denoting $w_n := u_n - v_n$, we have

$$P(u_n) - P(v_n) = P(w_n + v_n) - P(v_n) = 2\hat{P}(w_n, v_n) + P(w_n),$$

where $\hat{P}$ is the symmetric bilinear form associated to P. Let $\overline{P} : S \to S^*$ be the operator defined by $\overline{P}(x)(y) := \hat{P}(x, y)$. Since S has an unconditional basis and contains no copy of l_1, the space S^* has an unconditional basis and is weakly sequentially complete. Therefore every operator from S into S^* is weakly compact. Passing to a subsequence we can assume that (w_n) is uniformly weakly null. Since S has the wDPP, $\|\overline{P}(w_n)\| \to 0$. Hence, $\hat{P}(w_n, v_n) = \overline{P}(w_n)(v_n) \to 0$. Clearly, $P(w_n) \to 0$ and so $P(u_n) - P(v_n) \to 0$.

Proposition 2.4. *The space S enjoys property* (P).

Proof. Let $(u_n), (v_n) \subset S$ be bounded sequences such that $(u_n - v_n)$ is not $\mathcal{P}$-null. We wish to find $Q \in \mathcal{P}(^kS)$ for some k so that $(Q(u_n) - Q(v_n))$ does not tend to zero. By u_n^i and v_n^i we shall denote the ith coordinate of u_n and v_n, respectively.

If $(u_n - v_n)$ is not weakly null, then $\phi(u_n) - \phi(v_n) \not\to 0$ for some $\phi \in S^*$. It is enough to take $Q := \phi$.

If $(u_n - v_n)$ is weakly null, passing to a subsequence and perturbing it by a norm null sequence, we can assume that $(u_n - v_n)$ is a block basis

$$u_n - v_n = \sum_{i=k_n}^{l_n} a_i e_i.$$

Take p_n with $k_n \leq p_n \leq l_n$ and $|a_{p_n}| = \|u_n - v_n\|_\infty$. We know that $\|u_n - v_n\|_\infty$ does not go to zero. Passing to a subsequence, we may assume

$$v_n^{p_n} \longrightarrow v; \qquad u_n^{p_n} \longrightarrow u; \quad u \neq v.$$

Let $P \in \mathcal{P}(^2S)$ be given by

$$P(x) := \sum_{i=1}^{\infty} (x_{p_i})^2.$$

If $P(u_n) - P(v_n) \not\to 0$ we are done. If

$$\begin{aligned} 0 &= \lim[P(u_n) - P(v_n)] = \lim[(u_n^{p_n})^2 - (v_n^{p_n})^2] \\ &= u^2 - v^2 = (u - v)(u + v), \end{aligned}$$

we have $u = -v = \alpha$ for some $\alpha \neq 0$. Defining $Q \in \mathcal{P}(^3S)$ by

$$Q(x) := \sum_{i=1}^{\infty} (x_{p_i})^3,$$

we have $Q(u_n) - Q(v_n) \to 2\alpha^3 \neq 0$, and the proof is finished. $\square$

Proposition 2.5. *Let* $(x_n), (y_n) \subset S$ *be* $\mathcal{P}$*-null sequences. Then*

(a) *the set* $\{x_n \otimes y_n\}$ *is a Banach-Saks set in* $S \otimes_\pi S$;

(b) *the sequence* $(x_n \otimes y_n)$ *is* $\mathcal{P}$*-null in* $S \otimes_\pi S$.

Proof. (a) Since (x_n) and (y_n) have subsequences equivalent to the c_0-basis, it is enough to show that $(e_n \otimes e_n)$ is uniformly weakly null in $c_0 \otimes_\pi c_0$. Take $L \in (c_0 \otimes_\pi c_0)^*$, which may be viewed as an operator from c_0 into l_1. Since the series $\sum e_n$ is weakly unconditionally Cauchy, using [**14**, Theorem 2] we can find $C > 0$ such that $\sum |\langle Le_n, e_n\rangle| \leq C$ whenever $\|L\| \leq 1$. Therefore, given $\varepsilon > 0$, choosing $N \in \mathbf{N}$ with $N \geq C/\varepsilon$, we have

$$\operatorname{card}\{n \in \mathbf{N} : |\langle Le_n, e_n\rangle| \geq \varepsilon\} \leq N$$

if $\|L\| \leq 1$, and the result is proved.

(b) Since S has the polynomial DPP, part (b) follows from [**3**, Theorem 2.1]. $\square$

As a consequence, if $A, B \subset S$ are Banach-Saks sets, then $A \otimes B$ is a Banach-Saks set in $S \otimes_\pi S$.

REFERENCES

1. R.M. Aron, Y.S. Choi and J.G. Llavona, *Estimates by polynomials*, Bull. Austral. Math. Soc. **52** (1995), 475–486.

2. B. Beauzamy, *Banach-Saks properties and spreading models*, Math. Scand. **44** (1979), 357–384.

3. P. Biström, J.A. Jaramillo and M. Lindström, *Polynomial compactness in Banach spaces*, Rocky Mountain J. Math. **28** (1998), 1203–1226.

4. P.G. Casazza and T.J. Shura, *Tsirelson's space*, Lecture Notes in Math. **1363**, Springer-Verlag, Berlin, 1989.

5. J.M.F. Castillo, R. García and R. Gonzalo, *Banach spaces in which all multilinear forms are weakly sequentially continuous*, Studia Math. **136** (1999), 121–145.

6. J.M.F. Castillo and M. González, *An approach to Schreier's space*, Extracta Math. **6** (1991), 166–169.

7. ———, *The Dunford-Pettis property is not a three-space property*, Israel J. Math. **81** (1993), 297–299.

8. P. Cembranos, *The hereditary Dunford-Pettis property on* $C(K, E)$, Illinois J. Math. **31** (1987), 365–373.

9. Y.S. Choi and S.G. Kim, *Polynomial properties of Banach spaces*, J. Math. Anal. Appl. **190** (1995), 203–210.

10. J. Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Math. **92**, Springer, Berlin, 1984.

11. S. Dineen, *Complex analysis in locally convex spaces*, North-Holland Math. Stud. **57**, North-Holland, Amsterdam, 1981.

12. J.H. Elton, *Weakly null normalized sequences in Banach spaces*, Ph.D. Thesis, Yale University, 1978.

13. J. Farmer and W.B. Johnson, *Polynomial Schur and polynomial Dunford-Pettis properties*, Contemp. Math. **144** (1993),95–105.

14. M. González and J.M. Gutiérrez, *Unconditionally converging polynomials on Banach spaces*, Math. Proc. Cambridge Philos. Soc. **117** (1995), 321–331.

15. ———, *Gantmacher type theorems for holomorphic mappings*, Math. Nachr. **186** (1997), 131–145.

16. J.A. Jaramillo and A. Prieto, *Weak polynomial convergence on a Banach space*, Proc. Amer. Math. Soc. **118** (1993), 463–468.

17. D. Leung, *Uniform convergence of operators and Grothendieck spaces with the Dunford-Pettis property*, Math. Z. **197** (1988), 21–32.

18. S. Mercourakis, *On Cesaro summable sequences of continuous functions*, Mathematika **42** (1995), 87–104.

19. J. Mujica, *Complex analysis in Banach spaces*, North-Holland Math. Stud. **120**, North-Holland, Amsterdam, 1986.

20. M.I. Ostrovskii, *Three space problem for the weak Banach-Saks property*, Math. Notes **38** (1985), 905–907.

21. A. Pełczyński and W. Szlenk, *An example of a non-shrinking basis*, Rev. Roum. Math. Pures Appl. **10** (1965), 961–965.

22. J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Konvergenz*, Studia Math. **2** (1930), 58–62.

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain
E-mail address: gonzalem@ccaix3.unican.es

Departamento de Matemáticas, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid, Spain
E-mail address: gutierrezj@member.ams.org