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TURÁN INEQUALITIES FOR SYMMETRIC ASKEY-WILSON POLYNOMIALS

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1. Let $\{P_n(x) : n = 0, 1, \dots\}$ be a sequence of polynomials orthogonal on an interval $[a, b]$. The polynomials $\{P_n(x)\}$ are said to satisfy Turán's inequality if

$$P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geq 0, \quad a \leq x \leq b, \quad n = 0, 1, \dots . \tag{1.1}$$

Turán first observed that (1.1) is satisfied by Legendre polynomials [**9**] and Szego [**8**] gave two beautiful proofs of that fact. Various authors have generalized (1.1) to the classical orthogonal polynomials of Jacobi, Hermite, and Laguerre [**1**], [**5**]. Szasz [**7**] also proved a Turán inequality for ultraspherical polynomials and Bessel functions.

Bustoz and Ismail [**4**] applied a procedure first used by Szász [**7**] to prove Turán inequalities for an important class of nonclassical orthogonal polynomials; the symmetric Pollaczek polynomials as well as for modified Lommel polynomials, and for q-Bessel functions. Also in [**3**] Bustoz and Ismail proved a Turán inequality for continuous q-ultraspherical polynomials by using the Szász technique. In this paper we will apply the Szász technique to prove a Turán inequality for symmetric Askey-Wilson polynomials.

2. Askey-Wilson polynomials. The q-shifted factorial $(a;q)_n$ is defined by

$$(a;q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\cdots(1-aq^{n-1}) & n = 1, 2, \dots, \end{cases}$$

and for $|q| < 1$ we define

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j).$$

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For notational convenience we will write

$$(a_1, a_2, \dots, a_p; q)_n \doteq (a_1; q)_n (a_2; q)_n \cdots (a_p; q)_n,$$

with a similar convention when $n = \infty$.

The basic hypergeometric series ${}_r\phi_s$ is defined by

$$ {}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; q, z \right] $$
$$ \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_n; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n. $$

([**6**] is the fundamental reference on basic series.) The Askey-Wilson polynomials [**1**], [**6**], $P_n(x; a, b, c, d|q)$ may be expressed as a ${}_4\phi_3$ that terminates. This expression is, writing $x = \cos\theta$,

$$
\begin{aligned}
(2.1)\quad & P_n(x; a, b, c, d|q) \\
&= (ab, ac, ad; q)_n a^{-n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right].
\end{aligned}
$$

These polynomials are orthogonal for $-1 \le x \le 1$ and $\max(|a|, |b|, |c|, |d|) < 1$. They satisfy the recursion $2xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)$, $n \ge 0$, with $p_{-1}(x) = 0$, $p_0(x) = 1$, where

$$
\begin{aligned}
A_n &= \frac{1 - abcdq^{n-1}}{(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \\
C_n &= \frac{(1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})} \\
&\quad \cdot (1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1}),
\end{aligned}
$$

and

$$
\begin{aligned}
B_n = a + a^{-1} &- A_n a^{-1}(1 - abq^n)(1 - acq^n)(1 - adq^n) \\
&- C_n a/(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1}).
\end{aligned}
$$

Write

$$h(x; a, b, c, d|q) = (ae^{i\theta}, ae^{-i\theta}, be^{i\theta}, be^{-i\theta}, ce^{i\theta}, ce^{-i\theta}, de^{i\theta}, de^{-i\theta}; q)_\infty.$$

Then the weight function for the Askey-Wilson polynomials is

$$w(x;a,b,c,d|q)=\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty(1-x^2)^{-1/2}}{h(x;a,b,c,d|q)}.$$

The Askey-Wilson polynomials can be evaluated when $x=(a+a^{-1})/2$ by using (2.1). This evaluation is needed in what follows. Writing $a=e^{i\theta}$ in (2.1) gives $x=(a+a^{-1})/2=(e^{i\theta}+e^{-i\theta})/2$ (naturally, θ here is not a real number), we get that

$$(ae^{-i\theta};q)_n=\begin{cases}1 & n=0,\\ 0 & n=1,2,\dots,.\end{cases}$$

Hence when $a=e^{i\theta}$ the ${}_4\phi_3$ in (2.1) has only a single term and we get, after rewriting,

$$P_n\left(\frac{a+a^{-1}}{2};a,b,c,d|q\right)=a^{-n}(-a^2;q)_n(a^2b^2;q^2)_n. \tag{2.3}$$

When $c=-a$ and $d=-b$, the Askey-Wilson polynomials become symmetric, that is, $B_n=0$ in (2.2). We will write $S_n(x;a,b|q)\doteq S_n(x)$ for the symmetric Askey-Wilson polynomials. From (2.3) we have

$$S_n\left(\frac{a+a^{-1}}{2}\right)=a^{-n}(-a^2;q)_n(a^2b^2;q^2)_n. \tag{2.4}$$

3. A Turán inequality for symmetric Askey-Wilson polynomials. Renormalize the symmetric Askey-Wilson polynomials by

$$V_n(x;a,b\mid q)=S_n(x;a,b\mid q)\Big/(-a^2;q)_n(a^2b^2;q^2)_n.$$

Then $V_n^2(x)-V_{n+1}(x)V_{n-1}(x)=0$ when $x=(a+a^{-1})/2$. The $V_n(x)$ satisfy the recursion

$$\begin{aligned}&(1-a^2b^2q^{n-1})(1+a^2q^n)V_{n+1}(x)\\&\quad=2(1-a^2b^2q^{2n-1})xV_n(x)-(1-q^n)(1+b^2q^{n-1})V_{n-1}(x),\quad n\ge 0.\end{aligned}$$

Defining

$$D_n(x) = (1 - a^2b^2q^{2n-1})(1 - a^2b^2q^{n-2})(1 + a^2q^{n-1})V_n^2(x) \\ - (1 - a^2b^2q^{2n-3})(1 - a^2b^2q^{n-1})(1 + a^2q^n)V_{n+1}(x)V_{n-1}(x)$$

we get the following recurrence relation:
(3.1)

$$D_n(x) = \frac{(1-q^{n-1})(1+b^2q^{n-2})(1-a^2b^2q^{2n-1})}{(1-a^2b^2q^{2n-5})(1-a^2b^2q^{n-2})(1+a^2q^{n-1})}D_{n-1}(x) \\ + \frac{(1-a^2b^2q^{2n-3})}{(1-a^2b^2q^{2n-5})(1-a^2b^2q^{n-2})(1+a^2q^{n-1})}g_n(a,b,q)V_{n-1}^2(x), \\ n \geq 2$$

where

$$g_n(a,b,q) \\ = (1-a^2b^2q^{2n-5})(1-a^2b^2q^{n-2})(1+a^2q^{n-1})(1+b^2q^{n-1})(1-q^n) \\ - (1-a^2b^2q^{2n-1})(1-a^2b^2q^{n-3})(1+a^2q^{n-2})(1+b^2q^{n-2})(1-q^{n-1}), \\ n \geq 2.$$

Defining

$$\zeta_n(x) = \frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1 - a^2b^2q^{2n-1})(1 - a^2b^2q^{2n-3})}D_n(x)$$

we obtain, multiplying (3.1) by

$$\frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1 - a^2b^2q^{2n-1})(1 - a^2b^2q^{2n-3})}$$

the recurrence relation for $\zeta_n(x)$ valid for $n \geq 2$.

$$\zeta_n(x) = \zeta_{n-1}(x) \\ + \frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1-a^2b^2q^{2n-5})(1-a^2b^2q^{n-2})(1+a^2q^{n-1})} \\ \cdot g_n(a,b,q)V_{n-1}^2(x)$$

and, iterating,

$$\zeta_n(x) = \zeta_1(x) + \sum_{k=1}^{n-1} h_n(a,b,q) g_n(a,b,q) V_{n-1}^2(x) \tag{3.2}$$

where

$$h_n(a,b,q) = \frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1-a^2b^2q^{2n-5})(1-a^2b^2q^{n-2})(1+a^2q^{n-1})}$$

which is clearly positive.

The crucial step is now to determine the sign of $g_n(a,b,q)$ which will give monotonicity properties of the sequence $\{\zeta_n(x)\}$. The next result establishes sufficient conditions for negativity of $g_n(a,b,q)$.

Lemma 3.1. *If $0 < q \le 1/2$, $a^2 < q$, $b^2 < q$, $n \ge 2$, then $g_n(a,b,q) < 0$.*

Proof. $g_n(a,b,q)$ can be expressed as

$$g_n(a,b,q) = -(1-q)(\phi_1 + \phi_2 + \phi_3),$$

where

$$\begin{aligned}
\phi_1 &= a^4b^4q^{5n-q}(a^2-q)(b^2-q) + a^4b^4q^{4n-8}(q+1)(a^2+b^2) \\
&\quad + a^2b^2q^{4n-b}(q+1)(a^2+b^2) + a^2b^2q^{2n-5}(a^2+b^2), \\
\phi_2 &= q^{n-3}(a^2-q)(b^2-q) + q^{2n-3}(q+1)(a^2+b^2) \\
&\quad - q^{2n-5}(q+1)^3a^2b^2, \\
\phi_3 &= 2a^2b^2(q+1)^2q^{3n-7}(a^2-q)(b^2-q) - a^4b^4q^{4n-8}(q+1)^3.
\end{aligned}$$

Obviously, $\phi_1 > 0$. For ϕ_2 we have

$$\phi_2 = q^{n-3}[(a^2-q)(b^2-q) + (a^2+b^2)(q+1)q^n - a^2b^2(q+1)^3q^{n-2}].$$

In the above equality set $x = a^2 - q$, $y = b^2 - q$ to get $\phi_2 = q^{n-3}T_n(x,y,q)$, where

$$\begin{aligned}
T_n(x,y,q) &= xy - q^{n-2}(q+1)^3xy + q^{n-1}(q+1)(q^2+q+1)x \\
&\quad + q^{n-1}(q+1)(q^2+q+1)y - q^n(q+1)(q^2+1).
\end{aligned}$$

Note that $q(1-q) \leq x, y \leq q$. $T_n(x,y,q)$ satisfies $T_{xx} = T_{yy} = 0$. Hence $T_n(x,y,q)$ has no local extrema and thus the minimum of $T_n(x,y,q)$ occurs on the boundary of the rectangle $q(1-q) \leq x,\ y \leq q$. By symmetry of $T_n(x,y,q)$ we need only consider the line segments L_1 and L_2:

$$L_1: \quad y = q(1-q), \quad q(1-q) \leq x \leq q,$$
$$L_2: \quad x = q, \quad q(1-q) \leq y \leq q.$$

A simple calculation shows that

$$\min\{T_n(x,y,q) \mid (x,y) \in L_1\} = q^2[(1-q)^2 + q^n(1-q-3q^2-q^3)] > 0$$

for $0 < q \leq 1/2$, $n \geq 1$. On L_2 we have

$$T_n(x,y,q) = [q-(q+1)q^n]y + (q+1)q^{n+1}.$$

Since $q-(q+1)q^n > 0$ for $0 < q \leq 1/2$ if $n \geq 2$, we have that

$$\min\{T_n(x,y,q) \mid (x,y) \in L_2, n \geq 2\} = q^{n+3} + q^{n+2} - q^3 + q^2 > 0$$

for $0 < q \leq 1/2$. Thus $\phi_2 > 0$ for $0 < q \leq 1/2$, $a < q$, $b < q$, $n \geq 2$. ϕ_3 is dealt with in an identical manner and we find that $\phi_3 > 0$ for $0 < q \leq 1/2$, $a < q$, $b < q$, $n \geq 1$. This then proves the lemma.

In [**1**], Askey and Wilson proved, using connection coefficients, that

$$|p_n(x,a,-a,c,-c \mid q)| \leq |p_n(1,a,-a,c,-c|q)|$$

if $c \leq q^{1/2}$. The Askey-Wilson polynomials are symmetric in a,b,c,d; so we can exchange $-a$ and c to get

$$|p_n(x,a,c,-a,-c \mid q)| \leq |p_n(1,a,c,-a,-c \mid q)|$$

and hence

$$|S_n(x;a,b \mid q)| \leq |S_n(1;a,b \mid q)|$$

if $b \leq q^{1/2}$.

Now, all the roots of $s_n(x;a,b \mid q)$ are contained in $[-1,1]$ and so are the roots of the derivative. This gives that $s_n(x;a,b \mid q)$ is monotonic outside the interval $[-z_n, z_n]$ where z_n is the largest root of $S_n(x)$. From this we have

$$|s_n(x;a,b \mid q)| \leq \left| s_n\left(\frac{a+a^{-1}}{2};a,b \mid q\right)\right|,$$

if

$$|x| \leq \frac{a+a^{-1}}{2} \quad \text{and} \quad b \leq q^{1/2}.$$

Now, applying this inequality together with Lemma 1 and (3.2), we get, under the conditions $a^2 < q$, $b^2 < q$, $0 < q \leq 1/2$, the inequality:

$$D_n(x) \geq D_n\left(\frac{a+a^{-1}}{2}\right). \tag{3.3}$$

Now,

$$\begin{aligned} D_n(x) = {} & (1-a^2b^2q^{2n-3})(1-a^2b^2q^{n-1})(1+a^2q^n)[V_n^2 - V_{n+1}V_{n-1}] \\ & + a^2q^{n-2}(1-q)[b^2q^{n-1}(1-a^2)(q+1) \\ & \qquad + (q-b^2)(1+a^2b^2q^{2n-2})]V_n^2 \end{aligned}$$

and

$$\begin{aligned} D_n\left(\frac{a+a^{-1}}{2}\right) = {} & a^{-2n}a^2q^{n-2}(1-q)[b^2q^{n-1}(1-a^2)(q+1) \\ & + (q-b^2)(1+a^2b^2q^{2n-2})] \end{aligned}$$

and inequality (3.3) can be rewritten as:

Theorem 3.2. *If $a^2 < q$, $b^2 < q$, $0 < q \leq 1/2$, then, for $|x| \leq (a+a^{-1})/2$ we have the Turán-type inequality*

$$\begin{aligned} & V_n^2(x) - V_{n+1}(x)V_{n-1}(x) \\ & \geq \frac{a^{-2n+2}q^{n-2}(1-q)[b^2q^{n-1}(1-a^2)(q+1) + (q-b^2)(1+a^2b^2q^{2n-2})]}{(1-a^2b^2q^{2n-3})(1-a^2b^2q^{n-1})(1+a^2q^n)} \\ & \quad \cdot (1-a^{2n}V_n^2(x)) \geq 0. \end{aligned}$$

Note that, under the conditions of Lemma 1, the sequence $\{\zeta_n\}$ is decreasing in n, so we have

$$\zeta_n(x) \leq \zeta_1(x).$$

Now,

$$\zeta_n(x) = \frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1-a^2b^2q^{2n-1})(1-a^2b^2q^{2n-3})} D_n(x)$$
$$= \frac{(a^2b^2;q)_{n-1}(-a^2;q)_n}{(q;q)_{n-1}(-b^2;q)_{n-1}(1-a^2b^2q^{2n-1})(1-a^2b^2q^{2n-3})}$$
$$\cdot \{(1-a^2b^2q^{2n-3})(1-a^2b^2q^{n-1})(1+a^2q^n)[V_n^2(x) - V_{n+1}(x)V_{n-1}(x)] + t_n(a,b,q)V_n^2(x)\}$$

where

$$t_n(a,b,q) = a^2q^{n-2}(1-q)[b^2q^{n-1}(1-a^2)(q+1) + (q-b^2)(1+a^2b^2q^{2n-2}).$$

Noticing that $t_n(a,b,q)$ is positive if $b^2 < q$ we get the following upper bound for $V_n^2(x) - V_{n+1}(x)V_{n-1}(x)$, after evaluating

$$\zeta_1(x) = \frac{(1-q)(1+a^2)(1+b^2)}{(1-a^2b^2q)}.$$

Theorem 3.3. *If $a^2 < q$, $b^2 < q$, $0 \le q \le 1/2$, then for $|x| \le (a+a^{-1})/2$ we have the inequality:*

$$V_n^2(x) - V_{n+1}(x)V_{n-1}(x) \le \frac{(1-q)(1+a^2)(1+b^2)(q;q)_{n-1}(-b^2;q)_{n-1}(1-a^2b^2q^{2n-1})}{(1+a^2q^n)(1-a^2b^2q^{n-1})(1-a^2b^2q)(a^2b^2;q)_{n-1}(-a^2;q)_n}.$$

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