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THE SPECTRAL GEOMETRY OF RIEMANNIAN SUBMERSIONS FOR MANIFOLDS WITH BOUNDARY

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ABSTRACT. We study the spectral geometry of a Riemannian submersion $\pi : Z \to Y$ where Z and Y are compact Riemannian manifolds with smooth boundaries and where $\pi : \partial Z \to \partial Y$ is also a Riemannian submersion. We impose suitable boundary conditions and give necessary and sufficient conditions that π^* preserve all the eigenforms of the Laplacian. We also study when a single eigenvalue can change.

0. Introduction. All manifolds in this note are assumed to be compact, connected, orientable, smooth Riemannian manifolds with smooth boundaries. Let $\Delta_M^p := \delta_M d_M + d_M \delta_M$ be the Laplace Beltrami operator on the space of smooth p forms $C^\infty \Lambda^p M$ on such a manifold M. We must impose suitable boundary conditions $\mathcal{B}$ if ∂M is nonempty. Section 1 is devoted to a brief review of Dirichlet, Neumann, absolute and relative boundary conditions; these are the boundary conditions that we will consider. Let $\Delta_{M,\mathcal{B}}^p$ be the Laplacian on M with domain defined by the boundary condition $\mathcal{B}$. Denote the corresponding eigenspaces by

$$E(\lambda, \Delta_{M,\mathcal{B}}^p) := \{\Phi \in C^\infty(\Lambda^p M) : \Delta_M^p \Phi = \lambda \Phi \text{ and } \mathcal{B}\Phi = 0\}.$$

In Lemma 1.2 we show $\Delta_{M,\mathcal{B}}^p$ is self-adjoint. If $\mathcal{B}$ denotes Dirichlet, relative, or absolute boundary conditions, $\Delta_{M,\mathcal{B}}^p$ is a nonnegative operator. By contrast, if $\mathcal{B}$ denotes Neumann boundary conditions, then $\Delta_{M,\mathcal{B}}^p$ can have negative spectrum as we shall show in Theorem 4.4. The material of Section 1 is fairly well known; we have organized it for the convenience of the reader in later sections.

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Let $\pi : Z \to Y$ be a Riemannian submersion. We assume that Z and Y are compact manifolds with smooth boundaries and that the restriction of π from ∂Z to ∂Y is also a Riemannian submersion. Pull-back $\pi^* : C^\infty \Lambda^p Y \to C^\infty \Lambda^p Z$. In Section 2 we discuss the relationship between pull-back and the boundary conditions we are considering. In Theorem 2.4 we show that π^* always preserves Dirichlet and absolute boundary conditions, and we give necessary and sufficient conditions that π^* preserves Neumann and relative boundary conditions.

In Section 3, in Theorems 3.1 and 3.2, we recall previously known results relating to the eigenspaces for closed manifolds. In Section 4 we study when a single eigenvalue can change. In Theorem 4.1 we show that eigenvalues cannot change if $p = 0$. In Theorem 4.2 we show that eigenvalues cannot decrease for the boundary conditions $\mathcal{B} = \mathcal{B}_D \mathcal{B}_A$ or $\mathcal{B}_R$ and, in Theorem 4.3, we construct examples where eigenvalues increase for these boundary conditions if $2 \le p < \dim(M)$. In Theorem 4.4 we show that eigenvalues can decrease and even become negative if $1 \le p < \dim(Y)$ with Neumann boundary conditions. In Section 5 we give necessary and sufficient conditions that all the eigenvalues are preserved.

1. Boundary conditions. Let N_M and N_M^* be the inward unit normal vector and covector fields on the boundary ∂M. If $\xi \in T^*M$, let $\operatorname{ext}(\xi)$ denote exterior multiplication and let $\operatorname{int}(\xi)$ denote the dual, interior multiplication. The following assertions are well known; see, for example, Gilkey [**1**].

Lemma 1.1 (Green's formula).

(1) $(d_M\Phi, \Psi)_{L^2(M)} = (\Phi, \delta_M\Psi)_{L^2(M)} - (\operatorname{ext}_M(N_M^*)\Phi, \Psi)_{L^2(\partial M)}$.

(2) $(\Delta_M^p\Phi, \Psi)_{L^2(M)} = (d_M\Phi, d_M\Psi)_{L^2(M)} + (\delta_M\Phi, \delta_M\Psi)_{L^2(M)} + (\operatorname{int}_M(N_M^*)d_M\Phi, \Psi)_{L^2(\partial M)} - (\operatorname{ext}_M(N_M^*)\delta_M\Phi, \Psi)_{L^2(\partial M)}$.

Let $\Phi \in C^\infty \Lambda^p M$. We say Φ satisfies *Dirichlet boundary conditions* if $\Phi|_{\partial M} = 0$. Let ∇ be the Levi-Civita connection. We say Φ satisfies *Neumann boundary conditions* if $\nabla_{N_M}\Phi|_{\partial M} = 0$. Let $i_M : \partial M \to M$ be the inclusion of the boundary ∂M into M. Let i_M^* be the pull back from $\Lambda^p M$ to $\Lambda^p \partial M$. We say that Φ satisfies *absolute boundary conditions* $\mathcal{B}_A$ if $i^*\operatorname{int}_M(N_M^*)\Phi = 0$ and if $i^*\operatorname{int}_M(N_M^*)d_M\Phi = 0$. Equivalently, let

$y = (y^1, \ldots, y^{m-1})$ be a system of local coordinates on the boundary of M, and let $x = (y, r)$ where r is the geodesic distance to the boundary. Let $dy^I := dy^{i_1} \wedge \cdots \wedge dy^{i_q}$ where $1 \leq i_1 < \cdots < i_q \leq m-1$. Expand $\Phi = \Sigma \Phi_I dy^I + \tilde{\Phi}_J dr \wedge dy^J$. Then Φ satisfies absolute boundary conditions if $\tilde{\Phi}_J|_{\partial M} = 0$ and if $\partial_r \Phi_I|_{\partial M} = 0$. Let $\star$ be the Hodge operator. We say that Φ satisfies *relative boundary conditions* $\mathcal{B}_R$ if $\star\Phi$ satisfies absolute boundary conditions or equivalently if $i_M^* \Phi = 0$ and $i_M^* \delta_M \Phi = 0$, see Lemma 2.3 for details. Note that if $p = 0$,then absolute boundary conditions correspond to Neumann boundary conditions and relative boundary conditions correspond to Dirichlet boundary conditions. If $p = m$, the situation is reversed; absolute boundary conditions correspond to Dirichlet boundary conditions and relative boundary conditions correspond to Neumann boundary conditions. Let

$$E(\lambda, \Delta^p_{M,\mathcal{B}}) := \{\Phi \in C^\infty \Lambda^p M : \mathcal{B}\Phi = 0 \text{ and } \Delta^p_M \Phi = \lambda \Phi\}$$

be the associated eigenspaces. We may use the Hodge-de Rham theorem to identify the absolute and relative cohomology groups with the spaces of harmonic forms which satisfy the associated boundary conditions

$$E(0, \Delta^p_{M,\mathcal{B}_A}) = H^p(M; \mathbf{R}) \text{ and } E(0, \Delta^p_{M,\mathcal{B}_R}) = H^p(M, \partial M; \mathbf{R}).$$

If M is oriented, the Hodge $\star$ operator intertwines $\Delta^p_{M,\mathcal{B}_A}$ and $\Delta^{m-p}_{M,\mathcal{B}_R}$ and induces the Poincare duality isomorphism

$$H^p(M; \mathbf{R}) = E(0, \Delta^p_{M,\mathcal{B}_A}) \approx E(0, \Delta^{m-p}_{M,\mathcal{B}_R}) = H^{m-p}(M, \partial M; \mathbf{R}).$$

Relative and absolute boundary conditions are important in index theory; the Euler-Poincare characteristics are given analytically by

$$\chi(M) = \Sigma_p (-1)^p \dim E(0, \Delta^p_{M,\mathcal{B}_A}),$$

and

$$\chi(M, \partial M) = \Sigma_p (-1)^p \dim E(0, \Delta^p_{M,\mathcal{B}_R}).$$

We summarize the spectral theory of these operators as follows.

Lemma 1.2. *Let $\mathcal{B} = \mathcal{B}_D, \mathcal{B}_N, \mathcal{B}_A$ or $\mathcal{B}_R$.*

(1) $\Delta^p_{M,\mathcal{B}}$ *is self-adjoint.*

(2) *If* $\mathcal{B} \neq \mathcal{B}_N$, $\Delta^p_{M,\mathcal{B}}$ *is nonnegative.*

(3) *All the eigenspaces* $E(\lambda, \Delta^p_{M,\mathcal{B}})$ *are finite dimensional. They are nontrivial only for a countable set of eigenvalues* λ_ν *with* $\lambda_\nu \to \infty$. *The operator* $\Delta^p_{M,\mathcal{B}}$ *has a discrete spectral resolution* $L^2\Lambda^p M = \oplus_\lambda E(\lambda, \Delta^p_{M,\mathcal{B}})$.

Proof. We use Lemma 1.1 to see

$$
\begin{aligned}
(1.1) \quad & (\Delta^p_M \Phi, \Psi)_{L^2(M)} - (\Phi, \Delta^p_M \Psi)_{L^2(M)} \\
&= (\mathrm{int}_M(N^*_M) d_M \Phi, \Psi)_{L^2(\partial M)} - (\mathrm{ext}_M(N^*_M)\delta_M \Phi, \Psi)_{L^2(\partial M)} \\
&\quad - (\Phi, \mathrm{int}_M(N^*_M) d_M \Psi)_{L^2(\partial M)} + (\Phi, \mathrm{ext}_M(N^*_M)\delta_M \Psi)_{L^2(\partial M)}.
\end{aligned}
$$

To establish assertion (1) of Lemma 1.2, we must show that if Φ and Ψ satisfy the boundary conditions $\mathcal{B}$, then

$$
(1.2) \qquad (\Delta^p \Phi, \Psi)_{L^2(M)} = (\Phi, \Delta^p \Psi)_{L^2(M)}.
$$

We must also show that if we are given Ψ so that equation (1.2) holds for all Φ with $\mathcal{B}\Phi = 0$, then $\mathcal{B}\Psi = 0$.

Dirichlet boundary conditions. If Φ and Ψ satisfy Dirichlet boundary conditions, then the boundary terms in Lemma 1.1 (2) vanish and we have

$$
(1.3) \qquad (\Delta^p_M \Phi, \Psi)_{L^2(M)} = (d_M \Phi, d_M \Psi)_{L^2(M)} + (\delta_M \Phi, \delta_M \Psi)_{L^2(M)}.
$$

Equation (1.3) is symmetric in Φ and Ψ; we interchange the roles of Φ and Ψ to see that Lemma 1.2 (2) holds. Conversely, suppose that Ψ is given so that Lemma 1.2 (2) holds for all Φ with $\Phi|_{\partial M} = 0$. We use equation (1.1) to see that

$$
(1.4) \quad (\mathrm{int}_M(N^*_M) d_M \Phi, \Psi)_{L^2(\partial M)} - (\mathrm{ext}_M(N^*_M)\delta_M \Phi, \Psi)_{L^2(\partial M)} = 0
$$

for all Φ with $\Phi|_{\partial M} = 0$. Decompose $\Psi = \Psi_1 + N^*_M \wedge \Psi_2$ and $\Phi = \Phi_1 + N^*_M \wedge \Phi_2$ near the boundary. We assume $\Phi_i|_{\partial M} = 0$. Then equation (1.4) yields

$$
(\partial_m \Phi_1, \Psi_1)_{L^2(\partial M)} + (\partial_m \Phi_2, \Psi_2)_{L^2(\partial M)} = 0.
$$

Since we can specify the normal derivatives of Φ_i arbitrarily, this implies $\mathcal{B}_D\Psi = 0$ as desired. By taking $\Phi = \Psi$ in equation (1.3), we see $(\Delta_M^p\Phi, \Phi) \geq 0$ which establishes assertion (2) of Lemma 1.2 for Dirichlet boundary conditions.

Absolute boundary conditions. Note that

$$(\mathrm{ext}_M(N_M^*)\delta_M\Phi, \Psi) = (\delta_M\Phi, \mathrm{int}_M(N_M^*)\Psi).$$

If Φ and Ψ satisfy absolute boundary conditions, then $\mathrm{int}_M(N_M^*)d_M\Phi$ and $\mathrm{int}_M(N_M^*)\Psi$ vanish on the boundary. Thus the boundary terms in Lemma 1.1 (2) vanish and equation (1.3) holds. As for Dirichlet boundary conditions, this implies equation (1.2) holds and shows that assertion (2) of Lemma 1.2 holds. Conversely, suppose that Ψ is given so that Lemma 1.2 (2) holds for all Φ with $\mathcal{B}_A\Phi = 0$. We use equation (1.1) to see that

$$(\delta_M\Phi, \mathrm{int}_M(N_M^*)\Psi)_{L^2(\partial M)} + (\Phi, \mathrm{int}_M(N_M^*)d_M\Psi)_{L^2(\partial M)} = 0. \tag{1.5}$$

Take adapted coordinate systems $x = (y, r)$ so $dr = N_M^*$ and so that $\partial_r = N_M$; r is the geodesic distance to the boundary. Let $\Phi = \Phi_I^1(y)\,dy^I + r\Phi_J^2(y)N_M^* \wedge dy^J$ near the boundary of M. Then $r\Phi_J^2|_{\partial M} = 0$ and $\partial_r\Phi_I^1|_{\partial M} = 0$ so Φ satisfies absolute boundary conditions. Note that $\delta_M\Phi|_{\partial M} = -\Phi_J^2(y)\,dy^J + Q(\Phi_I^1)$ for some suitably chosen operator Q. Define Φ by the equations

$$\Phi_I^1 dy^I := \mathrm{int}_M(N_M^*)d_M\Psi|_{\partial M}$$

and

$$-\Phi_J^2 dy^J := \mathrm{int}_M(N_M^*)\Psi|_{\partial M} - Q(\Phi_I^1).$$

We use equation (1.5) to show that $\mathcal{B}_A\Psi = 0$ by computing

$$\begin{aligned} 0 &= (-\Phi_J^2 dy^J + Q(\Phi_I^1), \mathrm{int}_M(N_M^*)\Psi)_{L^2(\partial M)} \\ &\quad + (\Phi_I^1 dy^I, \mathrm{int}_M(N_M^*)d_M\Psi)_{L^2(\partial M)} \\ &= \|\mathrm{int}_M(N_M^*)\Psi\|_{L^2(\partial M)}^2 + \|\mathrm{int}_M(N_M^*)d_M\Psi\|_{L^2(\partial M)}^2. \end{aligned}$$

Relative boundary conditions. This case follows from absolute boundary conditions using the Hodge $\star$ operator.

Neumann boundary conditions. Let '$;$' denote multiple covariant differentiation with respect to a local orthonormal frame field. We adopt the Einstein convention and sum over repeated indices. We use the Weitzenböch formulas to express

$$\Delta_M^p \Phi = -\Phi_{;ii} + \mathcal{R}\Phi$$

where $\mathcal{R}$ is a self-adjoint endomorphism of the exterior algebra given by the curvature tensor; see, for example, Gilkey [**1**, Lemma 4.1.2]. For example, if $p = 0$, then $\mathcal{R} = 0$; if $p = 1$, then $\mathcal{R}$ is the Ricci tensor. We compute

$$\begin{aligned}(\Phi_{;ii}, \Psi)_{L^2(M)} &= (\Phi_{;i}, \Psi_{;i})_{L^2(M)} - (\Phi_{;m}, \Psi)_{L^2(\partial M)},\\ (\Delta_M^p \Phi, \Psi)_{L^2(M)} - (\Phi, \Delta_M^p \Psi)_{L^2(M)} &= (-\Phi_{;ii}, \Psi)_{L^2(M)} + (\Phi, \Psi_{;ii})_{L^2(M)}\\ &= -(\Phi_{;m}, \Psi)_{L^2(\partial M)} + (\Phi, \Psi_{;m})_{L^2(\partial M)}.\end{aligned}$$

The boundary correction terms vanish in the final equation if both Φ and Ψ satisfy Neumann boundary conditions; conversely, if these boundary correction terms vanish for all Φ satisfying Neumann boundary conditions, then Ψ satisfies Neumann boundary conditions.

The final assertion of the lemma now follows by standard elliptic theory from the previous assertions. □

2. The geometry of Riemannian submersions. We say that $\pi : Z \to Y$ is a *Riemannian submersion* if

(1) π is a smooth surjective map from Z to Y.

(2) For all $z \in Z$, $\pi_* : T_z Z \to T_{\pi z} Y$ is surjective.

(3) Let $\mathcal{V} := \ker(\pi_*)$ and $\mathcal{H} := \mathcal{V}^\perp$. Then $\pi_* : \mathcal{H}_z \to T_{\pi z} Y$ is an isometry.

If the boundaries of Z and Y are nonempty, we also assume that $\pi^{-1}\partial Y = \partial Z$ and that the restriction of π defines a Riemannian submersion from ∂Z to ∂Y.

Let $\pi : Z \to Y$ be a Riemannian submersion. We introduce the following notational conventions. Let indices i, j and k index local orthonormal frames $\{e_i\}$ and $\{e^i\}$ for the vertical distributions and co-distributions $\mathcal{V}$ and $\mathcal{V}^*$ of π. Let indices a, b, and c index local orthonormal frames $\{f_a\}$ and $\{f^a\}$ for the horizontal distributions and co-distributions $\mathcal{H}$ and $\mathcal{H}^*$ of π, and local orthonormal frames F_a and F^a for the tangent and cotangent bundles TY and T^*Y of Y. We use capital letters for fields on Y and lower case letters for their horizontal lifts to Z. Let

$$\theta := -g_Z([e_i, f_a], e_i)f^a \quad \text{and} \quad \omega_{abi} := \frac{1}{2}g_Z(e_i, [f_a, f_b]). \tag{2.1}$$

Then θ is the nonnormalized mean curvature co-vector of the fibers of π and ω is the curvature of the horizontal distribution. The fibers of π are minimal if and only if $\theta = 0$ or equivalently if π is a harmonic map. The horizontal distribution $\mathcal{H}$ is integrable if and only if $\omega = 0$.

Pull back π^* is a linear map from $C^\infty\Lambda^pY$ to $C^\infty\Lambda^pZ$ which commutes with the exterior derivative, i.e., $\pi^*d_Y = d_Z\pi^*$. However, π^* does not in general commute with the coderivative. We refer to [**4**] for the proof of the following result.

Lemma 2.1. *Let $\pi : Z \to Y$ be a Riemannian submersion. Define $\Xi := \mathrm{int}_Z(\theta) + \mathcal{E}$ where $\mathcal{E} := \omega_{abi}\mathrm{ext}_Z(e^i)\mathrm{int}_Z(f^a)\mathrm{int}_Z(f^b)$. Then we have $\delta_Z\pi^* - \pi^*\delta_Y = \Xi\pi^*$ and $\Delta_Z\pi^* - \pi^*\Delta_Y = (d_Z\Xi + \Xi d_Z)\pi^*$.*

Let $\pi_i : W_i \to Y$ be Riemannian submersions. Let

$$W = W(W_1, W_2) := \{w = (w_1, w_2) \in W_1 \times W_2 : \pi_1(w_1) = \pi_2(w_2)\}$$

be the fiber product. Let $\mathcal{H}_i$ and $\mathcal{V}_i$ be the horizontal and vertical distributions of π_i. We may identify the tangent bundle of the product $T(W_1 \times W_2)$ with the direct sum $T(W_1) \oplus T(W_2)$ to embed $T(W)$ in $T(W_1) \oplus T(W_2)$. Let

$$\begin{aligned}
&\pi_W(w) := \pi_1(w_1) = \pi_2(w_2) : W \to Y,\\
&\mathcal{V}_W(w) := \mathcal{V}_1(w_1) \oplus \mathcal{V}_2(w_2),\\
&\mathcal{H}_W(w) := \{(\xi, \eta) \in \mathcal{H}_1(w_1) \oplus \mathcal{H}_2(w_2) : (\pi_1)_*\xi = (\pi_2)_*\eta\}.
\end{aligned}$$

We define a metric on W by requiring that $\mathcal{H}_W$, $\mathcal{V}_1$ and $\mathcal{V}_2$ are orthogonal, that the metrics on $\mathcal{V}_1$ and $\mathcal{V}_2$ are induced from the metrics on W_1 and on W_2, and that $(\pi_W(w))_*$ is an isometry from $\mathcal{H}_W(w)$ to $TY(\pi(w))$. The metric on $\mathcal{H}_W$ differs from the subspace metric by a factor of $1/\sqrt{2}$; the diagonal in a right equilateral triangle has length

$\sqrt{2}$. Let $\sigma_1(w_1, w_2) = w_1$ and $\sigma_2(w_1, w_2) = w_2$. Then $\sigma_i : W \to W_i$ and $\pi_W : W \to Y$ are Riemannian submersions. We can express θ_W and $\mathcal{E}_W$ in terms of the corresponding tensors on W_1 and W_2 as follows. We refer to [**3**] for the proof of the following lemma.

Lemma 2.2. *We have* $\theta_W = \sigma_1^*\theta_{W_1} + \sigma_2^*\theta_{W_2}$ *and* $\mathcal{E}_W\pi^* = \sigma_1^*\mathcal{E}_{W_1}\pi_1^* + \sigma_2^*\mathcal{E}_{W_2}\pi_2^*$.

We now study the geometry near the boundary. Let F_m be the inward unit normal on the boundary of Y, and let F_a for $1 \leq a \leq m-1$ be a local orthonormal frame field for the tangent bundle of the boundary. Let L be the second fundamental form on Y; $L_{ab} = L(F_a, F_b) := g_Y(\nabla_{F_a}F_b, F_m)$; the second fundamental form on Z is defined similarly. Let i_Y and i_Z be the inclusions of ∂Y and ∂Z in Y and Z. We have $\pi \circ i_Z = i_Y \circ \pi$. Let $F \in T^*Y$. Since π is a Riemannian submersion,

$$\pi^* \circ \mathrm{int}_Y(F) = \mathrm{int}_Z(\pi^*F) \circ \pi^*$$

and

$$\pi^* \circ \mathrm{ext}_Y(F) = \mathrm{ext}_Z(\pi^*F) \circ \pi^*.$$

The following lemma summarizes some technical results that we shall need. Let Γ be the Christoffel symbols of the Levi-Civita connection.

Lemma 2.3. *Let* $\pi : Z \to Y$ *be a Riemannian submersion.*

(1) $L_{ab}^Y = L_{ab}^Z$ *and* $\Gamma_{mai}^Z = -2\omega_{ami} - L_{ai}$.

(2) $i_Y^*\mathrm{int}_Y(N_Y^*)\star_Y = \varepsilon(m,p)\star_{\partial Y} i_Y^*$ *on* $C^\infty\Lambda^pY$ *for* $\varepsilon(m,p) = \pm 1$.

(3) $\mathcal{B}_R\Phi = 0 \Leftrightarrow \mathcal{B}_A \star_Y \Phi = 0 \Leftrightarrow i_Y^*\Phi = 0$ *and* $i_Y^*\delta_Y\Phi = 0$.

(4) *We have* $i_Z^*(\delta_Z\pi^* - \pi^*\delta_Y) = 0$ *if and only if*

(a) *If* $p = 0$ *there is no condition on* θ. *If* $1 \leq p < m$, *then* $\theta = 0$. *If* $1 \leq p = m$, *then* $\theta_m = 0$.

(b) *If* $p = 0$ *or if* $p = 1$, *there is no condition on* ω. *If* $1 < p < m$, *then* $\omega = 0$. *If* $2 \leq p = m$, *then* $\omega_{amj} = 0$ *for all* a, j.

Proof. We prove the first assertion by computing

$$L_{ab}^Y = \Gamma_{abm}^Y = \Gamma_{abm}^Z = L_{ab}^Z,$$

and

$$\Gamma^Z_{mai} = \Gamma^Z_{mai} - \Gamma^Z_{ami} - \Gamma^Z_{aim} = 2\omega_{mai} - \Gamma^Z_{aim} = -2\omega_{ami} - L_{ai}.$$

Assertion (2) is an easy calculation once the orientations involved are taken into account. We use assertion (2) to prove assertion (3) by computing

$$\begin{aligned}
&\mathcal{B}_R\Phi = 0 \Longleftrightarrow \mathcal{B}_A \star_Y \Phi = 0\\
&\quad\Longleftrightarrow i^*_Y \mathrm{int}_Y(N^*_Y)\star_Y\Phi = 0 \quad\text{and}\quad i^*_Y\mathrm{int}_Y(N^*_Y)d\star_Y\Phi = 0\\
&\quad\Longleftrightarrow \star_{\partial Y} i^*_Y\Phi = 0 \quad\text{and}\quad \star_{\partial Y}\, i^*_Y\delta_Y\Phi = 0\\
&\quad\Longleftrightarrow i^*_Y\Phi = 0 \quad\text{and}\quad i^*_Y\delta_Y\Phi = 0.
\end{aligned}$$

By Lemma 2.1,

$$i^*_Z(\delta_Z\pi^* - \pi^*\delta_Y) = i^*_Z(\mathrm{int}_Z(\theta) + \omega_{abi}\mathrm{ext}_Z(e^i)\mathrm{int}_Z(f^a)\mathrm{int}_Z(f^b))\pi^*.$$

The condition $i^*_Z(\delta_Z\pi^* - \pi^*\delta_Y) = 0$ decouples; it is satisfied if and only if we have the pair of equations $i^*_Z\mathrm{int}_Z(\theta)\pi^* = 0$ and $i^*_Z\omega_{abi}\mathrm{int}_Z(f^a)\mathrm{int}_Z(f^b))\pi^* = 0$ for all i. If $p = 0$, θ and ω play no role. If $p = 1$, ω plays no role. If $p < m$, both the normal and tangential components of ω and θ play a role; if $p = m$, only the normal component plays a role. □

Let $\mathcal{B}^Y$ and $\mathcal{B}^Z$ denote the appropriate boundary conditions on Y and on Z.

Theorem 2.4. *Let $\pi : Z \to Y$ be a Riemannian submersion. Then*

(1) *If $\mathcal{B}^Y_D\Phi = 0$, then $\mathcal{B}^Z_D\pi^*\Phi = 0$.*

(2) *If $\mathcal{B}^Y_A\Phi = 0$, then $\mathcal{B}^Z_A\pi^*\Phi = 0$.*

(3) *Assume that $i^*_Z(\delta_Z\pi^* - \pi^*\delta_Y) = 0$. If $\mathcal{B}^Y_R\Phi = 0$, then $\mathcal{B}^Z_R\pi^*\Phi = 0$.*

(4) *If $p = 0$, Neumann boundary conditions are preserved. If $p > 0$, Neumann boundary conditions are preserved if and only if $\Gamma^Z_{mai} = 0$ for all i, a.*

Proof. Assertion (1) is immediate. Suppose that Φ satisfies absolute boundary conditions. Then

$$\begin{aligned}
&i_Y^*\mathrm{int}_Y(N_Y^*)\Phi = 0 \quad \text{and} \quad i_Y^*\mathrm{int}_Y(N_Y^*)d_Y\Phi = 0,\\
&\Longrightarrow \pi^* i_Y^*\mathrm{int}_Y(N_Y^*)\Phi = 0, \quad \text{and} \quad \pi^* i_Y^*\mathrm{int}_Y(N_Y^*)d_Y\Phi = 0,\\
&\Longrightarrow i_Z^*\pi^*\mathrm{int}_Y(N_Y^*)\Phi = 0, \quad \text{and} \quad i_Z^*\pi^*\mathrm{int}_Y(N_Y^*)d_Y\Phi = 0,\\
&\Longrightarrow i_Z^*\mathrm{int}_Z(N_Z^*)\pi^*\Phi = 0 \quad \text{and} \quad i_Z^*\mathrm{int}_Z(N_Z^*)d_Z\pi^*\Phi = 0,\\
&\Longrightarrow \pi^* \text{ satisfies absolute boundary conditions on } Z.
\end{aligned}$$

This proves assertion (2). If we assume $\delta_Z\pi^* = \pi^*\delta_Y$, the proof of assertion (3) is the same. We note that

$$\nabla_{N_Z}\pi^*\Phi - \pi^*\nabla_{N_Y}\Phi = \mathrm{ext}_Z(e^i)\mathrm{int}_Z(f^a)\Gamma_{mai}^Z\pi^*\Phi.$$

If $p = 0$, this vanishes automatically. If $p > 0$, this vanishes if and only if Γ_{mai} vanishes on the boundary of Z. □

3. Relating the eigenspaces for closed manifolds. If Y and Z are closed manifolds, we can give necessary and sufficient conditions that all the eigenspaces are preserved; eigenvalues cannot change in this situation. We refer to [**4**] for the proof of the following result; this extends previous work of Goldberg and Ishihara [**5**] and Watson [**8**].

Theorem 3.1. *Fix p with $0 \le p \le \dim_{\mathbf{R}} Y$. The following conditions are equivalent*:

(1) $\Delta_Z^p\pi^* = \pi^*\Delta_Y^P$.

(2) *For all $\lambda \ge 0$, there exists $\mu(\lambda) \ge 0$ so $\pi^*E(\lambda, \Delta_Y^p) \subset E(\mu(\lambda), \Delta_Z^p)$.*

(3) *The fibers of π are minimal and*

(a) *if $p = 0$, there is no further condition.*

(b) *if $p > 0$, the horizontal distribution is integrable.*

Theorem 3.1 shows that if all the eigenspaces are preserved, then eigenvalues cannot change. We refer to [**2**] for the proof of the following result.

Theorem 3.2.

(1) *Let $\pi : Z \to Y$ be a Riemannian submersion of closed Riemannian manifolds. If $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ and if $\pi^*\Phi \in E(\mu, \Delta_Z^p)$, then $\lambda \leq \mu$. If $p = 0$, then $\lambda = \mu$.*

(2) *Let $p \geq 2$, and let $0 \leq \lambda < \mu < \infty$ be given. There exists a Riemannian submersion $\pi : Z \to Y$ of closed manifolds and there exists $0 \neq \Phi \in E(\lambda, \Delta_Y^p)$ so that $\pi^*\Phi \in E(\mu, \Delta_Z^p)$.*

The proof of assertion (2) of Theorem 3.2 uses results of Muto [**6, 7**]; the proof of assertion (1) of Theorem 3.2 uses the fiber products described in Lemma 2.2. The case $p = 1$ is left open in Theorem 3.2; we do not know if eigenvalues can change if $p = 1$ for closed manifolds. We shall see in the next section that eigenvalues can change if $p = 1$ if $\partial Y \neq 0$ with Neumann boundary conditions.

4. When can eigenvalues change. Absolute and relative boundary conditions are Neumann and Dirichlet boundary conditions if $p = 0$; eigenvalues cannot change in this setting.

Theorem 4.1. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_N$ and let $p = 0$. If $0 \neq \Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$ and if $\pi^*\Phi \in E(\mu, \Delta^0_{Z,\mathcal{B}})$, then $\lambda = \mu$.*

Proof. Suppose that $0 \neq \Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$ and that $\pi^*\Phi \in E(\lambda + \varepsilon, \Delta^0_{Z,\mathcal{B}})$. We use Lemma 2.1 to see that

$$\varepsilon\pi^*\Phi = \Delta_Z^0\pi^*\Phi - \pi^*\Delta_Y^0 = \mathrm{int}_Z(\theta)d_Z\pi^*\Phi. \tag{4.1}$$

By replacing Φ by $-\Phi$ if necessary, we can assume the maximal value of Φ is positive; let this maximal value be attained at $y_0 \in Y$. If y_0 is in the interior of Y, then $d_Y\Phi(y_0) = 0$. Choose z_0 so $\pi(z_0) = y_0$. Then $d_Z\pi^*\Phi(z_0) = \pi^*d_Y\Phi(y_0) = 0$ so equation (4.1) implies $\varepsilon\pi^*\Phi(z_0) = 0$ and hence $\varepsilon = 0$. If $\mathcal{B} = \mathcal{B}_D$, then Φ cannot attain its maximum on the boundary ∂Y and the theorem follows. Let $\mathcal{B} = \mathcal{B}_N$. Since $y_0 \in \partial Y$, $d_{\partial Y}\Phi(y_0) = 0$. Since $N_Y(\Phi)(y_0) = 0$, $d_Y\Phi(y_0) = 0$ and $\varepsilon = 0$. $\square$

Next we study the case $p > 0$.

Theorem 4.2. *Let $\pi : Z \to Y$ be a Riemannian submersion. Let $\mathcal{B} = \mathcal{B}_D, \mathcal{B}_A$ or $\mathcal{B}_R$. If $0 \neq \Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}})$ and if $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}})$, then $\lambda \leq \mu$.*

Proof. We ignore the boundary conditions for the moment. Let $Z_0 := Z$. For $n \geq 1$, let $Z_n := W(Z_{n-1}, Z_{n-1})$ be the fiber product discussed in Section 2. Let $\pi_n : Z_n \to Y$ be the associated Riemannian submersion. We use Lemmas 2.1 and 2.2 to see that $\pi^*_{n-1}\Phi \in E(\lambda + \varepsilon_{n-1}, \Delta^p_{Z_n})$ implies $\pi^*_n\Phi \in E(\lambda + \varepsilon_n, \Delta^p_{Z_n})$ where $\varepsilon_n = 2\varepsilon_{n-1}$. Thus if $\varepsilon = \mu - \lambda$, we have $\pi^*_n\Phi \in E(\lambda + 2^n\varepsilon, \Delta^p_{Z_n})$. If $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_A$, then the boundary conditions are preserved automatically and $\pi^*_n\Phi \in E(\lambda + 2^n\varepsilon, \Delta^p_{Z_n,\mathcal{B}})$. Since this operator is nonnegative by Lemma 1.2 we see $\lambda + 2^n\varepsilon \geq 0$ for all n and thus $\varepsilon \geq 0$ so $\lambda \leq \mu$ as desired.

If $\mathcal{B} = \mathcal{B}_R$, then by assumption $\mathcal{B}_R\Phi = 0$ and $\mathcal{B}_R\pi^*\Phi = 0$. Thus

$$i^*_Z(\delta_Z\pi^* - \pi^*\delta_Y)\Phi = 0$$

so we have $i^*_Z\mathrm{int}_Z(\theta)\pi^*\Phi = 0$ and $i^*_Z\mathcal{E}\pi^*\Phi = 0$. Lemma 2.2 then shows inductively $i^*_{Z_n}\mathrm{int}_{Z_n}(\theta)\pi^*_n\Phi = 0$ and $i^*_{Z_n}\mathcal{E}\pi^*_n\Phi = 0$ from which it follows that $\mathcal{B}_R\pi^*_n\Phi = 0$ so $\pi^*_n\Phi$ satisfies the given boundary conditions. The remainder of the argument is the same as that given above for $\mathcal{B} = \mathcal{B}_D$ or $\mathcal{B} = \mathcal{B}_A$. $\square$

We show that Theorem 4.2 is sharp in certain cases:

Theorem 4.3. *Let $\mathcal{B} \in \{\mathcal{B}_D, \mathcal{B}_N, \mathcal{B}_A, \mathcal{B}_R\}$, let $2 \leq p$, and let $0 < \lambda < \mu < \infty$ be given. There exists a Riemannian submersion $\pi : Z \to Y$ and there exists $0 \neq \Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}})$ so that $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}})$.*

Proof. Let $W := [0, \varepsilon]$. For suitably chosen ε, we may find $0 \neq \Psi \in E(\lambda, \Delta^p_{W,\mathcal{B}})$. By Theorem 3.2, we may find a Riemannian submersion $\bar{\pi} : \bar{Z} \to \bar{Y}$ of closed manifolds and $\bar{\Phi} \in E(0, \Delta^p_{\bar{Y}})$ so that $\bar{\pi}^*\bar{\Phi} \in E(\mu - \lambda, \Delta^p_{\bar{Z}})$. Let $Z = \bar{Z} \times W$, $Y = \bar{Y} \times W$, and let $\pi(\bar{z}, w) = (\bar{\pi}\bar{z}, w)$. Let $\Phi = \Psi\bar{\Phi}$. Then $0 \neq \Phi \in E(\lambda, \Delta^p_Y)$ and $\pi^*\Phi \in E(\mu, \Delta^p_Z)$. Furthermore, we check directly that $\mathcal{B}^W\Psi = 0$ implies that $\mathcal{B}^Y\Phi = 0$ and that $\mathcal{B}^Z\pi^*\Phi = 0$. $\square$

Theorem 4.2 fails with Neumann boundary conditions; eigenvalues can decrease. The following result shows the eigenvalues of the Neumann Laplacian can be negative. It also shows that eigenvalues can change if $p = 1$.

Theorem 4.4. *Let* $0 < p$*, and let* $\lambda, \mu \in \mathbf{R}$ *be given. There exists a compact Riemannian manifold* Y *with smooth boundary, there exists a Riemannian submersion* $\pi : Z \to Y$ *and there exists* $0 \neq \Phi \in E(\lambda, \Delta^p_{Y,\mathcal{B}_N})$ *so that* $\pi^*\Phi \in E(\mu, \Delta^p_{Z,\mathcal{B}_N})$.

Proof. We suppose $p = 1$, $\lambda = 0$ and $m = 2$; the general case can be dealt with by taking Riemannian products. Let $Y := [0,1]$ with parameter y, and let $\Phi := dy$; Φ satisfies Neumann boundary conditions and $\Delta^1_Y\Phi = 0$. Let $Z := [0,1] \times S^1$, and let t be the usual periodic parameter on the circle. We consider a metric of the form $ds^2 := dy^2 + e^{2f(y)}\,dt^2$ where $f(y) := -(\mu/2)y^2$. By [**3**, Lemma 4.2], we have that $\theta = -df = \mu y\,dy$. Since $\dim(Y) = 1$, the horizontal distribution $\mathcal{H}$ is integrable and $\omega = 0$. Since $\Delta_Y\Phi = 0$, Lemma 2.1 shows that

$$\Delta^1_Z\pi^*\Phi = d\,\mathrm{int}_Z(\theta)\pi^*\Phi = \mu\pi^*\,dy = \mu\pi^*\Phi.$$

Consequently, $\pi^*\Phi \in E(\mu, \Delta^1_{Z,\mathcal{B}_N})$. For fixed t, the curves $y \to (y,t)$ are unit speed geodesics which are normal to the boundary. We check dy satisfies Neumann boundary conditions by computing

$$\nabla_{\partial y}\partial_y = 0, \qquad \nabla_{\partial y}\partial_t = \frac{1}{2}\frac{\partial f}{\partial y}\partial_t, \qquad \nabla_{\partial y}\,dy = 0. \qquad \square$$

5. When all eigenfunctions are preserved. If π^* intertwines the operators Δ^p_Y and Δ^p_Z and intertwines the boundary conditions defined by $\mathcal{B}$ on Y with those on Z, we say that $\Delta^p_{Z,\mathcal{B}}\pi^* = \pi^*\Delta^p_{Y,\mathcal{B}}$.

Theorem 5.1. *Let* $0 \le p \le \dim_{\mathbf{R}} Y$*. Let* $\mathcal{B} = \mathcal{B}_D, \mathcal{B}_R$ *or* $\mathcal{B}_A$*. Let* $\pi : Z \to Y$ *be a Riemannian submersion. The following conditions are equivalent*:

(1) $\Delta^p_{Z,\mathcal{B}}\pi^* = \pi^*\Delta^p_{Y,\mathcal{B}}$.

(2) *For all* $\lambda \geq 0$, *there exists* $\mu(\lambda) \geq 0$ *so* $\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E(\mu(\lambda), \Delta^p_{Z,\mathcal{B}})$.

(3) *The fibers of* π *are minimal and*

(a) *If* $p = 0$, *there is no further condition.*

(b) *Suppose* $p > 0$. *Then the horizontal distribution is integrable.*

Proof. Suppose that condition (3) holds. This means that $\theta = 0$ and that if $p > 0$ that $\omega = 0$. We use Lemma 2.1 to see that $\Delta^p_Z \pi^* = \pi^* \Delta^p_Y$; ω plays not role if $p = 0$. We use assertion (1) of Theorem 2.4 to see that Dirichlet boundary conditions are preserved and assertion (2) of Theorem 2.4 to see that absolute boundary conditions are preserved. We use Lemma 2.1 to see that $\delta_Z \pi^* - \pi^* \delta_Y = 0$ and hence assertion (3) of Theorem 2.4 shows that relative boundary conditions are preserved. This shows that assertion (3) implies assertion (1). It is immediate that assertion (1) implies assertion (2).

Assume that assertion (2) holds. Let $0 \neq \Phi \in E(\lambda, \Delta^0_{Y,\mathcal{B}})$, and let $\phi := \pi^* \Phi$. We use Lemma 2.1 to see that

$$(\mu - \lambda)\phi = \{d_Z(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})d_Z\}\pi^*\Phi. \tag{5.1}$$

Suppose first that $p = 0$. By Theorem 4.1 we have $\mu = \lambda$, and this equation yields $\mathrm{int}_Z(\theta)\pi^* d_Y \Phi = 0$. Let $\Psi \in C^\infty_0(Y)$ be a smooth function on Y with compact support. We can uniformly approximate Ψ in the C^∞ topology by finite sums of eigenfunctions. Thus we have $\mathrm{int}_Z(\theta)\pi^* d_Y \Psi = 0$ on $C^\infty_0(Y)$. Since θ is a horizontal co-vector, this implies $\theta = 0$ on the interior of Z. Continuity then yields $\theta = 0$ on the boundary as well. This completes the proof of the theorem if $p = 0$.

Let $\rho_{\mathcal{H}}$ be an orthogonal projection from $\Lambda^p Z$ to $\Lambda^p \mathcal{H}$. Let $\Phi \in E(\lambda, \Delta^p_{Z,\mathcal{B}})$. We apply $(1 - \rho_{\mathcal{H}})$ to equation (5.1) to see

$$0 = (1 - \rho_{\mathcal{H}})\{d_Z(\mathrm{int}_Z(\theta) + \mathcal{E}) + (\mathrm{int}_Z(\theta) + \mathcal{E})d_Z\}\pi^*\Phi. \tag{5.2}$$

Since the span of the eigenspaces is dense in the space $C^\infty_0(\Lambda^p Y)$ of differential forms which are compactly supported in the interior, this identity extends to $C^\infty_0 \Lambda^p Y$ by continuity. Fix a point $z_0 \in Z$ and let $y_0 = \pi z_0$. Choose $F \in C^\infty Y$ so that $F(y_0) = 0$. Let $\xi := dF(y_0)$. Since $\mathrm{int}_Z(\theta) + \mathcal{E}$ is a 0th order operator, we apply equation (5.2) to

$F\Phi$ and evaluate at z_0 to see

$$0 = (1-\rho_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)(\mathrm{int}_Z(\theta)+\mathcal{E}) + (\mathrm{int}_Z(\theta)+\mathcal{E})\mathrm{ext}_Z(\pi^*\xi)\}\pi^*\{\Phi(y_0)\}.$$

Since $0 = (1-\rho_{\mathcal{H}})\{\mathrm{ext}_Z(\pi^*\xi)\mathrm{int}_Z(\theta)+\mathrm{int}_Z(\theta)\mathrm{ext}_Z(\pi^*\xi)\}\pi^*$, and since $\mathcal{E}$ always introduces a vertical covector, we conclude

$$0 = \{\mathrm{ext}_Z(\pi^*\xi)\mathcal{E} + \mathcal{E}\mathrm{ext}_Z(\pi^*\xi)\}\pi^*.$$

We set $\pi^*\xi = f^c$. We set $e(i) := \mathrm{ext}_Z(e^i)$, $e(a) := \mathrm{ext}_Z(f^a)$ and $i(a) := \mathrm{int}_Z(f^a)$ in the following computation in the interests of brevity

$$\begin{aligned} 0 &= \omega_{abi}\{e(c)e(i)i(a)i(b) + e(i)i(a)i(b)e(c)\} \\ &= \omega_{abi}e(i)\{-e(c)i(a)i(b) + i(a)i(b)e(c)\} \\ &= \omega_{abi}e(i)\{i(a)e(c)i(b) + i(a)i(b)e(c) - \delta_{ac}i(b)\} \\ &= \omega_{abi}e(i)\{-i(a)i(b)e(c) + i(a)i(b)e(c) - \delta_{ac}i(b) + \delta_{bc}i(a)\} \\ &= -2\omega_{cbi}e(i)i(b). \end{aligned}$$

Since $p \geq 1$, this implies $\omega = 0$ on the interior of Y; continuity then implies $\omega = 0$ on the boundary as well. This shows that $\mathcal{H}$ is integrable.

We now recall a bit of the geometry of Riemannian submersions with integrable horizontal distributions. We refer to [**3**] for the proof of the following result:

Lemma 5.2. *Let X be the fiber of a Riemannian submersion $\pi : Z \to Y$. Assume the horizontal distribution of π is integrable. Then we can find local coordinates $z = (x, y)$ on Z so $\pi(x, y) = y$ and so $ds^2_Z = g_{ij}(x, y)dx^i \circ dx^j + h_{ab}(y)dy^a \circ dy^b$. If we set $g_X := \det(g_{ij})^{1/2}$, then $\theta = -d_Y \ln(g_X)$.*

Let d_X denote exterior differentiation along the fiber. We set $\mathcal{E} = 0$ and use equation (5.2) to see

$$0 = d_X \mathrm{int}_Z(\theta)\pi^* \quad \text{on} \quad C_0^\infty \Lambda^p Y. \tag{5.3}$$

This implies θ is constant on the fibers so $\theta = \pi^*\Theta$ is the pull back of a globally defined 1-form on the base. Since $\mathcal{H}$ is integrable, we use Lemma 5.2 to give a local decomposition of Z so that we have $\theta = \pi^*\Theta = -d_Y \ln(g_X)$. Let $\psi(y)$ be the volume of the fibers. Let $d\nu_x^e$ be the Euclidean measure. Then

$$\begin{aligned} d_Y\psi(y) &= d_Y \int_X g_X(x,y)d\nu_x^e = \int_X (g_X g_X^{-1} d_Y g_X)(x,y)d\nu_x^e \\ &= -\int_X g_X(x,y)\theta(x,y)d\nu_x^e = -\Theta(y)\int_X g_X(x,y)d\nu_x^e \\ &= -\Theta(y)\psi(y). \end{aligned}$$

Thus $\theta = -\pi^* d_Y \ln\psi$ where $\psi \in C^\infty(Y)$ is globally defined.

Let $g(t)_Z = \psi^{2t} ds_{\mathcal{V}}^2 + ds_{\mathcal{H}}^2$ define a conformal variation of the metric on the vertical distribution and leave the metric on the horizontal distribution unchanged. Then $\pi : Z(t) \to Y$ is a Riemannian submersion with integrable horizontal distribution. We use Lemma 5.2 to see $\theta(t) = (1 + t\dim(X))\theta$ and thus

$$\begin{aligned} \Delta^p_{Z(t)}\pi^* - \pi^*\Delta^p_Y &= (1 + t\dim(X))(d_Z \mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)d_Z)\pi^* \\ &= (1 + t\dim(X))(\Delta^p_Z\pi^* - \pi^*\Delta^p_Y). \end{aligned} \tag{5.4}$$

Dirichlet or absolute boundary conditions are preserved by π^*. Therefore, if $\mathcal{B} = \mathcal{B}_D$ or if $\mathcal{B} = \mathcal{B}_A$, equation (4.5) implies that

$$\pi^* E(\lambda, \Delta^p_{Y,\mathcal{B}}) \subset E\big(\lambda + (1 + t\dim(X))\varepsilon(\lambda), \Delta^p_{Z(t),\mathcal{B}}\big). \tag{5.5}$$

By Lemma 1.2, $\Delta^p_{Z(t),\mathcal{B}}$ is a nonnegative operator. Thus $\lambda + (1 + t\dim(X))\varepsilon(\lambda) \geq 0$. Since t is arbitrary, $\varepsilon(\lambda) = 0$. Thus

$$(d_Z \mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)d_Z)\pi^* = 0 \tag{5.6}$$

on $E(\lambda, \Delta^p_{Y,\mathcal{B}})$. Since these eigenspaces are dense in $C_0^\infty\Lambda^p Y$, equation (5.6) continues to be valid on $C_0^\infty\Lambda^p Y$. Choose $\Psi \in C_0^\infty\Lambda^p Y$. Let $f \in \mathcal{H}(z_0)$ where z_0 is in the interior of Z. Let $y_0 := \pi(z_0)$ and $\Phi \in C_0^\infty\Lambda^p Y$ so that $\Phi(y_0) = 0$ and $\pi^* d\Phi(y_0) = f$. We apply equation (5.6) to $\Phi\Psi$ and evaluate at y_0 to see

$$(\mathrm{ext}_Z(f)\mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)\mathrm{ext}_Z(f))\pi^*\{\Psi(y_0)\} = 0.$$

Since $\mathrm{ext}_Z(f)\mathrm{int}_Z(\theta) + \mathrm{int}_Z(\theta)\mathrm{ext}_Z(f) = g_Z(f,\theta)$, this implies $g_Z(f,\theta)(z_0) = 0$. Since θ is horizontal, we conclude θ vanishes on the interior of Z and hence, by continuity, we have $\theta = 0$. This completes the proof of Theorem 5.1 if $\mathcal{B}$ denotes Dirichlet or absolute boundary conditions.

Suppose $\mathcal{B} = \mathcal{B}_R$ denotes relative boundary conditions. We suppose $\mathcal{B}_R\Phi = 0$. Since $i_Z^*\pi^* = \pi^* i_Y^*$, we have $i_Z^*\pi^*\Phi = 0$. Since $\mathcal{E} = 0$, we then have

$$\mathcal{B}_R\pi^*\Phi = 0 \Longleftrightarrow i_Z^*\delta_Z\pi^*\Phi = 0 \Longleftrightarrow i_Z^*\mathrm{int}_Z(\theta)\pi^*\Phi = 0.$$

Since $\theta(t) = (1+t\dim(X))\theta$, this condition is preserved. Thus equation (4.6) holds and the argument given above completes the proof. □

Remark 5.3. If $p > 0$, we conjecture that an analogue of Theorem 5.1 holds for Neumann boundary conditions. However, the proof that we have given uses in an essential fashion the positivity of the operator involved and therefore does not extend directly.

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