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LOCALNESS OF THE CENTRALIZER NEARRING DETERMINED BY End G

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ABSTRACT. For G a finite p-group, we investigate the localness of the nearring $M_E(G) = \{f : G \to G \mid f\sigma = \sigma f$ for every $\sigma \in \operatorname{End} G\}$. Examples of groups which make $M_E(G)$ local are provided.

1. Introduction. Let G be a group written additively but not necessarily abelian, and let S be a subsemigroup of $\operatorname{End} G$. The set $M_S(G) = \{f : G \to G \mid f\sigma = \sigma f$ for every $\sigma \in S\}$ forms a nearring under pointwise addition and function composition and is called the *centralizer nearring determined by S and G*. These nearrings are very general since every nearring with identity is isomorphic to an $M_S(G)$ for some pair S and G [**5**, 14.3]. Therefore, it is difficult to investigate these nearrings without some restriction on either G or S. In particular, much attention has been focused on the case where S is a group of automorphisms of G (e.g., see [**6**] or [**9**]).

If S consists of only the identity function on G, then $M_S(G) = M(G)$, the set of all functions from G to G. Similarly, if S consists of only the zero function on G, then $M_S(G) = M_0(G)$, the set of all zero-preserving functions from G to G. The structure of these nearrings is well known (see [**5, 11**] or [**12**] for information and for other general results about nearrings). In this paper which contains results from the author's doctoral dissertation [**2**], we are interested in the structure of the other extreme situation, in other words, when $S = \operatorname{End} G = E$. We call $M_E(G)$ the *centralizer nearring determined by* $\operatorname{End} G$. Since $\operatorname{End} G$ contains the zero function, $M_E(G)$ will be a zero-symmetric nearring.

We recall that a nearring N is local if the set of nonunits in N forms an additive subgroup. If N is finite, this condition is equivalent to saying that every element of N is either invertible or nilpotent [**10**]. This provides further motivation for studying $M_E(G)$, for if G is finite and $M_E(G)$ is not local, then $M_S(G)$ cannot be local for any subsemigroup

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S of $\operatorname{End} G$ since $M_E(G)$ is a subnearring of $M_S(G)$. Studying these "smallest" nearrings, therefore, provides information about general local nearrings with identity.

In [**3**] it was shown that if G is a finite group and $M_E(G)$ is local, then G is a p-group for some prime p. Hence, we will always assume in this paper that G is a finite p-group.

In the next section we initiate the study of the localness of $M_E(G)$ in terms of group properties. In particular we focus on the nilpotency of elements in $M_E(G)$ and give conditions on G to ensure that certain noninvertible elements are nilpotent. In the subsequent section we show that if G is an extra special p-group, then $M_E(G)$ is a local nearring and explicitly describe all functions in $M_E(G)$. In the last two sections we provide two more classes of groups that give rise to local nearrings.

Throughout the paper, we define Y to be the set of all elements of order p in G, denote the center of the group G by $Z(G)$ and the exponent of G by $\exp G$, and let the identity function on G be represented by *id*. In constructing endomorphisms the *commutator subgroup* of $G, G' = \langle\{-x_1 - x_2 + x_1 + x_2 \mid x_1, x_2 \in G\}\rangle$ and the *Frattini subgroup* of G, $\Phi(G) = \{\cap H_i \mid H_i$ is a maximal subgroup of $G\}$ will be useful.

2. General results. The first lemma collects some well-known results about p-groups [**13**].

Lemma 1. *Let G be a finite p-group.*

(i) *G' is a normal subgroup of G and G/G' is abelian*;

(ii) *If $H \triangleleft G$, then G/H is abelian if and only if $G' < H$*;

(iii) *$\Phi(G)$ is a normal subgroup of G and $G/\Phi(G)$ is elementary abelian*;

(iv) *$\Phi(G) = \{0\}$ if and only if G is elementary abelian*;

(v) *If $H \triangleleft G$, then G/H is elementary abelian if and only if $\Phi(G) < H$*;

(vi) *$px \in \Phi(G)$ for every $x \in G$*;

(vii) *All maximal subgroups of G are normal in G and are of index p*;

(viii) *If* $\{0\} \neq H \lhd G$, *then* $H \cap Z(G) \neq \{0\}$.

We will use the next result in the sequel without explicitly referencing it. The proof relies on the inclusion of the inner automorphisms in E.

Lemma 2 [**3**]. *$M_E(G)$ is an abelian nearring, i.e., $(M_E(G), +)$ is an abelian group.*

Lemma 3 [**4**]. *Let $\varphi \in \operatorname{End} G$ and $f \in M_E(G)$. Then*

(i) $f(\operatorname{Ker} \varphi) \subseteq \operatorname{Ker} \varphi$;

(ii) $f(\operatorname{Im} \varphi) \subseteq \operatorname{Im} \varphi$.

Proof. (i) Let $x \in \operatorname{Ker} \varphi$ and $f \in M_E(G)$. Then $\varphi(x) = 0$ and $\varphi f(x) = f\varphi(x)$ imply that $\varphi f(x) = f(0) = 0$. Hence, $f(x) \in \operatorname{Ker} \varphi$ and we have the result.

(ii) Let $x \in \operatorname{Im} \varphi$ and $f \in M_E(G)$. Then there exists an element $w \in G$ such that $\varphi(w) = x$. Thus $f\varphi(w) = f(x)$ and $\varphi f(w) = f(x)$. So $f(x) \in \operatorname{Im} \varphi$ and the proof is complete. □

In the next sequence of results, we determine the action of $f \in M_E(G)$ on the elements of order p in G.

Lemma 4. *Let $y \in Y$ and $f \in M_E(G)$. Then $f(y) = ky$ for some integer $0 \leq k < p$.*

Proof. Let $y \in Y$, and let H be a maximal (normal) subgroup of G. Then the map $\varphi : G \to G/H \to \langle y \rangle$ is an endomorphism of G with $\operatorname{Im} \varphi = \langle y \rangle$. By the previous lemma, $f(\langle y \rangle) \subseteq \langle y \rangle$. So $f(y) = ky$ for some integer $0 \leq k < p$. □

Lemma 5. *Let $f \in M_E(G)$, $x \in G$, and fix y in Y. Assume that $f(y) = ky$ for some integer $0 \leq k < p$. Then $f(x) = a + kx$ for some $a \in \Phi(G)$.*

Proof. Let $x \in G$, and let H be an arbitrary maximal subgroup of G. If $x \notin H$, then as above we construct the endomorphism $\varphi : G \to G/H \to \langle y \rangle$ such that $\varphi(x) = y$. Since φ is an endomorphism, $\varphi(f(x)-kx) = \varphi f(x)-\varphi(kx) = f\varphi(x)-k\varphi(x) = f(y)-ky = ky-ky = 0$ and $f(x) - kx \in \operatorname{Ker}\varphi = H$. If $x \in H$, then using the endomorphism $\psi : G \to G/H \to \langle y \rangle$ yields $\psi(f(x) - kx) = \psi f(x) - \psi(kx) = f\psi(x) - k\psi(x) = f(0) - k0 = 0$ and $f(x) - kx \in \operatorname{Ker}\psi = H$. In either case we have that $f(x) - kx \in H$. Since H is chosen arbitrarily as a maximal subgroup of G, then $f(x) - kx$ is in the intersection of all such maximal subgroups H, i.e., $f(x) - kx \in \Phi(G)$. Therefore, $f(x) = a + kx$ for some element $a \in \Phi(G)$. □

Theorem 6. *Let $f \in M_E(G)$. Then $f|_Y = k \cdot id$ for some integer $0 \leq k < p$.*

Proof. Let $f \in M_E(G)$ and $x \in G \backslash \Phi(G)$. Fix $y \in Y$ and assume that $f(y) = ky$ for some integer $0 \leq k < p$. Then, by Lemma 5, $f(x) = a + kx$ for some $a \in \Phi(G)$. Now let $y_1 \in Y$ and assume by Lemma 4 that $f(y_1) = k_1 y_1$. Since $x \notin \Phi(G)$, then there exists a maximal normal subgroup H of G with $x \notin H$. Let $\varphi \in \operatorname{End} G$ be given by $\varphi : G \to G/H \to \langle y_1 \rangle$ with $\varphi(x) = y_1$. Then $\varphi f(x) = \varphi(a + kx) = \varphi(a)+k\varphi(x) = 0+ky_1$. Also, $f\varphi(x) = f(y_1) = k_1 y_1$ so that $ky_1 = k_1 y_1$ and $k = k_1$. Since $y_1 \in Y$ is chosen arbitrarily, $f|_Y = k \cdot id$. □

Corollary 7. *Let G be a p-group of exponent p. Then $M_E(G) \cong Z_p$.*

Proof. Let $f \in M_E(G)$. Since G is of exponent p, we have that $Y = G$. So $f = f|_Y = k \cdot id$ for some $0 \leq k < p$; thus, $f \in \langle id \rangle$. Since f is arbitrary, $M_E(G) = \langle id \rangle$ and $M_E(G) \cong Z_p$. □

Define the *annihilator* of Y to be $\operatorname{Ann} Y = \{f \in M_E(G) \mid f(Y) = 0\}$. We now focus on functions in $\operatorname{Ann} Y$ and their relationship to other functions in $M_E(G)$.

Corollary 8. *Let $f \in M_E(G)$. Then $f = k \cdot id + h$ where $f|_Y = k \cdot id$ and $h \in \operatorname{Ann} Y$.*

Proof. If $f \in M_E(G)$, then $f|_Y = k \cdot id$ for some $0 \leq k < p$. Hence $-k \cdot id + f \in \operatorname{Ann} Y$. It follows that $f = k \cdot id + h$ for some $h \in \operatorname{Ann} Y$. □

Lemma 9. *Let $h \in \operatorname{Ann} Y$. If $k \cdot id + h \in M_E(G) \backslash \operatorname{Ann} Y$ is idempotent for $1 \leq k < p$, then $k = 1$.*

Proof. Suppose $(k \cdot id + h)^2 = k \cdot id + h$. Then $(k \cdot id + h)(k \cdot id + h) = k(k \cdot id + h) + h(k \cdot id + h) = k^2 \cdot id + kh + h(k \cdot id + h) = k \cdot id + h$. Therefore $(k^2 - k) \cdot id = h - kh - h(k \cdot id + h) \in \operatorname{Ann} Y$. So $k^2 - k = 0$ and $k^2 = k$. Since $k \in Z_p$, $k = 1$ and we reach the desired conclusion. □

In view of the previous lemma, henceforth all idempotents considered in $M_E(G) \backslash \operatorname{Ann} Y$ will be of the form $id + h$ where $h \in \operatorname{Ann} Y$.

Lemma 10. *Let $h \in \operatorname{Ann} Y$. Then $id + h$ is idempotent if and only if $h(id + h) = 0$.*

Proof. If $id + h$ is idempotent, then $(id + h)(id + h) = id(id + h) + h(id + h) = id + h + h(id + h) = id + h$ and $h(id + h) = 0$. The converse is now clear. □

Theorem 11. *$\operatorname{Ann} Y$ is a maximal ideal of $M_E(G)$ and $M_E(G)/\operatorname{Ann} Y \cong Z_p$.*

Proof. Define $\Psi : M_E(G) \to Z_p$ by $\Psi(f) = k$ where $f|_Y = k \cdot id$ and $0 \leq k < p$. It is straightforward to show that Ψ is a nearring epimorphism. Clearly $\operatorname{Ann} Y$ is the kernel of Ψ so that $\operatorname{Ann} Y$ is an ideal of $M_E(G)$. Since the image of Ψ is Z_p, by the first isomorphism theorem, $M_E(G)/\operatorname{Ann} Y \cong Z_p$. Since Z_p is simple, it follows that $\operatorname{Ann} Y$ is a maximal ideal [**12**, 1.40]. □

Since $\operatorname{Ann} Y$ being nil is a necessary condition for $M_E(G)$ to be local, we make this assumption in our next results.

Theorem 12. *If* $\operatorname{Ann} Y$ *is nil, then* $\operatorname{Ann} Y$ *is the unique maximal right* $M_E(G)$*-subgroup.*

Proof. It is clear that $\operatorname{Ann} Y$ is a right $M_E(G)$-subgroup. Let $f \in M_E(G)\backslash\operatorname{Ann} Y$. By Corollary 8, $f = k\cdot id + h$ where $0 < k < p$ and $h \in \operatorname{Ann} Y$. By hypothesis, $h^n = 0$ for some integer $n \geq 1$. Let I be the right $M_E(G)$-subgroup generated by f. Then $f \circ h^m = (k\cdot id+h)\circ h^m = kh^m + h^{m+1} \in I$ for all $m \geq 0$. We use this fact to prove by induction that $k^{2i}h - h^{2i+1} \in I$ for all $i \geq 1$.

Using $m = 1$ and $m = 2$ in the fact above, we see that $kh + h^2$, $kh^2 + h^3 \in I$. Since $M_E(G)$ is abelian, then $k(kh + h^2) - (kh^2 + h^3) = k^2h + kh^2 - kh^2 - h^3 = k^2h - h^3 \in I$. This shows our induction result is true for $i = 1$.

Assume $k^{2i}h - h^{2i+1} \in I$. Using $m = 2i + 1$ in the fact yields $k(k^{2i}h - h^{2i+1}) + (kh^{2i+1} + h^{2i+2}) = k^{2i+1}h + h^{2i+2} \in I$ while using $m = 2i+2$ yields $k(k^{2i+1}h+h^{2i+2})-(kh^{2i+2}+h^{2i+3}) = k^{2i+2}h-h^{2i+3} = k^{2(i+1)}h - h^{2(i+1)+1} \in I$. Hence the result holds for $i + 1$ and the induction proof is complete.

Let r be any integer greater than $(1/2)(n - 3)$. Then $2r + 3 > n$ and $h^{2r+3} = 0$. Letting $i = r + 1$ in the induction result implies that $k^{2(r+1)}h - h^{2(r+1)+1} = k^{2r+2}h - h^{2r+3} = k^{2r+2}h \in I$. Since $k \neq 0$, k is invertible in Z_p. Hence $h \in I$ and $f - h = (k \cdot id + h) - h = k \cdot id \in I$. It follows that $id \in I$ and $I = M_E(G)$.

Thus, since any function not in $\operatorname{Ann} Y$ generates $M_E(G)$ as a right $M_E(G)$-subgroup, then $\operatorname{Ann} Y$ is the unique maximal right $M_E(G)$-subgroup. □

Corollary 13. *If* $\operatorname{Ann} Y$ *is nil, then* $\operatorname{Ann} Y$ *is the unique maximal ideal of* $M_E(G)$*.*

Proof. The result follows immediately from the theorem since every ideal of $M_E(G)$ is a right $M_E(G)$-subgroup. □

Recall from [**8**] that a nearring N is *completely primary* if $N/J_2(N)$ is a nearfield. An element n of a nearring N is called *quasi-regular* if $m(1-n) = 1$ for some element $m \in N$. A (left) N-subgroup A is *quasi-*

regular if each element of A is quasi-regular. We use these definitions to characterize when $M_E(G)$ is local in terms of $\operatorname{Ann} Y$.

Lemma 14. *If* $\operatorname{Ann} Y$ *is nil, then* $\operatorname{Ann} Y = J_2(M_E(G))$*, and* $M_E(G)$ *is a completely primary nearring.*

Proof. If $\operatorname{Ann} Y$ is nil, then by Corollary 13, $\operatorname{Ann} Y$ is the unique maximal ideal of $M_E(G)$. Since $J_2(M_E(G))$ is the intersection of all maximal ideals of $M_E(G)$ [**12**, 5.42], we have that $\operatorname{Ann} Y = J_2(M_E(G))$. So, by Theorem 11, $M_E(G)$ is completely primary. □

Theorem 15. *The following are equivalent*:

(i) $M_E(G)$ *is local*;

(ii) $\operatorname{Ann} Y$ *is nil and quasi-regular*;

(iii) $\operatorname{Ann} Y$ *is nil and for every* $h \in \operatorname{Ann} Y$, $h(id + h) = 0$ *implies that* $h = 0$.

Proof. If $M_E(G)$ is local, then $\operatorname{Ann} Y$ is nil and $M_E(G)$ is completely primary by Lemma 14. By Theorem 3.3 of [**8**], $\operatorname{Ann} Y = J_2(M_E(G))$ is quasi-regular. Conversely, if $\operatorname{Ann} Y$ is nil, then $\operatorname{Ann} Y = J_2(M_E(G))$ and $M_E(G)$ is completely primary. Again, from Theorem 3.3 of [**8**], we conclude that $M_E(G)$ is local. This shows the equivalence of (i) and (ii).

To show (iii) implies (i), assume that $f \in M_E(G)$ is idempotent. If $f \in \operatorname{Ann} Y$, then f is nilpotent by hypothesis; hence, $f = 0$. If $f \in M_E(G) \backslash \operatorname{Ann} Y$, then $f = id + h$ for some $h \in \operatorname{Ann} Y$. By Lemma 10, $h(id + h) = 0$, and so by hypothesis, $h = 0$. It follows that $f = id$. Therefore, $M_E(G)$ has no nontrivial idempotents and is local. Reversing the above steps yields that (i) implies (iii), and the proof is complete. □

We now focus on the action of functions in $M_E(G)$ on various subgroups of G.

Lemma 16. *Let* H *be a subgroup of* G *with* $G' \leq H$*. If* $f \in M_E(G)$*,* *then* $f(H) \subseteq H$.

Proof. Let $\psi : G \to G/H \to Z_{p^{n_1}} \times Z_{p^{n_2}} \times \cdots \times Z_{p^{n_t}}$ be an epimorphism. Let $\pi_i : Z_{p^{n_1}} \times Z_{p^{n_2}} \times \cdots \times Z_{p^{n_t}} \to Z_{p^{n_i}}$ denote the projection homomorphism onto the ith component. We wish to show that $\cap_{i=1}^{t} \operatorname{Ker} \pi_i \psi = H$. Clearly $H \subseteq \cap_{i=1}^{t} \operatorname{Ker} \pi_i \psi$ since $H \subseteq \operatorname{Ker} \psi$. Let $x \in \cap_{i=1}^{t} \operatorname{Ker} \pi_i \psi$ and suppose that $x \notin H$. Then $\psi(x) \neq 0$ and $\pi_i(\psi(x)) \neq 0$ for some i. Hence $x \notin \operatorname{Ker} \pi_i \psi$ which is a contradiction. Thus $\cap_{i=1}^{t} \operatorname{Ker} \pi_i \psi = H$.

Let $w \in G$ be an element of maximum order. Then, for every i there is an integer m_i such that $\langle m_i w \rangle \cong Z_{p^{n_i}}$. So we can create an endomorphism of G via $\varphi_i \pi_i \psi \in \operatorname{End} G$ where φ_i denotes the isomorphism between $Z_{p^{n_i}}$ and $\langle m_i w \rangle$. By Lemma 3, $f(\operatorname{Ker} \varphi_i \pi_i \psi) \subseteq \operatorname{Ker} \varphi_i \pi_i \psi$. Hence $f(\cap_{i=1}^{t} \operatorname{Ker} \varphi_i \pi_i \psi) \subseteq \cap_{i=1}^{t} \operatorname{Ker} \varphi_i \pi_i \psi$. But $\operatorname{Ker} \varphi_i \pi_i \psi = \operatorname{Ker} \pi_i \psi$, so that $f(H) \subseteq H$. $\square$

Lemma 17. *Let* $\varphi \in \operatorname{End} G$, *and let* $a \in G$ *with* $\varphi(a) \in Y \cup \{0\}$. *If* $h \in \operatorname{Ann} Y$, *then* $h(a) \in \operatorname{Ker} \varphi$.

Proof. Since $\varphi(a) \in Y \cup \{0\}$ and $h \in \operatorname{Ann} Y$, then $\varphi h(a) = h\varphi(a) = 0$. Hence $h(a) \in \operatorname{Ker} \varphi$. $\square$

Corollary 18. *If* $h \in \operatorname{Ann} Y$, *then* $h(G) \subseteq \Phi(G)$.

Proof. Let $h \in \operatorname{Ann} Y$, and let H be a maximal subgroup of G. Then we can construct the endomorphism $\varphi : G \to G/H \to \langle y \rangle$ where $y \in Y$. Since $\operatorname{Im} \varphi \subseteq Y \cup \{0\}$ and $\operatorname{Ker} \varphi = H$, by the previous lemma we conclude that $h(G) \subseteq H$. Since H is chosen arbitrarily as a maximal subgroup of G, then $h(G)$ is contained in the intersection of all such maximal subgroups, i.e., $h(G) \subseteq \Phi(G)$. $\square$

Theorem 19. *Let* $\varphi \in \operatorname{End} G \backslash \operatorname{Aut} G$ *such that* $G' \subseteq \operatorname{Ker} \varphi$, *and let* $h \in \operatorname{Ann} Y$. *Then* $h^n(G) \subseteq \operatorname{Ker} \varphi$ *for some integer* n.

Proof. Since G is finite and $G' \subseteq \operatorname{Ker} \varphi$, by Lemma 1, $\operatorname{Im} \varphi$ is abelian, say $\operatorname{Im} \varphi = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_r \rangle$ for some nonzero $x_1, x_2, \ldots, x_r \in G$.

Assume $|x_1| = p^m$. Since $\langle x_1\rangle$ is abelian, $\langle px_1\rangle \triangleleft \langle x_1\rangle$ and $|\langle x_1\rangle/\langle px_1\rangle| = p$. The endomorphism $\varphi_1 : G \to G/\mathrm{Ker}\,\varphi \to \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_r\rangle \to \langle x_1\rangle/\langle px_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_r\rangle \to \langle p^{m-1}x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_r\rangle$ has the property that $|\mathrm{Ker}\,\varphi_1/\mathrm{Ker}\,\varphi| = p$. Continuing in this manner, we get a chain of normal subgroups, $\mathrm{Ker}\,\varphi = H_0 \triangleleft H_1 \triangleleft \cdots \triangleleft H_t \triangleleft H_{t+1} = G$ with $|H_i/H_{i-1}| = p$ and $H_i = \mathrm{Ker}\,\varphi_i$ for some $\varphi_i \in \mathrm{End}\,G$, $i = 1, 2, \ldots, t{+}1$.

For some $y \in Y$, $\varphi_t : G \to G/H_t \to \langle y\rangle$ and $\mathrm{Im}\,\varphi_t \subseteq Y \cup \{0\}$. Thus $h(G) \subseteq H_t$ by the previous lemma. Since $|H_t/H_{t-1}| = p$, then $pH_t \subseteq H_{t-1}$ and $\varphi_{t-1}(H_t) \subseteq Y \cup \{0\}$. By the previous lemma, $h(H_t) \subseteq H_{t-1}$. Thus $h^2(G) = h(h(G)) \subseteq h(H_t) \subseteq H_{t-1}$. Continuing in this manner yields the result. □

Theorem 20. *Let $h \in \mathrm{Ann}\,Y$. Then $h^n(G) \subseteq G'$ for some integer n.*

Proof. We use the notation introduced in Lemma 16 with $H = G'$. By the previous theorem, for each i there exists an integer n_i such that $h^{n_i}(G) \subseteq \mathrm{Ker}\,\varphi_i\pi_i\psi$. Let n be the maximum of all such n_i. Then $h^n(G) \subseteq \mathrm{Ker}\,\varphi_i\pi_i\psi$ for every i, i.e., $h^n(G) \subseteq \cap_{i=1}^t \mathrm{Ker}\,\varphi_i\pi_i\psi = G'$. □

Corollary 21. *If $\exp G' = p$, then $\mathrm{Ann}\,Y$ is nil.*

Proof. Let $h \in \mathrm{Ann}\,Y$. By Theorem 20, $h^n(G) \subseteq G'$ for some integer n. But since $G' \subseteq Y$, then $h(G') = 0$. Thus $h^{(n+1)}(G) = h(h^n(G)) \subseteq h(G') = 0$. Hence h is nilpotent and $\mathrm{Ann}\,Y$ is nil. □

Lemma 22. *Let $\varphi \in \mathrm{End}\,G$. If $h \in \mathrm{Ann}\,Y$, then there exists an integer n such that $h^n(\mathrm{Im}\,\varphi) \subseteq (\mathrm{Im}\,\varphi)'$.*

Proof. Let $h \in \mathrm{Ann}\,Y$ and $w \in \mathrm{Im}\,\varphi$. Then there exists an element $x \in G$ such that $\varphi(x) = w$ and an integer n such that $h^n(G) \subseteq G'$ by Theorem 20. So $h^n(w) = h^n\varphi(x) = \varphi h^n(x)$. Since $h^n(x) \in G'$ and $\varphi(G') \subseteq (\mathrm{Im}\,\varphi)'$, then $h^n(w) = \varphi(h^n(x)) \in \varphi(G') \subseteq (\mathrm{Im}\,\varphi)'$. Since $w \in \mathrm{Im}\,\varphi$ is arbitrary, $h^n(\mathrm{Im}\,\varphi) \subseteq (\mathrm{Im}\,\varphi)'$. □

Lemma 23. *Let $W = \{w \in G \mid |w| \leq \exp G/G'\}$. If $h \in \operatorname{Ann} Y$, then there exists an integer n such that $h^n(W) = 0$.*

Proof. Let $w \in W$. We can get an endomorphism $\varphi : G \to G/G' \to Z_{p^{t_1}} \times \cdots \times Z_{p^{t_k}} \to \langle w \rangle$. Since $\operatorname{Im} \varphi = \langle w \rangle$ is abelian, then $(\operatorname{Im} \varphi)' = \{0\}$, and the above lemma implies that for $h \in \operatorname{Ann} Y$, there exists an integer n_w such that $h^{n_w}(\langle w \rangle) = 0$. Let $n = \operatorname{Max} \{n_w \mid w \in W\}$. Then $h^n(W) = 0$ and the proof is complete. $\square$

Corollary 24. *If $\exp G' \leq \exp G/G'$, then $\operatorname{Ann} Y$ is nil.*

Proof. Let $h \in \operatorname{Ann} Y$. By Theorem 20, there is an integer n_1 such that $h^{n_1}(G) \subseteq G'$. By Lemma 23 there is an integer n_2 such that $h^{n_2}(W) = 0$. Since, by hypothesis, $G' \subseteq W$, then $h^{n_2+n_1}(G) = h^{n_2}(h^{n_1}(G)) \subseteq h^{n_2}(G') = 0$. Therefore h is nilpotent and $\operatorname{Ann} Y$ is nil. $\square$

We can get a more definitive result if $\exp G = \exp G/G'$.

Theorem 25. *If $\exp G = \exp G/G' = p^n$, then $M_E(G) \cong Z_{p^n}$ and, hence, is local.*

Proof. Assume that $G/G' \cong Z_{p^{n_1}} \times \cdots \times Z_{p^{n_m}}$ where $\exp G = p^{n_1}$. Since $\exp G = \exp G/G'$, then there exists an element $x \in G \backslash G'$ such that $|x| = p^{n_1}$. Let $f \in M_E(G)$. Creating the endomorphism $\varphi : G \to G/G' \to Z_{p^{n_1}} \times \cdots \times Z_{p^{n_m}} \to Z_{p^{n_1}} \to \langle x \rangle$ yields $f(x) = kx$ for some integer k by Lemma 3 since $\operatorname{Im} \varphi = \langle x \rangle$.

Let $x_1 \in G$. Then $|x| \geq |x_1| = p^{n_t}$ and we can create an endomorphism $\psi : G \to G/G' \to Z_{p^{n_1}} \times \cdots \times Z_{p^{n_m}} \to Z_{p^{n_1}} \to Z_{p^{n_t}} \to \langle x_1 \rangle$ with $\psi(x) = x_1$. Hence, $f(x_1) = f\psi(x) = \psi f(x) = \psi(kx) = k\psi(x) = kx_1$. Since $x_1 \in G$ is arbitrary, $f = k \cdot id$, $M_E(G) = \langle id \rangle$, and the result follows. $\square$

The converse to the theorem is false. If we let Q denote the quaternion group of order 8, we know from Lemma 6.1 of [**3**] that $M_E(Q) \cong Z_4$ and is, thus, local. But $Q' = Z(Q)$ and $|Q'| = |Z(Q)| = 2$. So $|Q/Q'| = 4$

and $Q/Q' \cong Z_2 \times Z_2$ since, otherwise, we would have that Q is abelian [**13**]. Hence $\exp Q/Q' = 2$ and $\exp Q = 4$.

Using the previous theorem we can construct numerous examples of groups that give rise to local nearrings. Let H be a finite p-group with $\exp H = p^m$. Consider $G = H \times Z_{p^n}$ where $n \geq m$. Then $G' = H' \times \{0\}$ and $G/G' \cong H/H' \times Z_{p^n}$. So $\exp G/G' = p^n = \exp G$. Hence $M_E(G) \cong Z_{p^n}$ by the previous theorem.

Furthermore, given the group Z_{p^n} we can find a nonabelian p-group G such that $M_E(G) \cong Z_{p^n}$. This is accomplished by letting $G = H \times Z_{p^n}$ where H is a nonabelian p-group of exponent p. As above, $\exp G/G' = p^n = \exp G$, and $M_E(G) \cong Z_{p^n}$. In particular, let

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; 1, a, b, c \in Z_p, p \text{ an odd prime} \right\}.$$

Under matrix multiplication, H is a nonabelian group of order p^3 and of exponent p and can be used in the above construction.

3. Extra special p-groups. Throughout this section we let G be a finite *extra special p-group*, i.e., a nonabelian p-group whose center, commutator subgroup, and Frattini subgroup all coincide and are of order p. If G is an extra special p-group, we know that $\operatorname{Ann} Y$ is nil by Corollary 21. We will, however, explicitly determine the functions in $\operatorname{Ann} Y$. The next lemma appears in [**7**].

Lemma 26. *If G is finite, extra special p-group, then* $\exp G = p$ *or* $\exp G = p^2$.

Since we know that $M_E(G) \cong Z_p$ when $\exp G = p$ by Corollary 7, we only consider when $\exp G = p^2$. For notation, let $O_1, \dots, O_r$ be the orbits of the elements of order p^2 under the action of $\operatorname{Aut} G$ on G. Furthermore, let $o_i \in O_i$, $i = 1, \dots, r$, be fixed orbit representatives.

Let A be a group of automorphisms of G and let $x \in G$. We define the *stabilizer* of x in A as $\operatorname{Stab}(x) = \{\sigma \in A \mid \sigma(x) = x\}$. Stabilizers play an important role in the study of centralizer nearrings determined by automorphisms via the well-known result of Betsch [**12**].

(Betsch's) Lemma 27. *Let $x_1, x_2 \in G$, and let A be a group of automorphisms of G. Then there is an $f \in M_A(G)$ with $f(x_1) = x_2$ if and only if* $\operatorname{Stab}(x_1) \subseteq \operatorname{Stab}(x_2)$.

Lemma 28. *Let G be a finite, extra special p-group of exponent p^2. Let $x \in G$ with $|x| = p^2$, and let $z \in \Phi(G)$. Then* $\operatorname{Stab}(x) \subseteq \operatorname{Stab}(z)$.

Proof. Let $\varphi \in \operatorname{Stab}(x)$. Then $\varphi(x) = x$ and $\varphi(px) = p\varphi(x) = px$. But $0 \neq px \in \Phi(G)$ by Lemma 1, so $z = m(px)$ for some $0 < m < p$ and $\varphi(z) = \varphi(m(px)) = (mp)\varphi(x) = mpx = z$. Hence $\varphi \in \operatorname{Stab}(z)$. □

Let $a = (a_1, a_2, \ldots, a_r) \in \Phi(G)^r$, and let $h_a : G \to G$ be defined by extending (via automorphisms)

$$h_a(x) = \begin{cases} a_i & \text{if } x = o_i \\ 0 & \text{if } x \in Y \cup \{0\} \end{cases},$$

i.e., if $x \in O_i$ and $\varphi \in \operatorname{Aut} G$ with $\varphi(o_i) = x$, then $h_a(x) = h_a(\varphi(o_i)) = \varphi h_a(o_i) = \varphi(a_i)$. This function is well-defined by Betsch's lemma and Lemma 28.

Lemma 29. *If G is a finite, extra special p-group, then* $\operatorname{Ann} Y = \{h_a \mid a \in \Phi(G)^r\}$.

Proof. Let $h_a \in \{h_a \mid a \in \Phi(G)^r\}$, and let $\varphi \in \operatorname{End} G$. If $\varphi \in \operatorname{Aut} G$, then $\varphi h_a = h_a \varphi$ by definition of h_a. If $\varphi \in \operatorname{End} G \backslash \operatorname{Aut} G$, then $\varphi(\Phi(G)) = 0$ because $\operatorname{Ker} \varphi \cap Z(G) \neq \varnothing$ by Lemma 1 and $|Z(G)| = |\Phi(G)| = p$. Since $h_a(G) \subseteq \Phi(G)$ by definition, then $\varphi h_a(G) = 0$. Also, $\varphi(G) \subseteq Y \cup \{0\}$ since $\Phi(G) \subseteq \operatorname{Ker} \varphi$ and $G/\operatorname{Ker} \varphi$ is elementary abelian by Lemma 1 so that $h_a \varphi(G) = 0$. Hence $\varphi h_a = h_a \varphi$ and $h_a \in M_E(G)$. That $h_a \in \operatorname{Ann} Y$ is clear and, therefore, $\{h_a \mid a \in \Phi(G)^r\} \subseteq \operatorname{Ann} Y$.

Let $h \in \operatorname{Ann} Y$. Then by Corollary 18, $h(G) \subseteq \Phi(G)$. In particular, $h(o_i) = a_i$ for every $i = 1, 2, \ldots, r$ and some $a_i \in \Phi(G)$. So $h = h_a$ where $a = (a_1, a_2, \ldots, a_r)$. Thus $\operatorname{Ann} Y \subseteq \{h_a \mid a \in \Phi(G)^r\}$ and the result follows. □

Corollary 30. *If G is a finite, extra special p-group and $h_a, h_b \in \operatorname{Ann} Y$, then $h_a h_b = 0$.*

Proof. Since $\operatorname{Im} h_b \subseteq Y \cup \{0\}$ and $h_a(Y \cup \{0\}) = 0$, the result is immediate. □

Corollary 31. *If G is a finite, extra special p-group of exponent p^2, then $|M_E(G)| = p^{r+1}$. In particular, if $r = 1$, $M_E(G) \cong Z_{p^2}$.*

Proof. Let $h \in \operatorname{Ann} Y$. Since $|\Phi(G)| = p$, there are p choices for $h(o_i)$ for each $i = 1, 2, \ldots, r$. So $|\operatorname{Ann} Y| = p^r$. Since $M_E(G) = \{k \cdot id + h \mid k \in Z_p, h \in \operatorname{Ann} Y\}$, the first statement follows. If $r = 1$, then $|M_E(G)| = p^2$. Since $\exp G = p^2$, then $M_E(G) = \langle id \rangle \cong Z_{p^2}$. □

Lemma 32. *Let G be a finite extra special p-group. Let $x \in G \backslash Z(G)$ and $mz \in Z(G)$ where $0 \leq m < p$ and $\langle z \rangle = Z(G)$. Then there exists a map $\varphi \in \operatorname{Aut} G$ such that $\varphi(x) = x + mz$.*

Proof. In [**14**], Winter states that if G is an extra special p-group, then $|G| = p^{(2n+1)}$ for some positive integer n and has generators $x_1, x_2, \ldots, x_{2n}$ satisfying the following relations once a suitable generator for the center, say $z \in Z(G)$, is chosen:

(i) $[x_{2i-1}, x_{2i}] = z$ for $i = 1, 2, \ldots, n$;

(ii) $[x_j, x_k] = 0$ unless $\{j, k\}$ is one of the pairs $\{2i-1, 2i\}$ or $\{2i, 2i-1\}$ for some $1 \leq i \leq n$;

(iii) $px_i \in \langle z \rangle = Z(G)$ for $i = 1, 2, \ldots, 2n$;

(iv) $pz = 0$.

Let $x \in G \backslash Z(G)$. By Lemma (3D) of [**14**], $x = a_1 x_1 + a_2 x_2 + \cdots + a_{2n} x_{2n} + cz$ where $0 \leq a_i,\ c < p$ for $i = 1, 2, \ldots, 2n$, and the coefficients are uniquely determined. Since $x \notin Z(G)$, at least one of the elements of $\{a_1, \ldots, a_{2n}\}$ is nonzero, say $a_j \neq 0$. Consider the map $\varphi : G \to G$ determined by linearly extending $\varphi(x_i) = x_i$ if $i \neq j$, $\varphi(x_j) = x_j + (a_j^{-1} m) z$ and $\varphi(z) = z$. By Lemma (3C) of [**14**], φ is an automorphism of G.

Then $\varphi(x) = \varphi(a_1x_1 + \cdots + a_{2n}x_{2n} + cz) = a_1\varphi(x_1) + \cdots + a_{j-1}\varphi(x_{j-1}) + a_j\varphi(x_j) + a_{j+1}\varphi(x_{j+1}) + \cdots + a_{2n}\varphi(x_{2n}) + c\varphi(z) = a_1x_1+\cdots+a_{j-1}x_{j-1}+a_j(x_j+(a_j^{-1}m)z)+a_{j+1}x_{j+1}+\cdots+a_{2n}x_{2n}+cz = a_1x_1 + \cdots + a_{2n}x_{2n} + cz + (a_ja_j^{-1}m)z = x + mz$. Hence φ is the automorphism we desire. □

Theorem 33. *If G is a finite extra special p-group, then $M_E(G)$ is local.*

Proof. If $\exp G = p$, then $M_E(G) \cong Z_p$ by Corollary 7. So we assume that $\exp G = p^2$. Let $h_a \in \operatorname{Ann} Y$ and suppose $h_a(id+h_a) = 0$. Then $h_a(id + h_a)(o_i) = h_a(o_i + a_i) = 0$ for every $i = 1, 2, \ldots, r$. By Lemma 32, there exists $\varphi_i \in \operatorname{Aut} G$ such that $\varphi_i(o_i) = o_i + a_i$ for each i. Thus $\varphi_i(a_i) = \varphi_i h_a(o_i) = h_a\varphi_i(o_i) = h_a(o_i + a_i) = 0$. Hence $a_i = 0$ since each φ_i is an automorphism, and we conclude that $h_a = 0$. From Corollary 21, $\operatorname{Ann} Y$ is nil. Applying Theorem 15 yields that $M_E(G)$ is local. □

4. Another example of a local centralizer nearring. In this section we investigate a nonextra special p-group such that $M_E(G)$ is local, but $M_E(G)$ is not isomorphic to any Z_m. Let p be an odd prime, and let G be the group with presentation $\langle a, b, c \mid p^2a = pb = pc = 0, -c + b + c = b + pa, a + b = b + a, a + c = c + a\rangle$. Then G is nonabelian, $\exp G = p^2$ and $|G| = p^4$ [**1**, p. 145]. Throughout this section, G will denote this group.

Lemma 34. *$Z(G) = \langle a\rangle$ and $G' = \Phi(G) = \langle pa\rangle$.*

Proof. Since $b + a = a + b$ and $a + c = c + a$, we have that $\langle a\rangle \subseteq Z(G)$. But, because G is nonabelian, then $G/Z(G)$ is not cyclic [**13**, 3.2.8]. Therefore, $|Z(G)| \le p^2$. Since $|a| = p^2$, we conclude that $Z(G) = \langle a\rangle$. It follows that $\langle a, b\rangle = \langle a\rangle \times \langle b\rangle \cong Z_{p^2} \times Z_p$ and $\langle a, c\rangle = \langle a\rangle \times \langle c\rangle \cong Z_{p^2} \times Z_p$ are maximal subgroups of G. Also $\langle b, c\rangle$ is a maximal subgroup of G, for $b + c = c + b + pa$ implies that $pa \in \langle b, c\rangle$ and $\langle b, c\rangle$ is a subgroup of order p^3. Therefore $\Phi(G) \subseteq \langle a, b\rangle \cap \langle a, c\rangle \cap \langle b, c\rangle = \langle pa\rangle$. Since G is nonabelian, Lemma 1 (iv) guarantees that $\{0\} \neq \Phi(G)$. Thus, $|pa| = p$ implies that $\Phi(G) = \langle pa\rangle$.

Since $\{0\} \neq G' \leq \Phi(G)$, the result follows. □

We note that G is not an extra special p-group since $|Z(G)| = p^2$. Furthermore, $\exp G = p^2 \neq p = \exp G/G'$, so Theorem 25 does not apply in this case.

Lemma 35. *For $0 \leq n < p$, define $\varphi_n : G \to G$ by $\varphi_n(ia+jb+kc) = (np+1)ia + jb + kc$. Then φ_n is an automorphism of G.*

Sketch of proof. We first show $c + ib = (-ip)a + ib + c$ for $i \geq 0$ by induction on i. The equation holds for $i = 0$; assume the equation holds for some nonnegative integer i and consider $c + (i + 1)b$. Then $c+(i+1)b = (c+ib)+b = ((-ip)a+ib+c)+b = (-ip)a+ib+b+c-pa = (-ip)a - pa + (i+1)b + c = (-(i+1)p)a + (i+1)b + c$ and the result holds for $i+1$, and hence for all $i \geq 0$.

Let k be a negative integer. Then, for $i = -k$, $c+ib = (-ip)a+ib+c$ implies that $c - kb = (kp)a - kb + c$. Taking inverses of both sides and using that $Z(G) = \langle a \rangle$ yields $kb - c = (-kp)a - c + kb$. Rearranging terms gives $c + kb = (-kp)a + kb + c$. Hence the result holds for all integers i.

One can now show by induction that $jc + ib = (-jip)a + ib + jc$ for all integers i and j. Using this fact it can be shown with tedious calculations that the function φ_n is an endomorphism of G. Since $\varphi_n(a) = (np + 1)a$ and $a \in \langle (np + 1)a \rangle$, then $a \in \operatorname{Im} \varphi_n$. Also, $\varphi_n(b) = b$ and $\varphi_n(c) = c$ so that $\langle a, b, c \rangle = G \subseteq \operatorname{Im} \varphi_n$. Hence φ_n is an automorphism of G. □

We use the same notation as in the previous section, namely, letting O_i, $i = 1, 2, \ldots, r$, denote the orbits of the elements of order p^2 under the action of $\operatorname{Aut} G$ on G and $o_i \in O_i$, $i = 1, 2, \ldots, r$, be fixed orbit representatives. We also continue to let $h_a : G \to G$ denote the function described just before Lemma 29.

Lemma 36. $\operatorname{Ann} Y = \{h_a \mid a \in \Phi(G)^r\}$.

Proof. Since $|\Phi(G)| = p$, the proof is similar to those of Lemma 28 and Lemma 29. □

Lemma 37. *Let $ia+jb+kc \in G$. Then $|ia+jb+kc| = p^2$ if and only if $i \pmod p \neq 0$.*

Proof. We note that, since b and c commute modulo pa, then $p(jb+kc) = 0$. Thus $a \in Z(G)$ implies that $p(ia+jb+kc) = p(ia)+p(jb+kc) = i(pa)+0 = i(pa)$. So if $ia+jb+kc$ is of order p^2, then $i \pmod p \neq 0$ and conversely. □

Lemma 38. *Let $x \in G$ be such that $|x| = p^2$, and let $m(pa) \in \Phi(G)$ with $0 \leq m < p$. Then there exists a map $\varphi \in \operatorname{Aut} G$ such that $\varphi(x) = x+m(pa)$.*

Proof. By the previous lemma, $x = ia+jb+kc$ where $i \pmod p \neq 0$. Hence $i \pmod p$ is invertible in Z_p. Then for $n = i^{-1}m$ in Lemma 35 we have that $\varphi_n(x) = \varphi_n(ia+jb+kc) = (np+1)ia+jb+kc = (i^{-1}mp)ia+ia+jb+kc = m(pa)+ia+jb+kc = x+m(pa)$. So φ_n is the automorphism we desire. □

Theorem 39. *Let G be the group given before Lemma 34. Then $M_E(G)$ is local and $M_E(G) \ncong Z_m$ for every positive integer m.*

Proof. The proof of the first statement is similar to that of Theorem 33. To prove the second statement, first note that a and $a+b$ are in different orbits since $a \in Z(G)$ and $b \notin Z(G)$, say $a \in O_1$ and $a+b \in O_2$. Let $d = (pa, 0, 0, \ldots, 0) \in \Phi(G)^r$. Then $h_d \in \operatorname{Ann} Y$ by Lemma 36. Therefore $M_E(G) \neq \langle id \rangle$ since $h_d(a) = pa$ and $h_d(a+b) = 0 \neq p(a+b)$. The result now follows. □

Recall from Corollary 31 that, if G is a finite extra special p-group of exponent p^2 and the number of orbits of elements of order p^2 under the action of $\operatorname{Aut} G$ on G, r, is one, then $M_E(G) \cong Z_{p^2}$. We remark, however, that if $r > 1$, then a proof similar to the one above shows that $M_E(G) \ncong Z_m$ for every positive integer m.

5. A final example of a local centralizer nearring. In this section we present another example of a finite group G such that $M_E(G)$ is local. This example, in general, is not extra special and $\exp G/G' \neq \exp G$.

Let G be the group with presentation $\langle a, b \mid p^{m-1}a = pb = 0, a + b = b + (1 + p^{m-2})a\rangle$ where $m \geq 3$ and p is an odd prime. Throughout this section G will denote this nonabelian group.

Lemma 40. *The following statements are true for the group G and integers i, j and n.*

(i) $|G| = p^m$ *and* $\exp G = p^{m-1}$;

(ii) $ia + jb = jb + (i + ijp^{m-2})a$;

(iii) $jb + ia = (i - ijp^{m-2})a + jb$;

(iv) $n(jb + ia) = (nj)b + (ni + (1/2)ijn^2p^{m-2} - (1/2)ijnp^{m-2})a$;

(v) $p(jb + ia) = (ip)a$;

(vi) $Z(G) = \langle pa\rangle = \Phi(G)$;

(vii) $G' = \langle p^{m-2}a\rangle$;

(viii) *For any two fixed integers r_1, r_2, the map $\varphi(jb + ia) = jb + i(r_2b + r_1a)$ is an endomorphism of G.*

Proof. Parts (i), (ii), (iv) and (v) are given by Burnside [**1**, p. 135]. Using (ii) we get $-(jb+ia) = -ia-jb = -jb+(-i+(-i)(-j)p^{m-2})a = -jb + (-i + ijp^{m-2})a$. Taking inverses of both sides yields (iii).

To show (vi) we first note that $ia+jb+pa = ia+(p-pjp^{m-2})a+jb = ia + pa - jp^{m-1}a + jb = ia + pa + jb = pa + ia + jb$ so that $pa \in Z(G)$. Because G is nonabelian, then $G/Z(G)$ is not cyclic [**13**, 3.2.8]; thus, $|Z(G)| \leq p^{m-2}$. But $|pa| = p^{m-2}$ implies that $Z(G) = \langle pa\rangle$.

Let $jb+ia+Z(G) \in G/Z(G)$. By (v), $p(jb+ia+Z(G)) = ipa+Z(G) = Z(G)$. Therefore, $\exp(G/Z(G)) = p$ and $\Phi(G) \subseteq Z(G)$ by Lemma 1 (v). By Lemma 1 (vi), $pa \in \Phi(G)$ so that $\Phi(G) = \langle pa\rangle$ and the proof of (vi) is complete.

By (iii), $-a - b + a + b = -a + ((1 + p^{m-2})a - b) + b = -a + a + p^{m-2}a - b + b = p^{m-2}a \in G'$. So $\langle p^{m-2}a\rangle \subseteq G'$. But $a + b + \langle p^{m-2}a\rangle = b + (1 + p^{m-2})a + \langle p^{m-2}a\rangle = b + a + p^{m-2}a + \langle p^{m-2}a\rangle = b + a + \langle p^{m-2}a\rangle$.

Since the generators a and b of G commute modulo $\langle p^{m-2}a\rangle$, then $G/\langle p^{m-2}a\rangle$ is abelian. So $G' \subseteq \langle p^{m-2}a\rangle$ by Lemma 1 (ii) and part (vii) now follows.

Part (viii) involves tedious calculations which we leave to the reader. □

If $m > 3$, then G is not extra special by parts (vi) and (vii) of the lemma. Furthermore, $p^{m-2}(jb + ia + G') = p^{m-2}jb + (p^{m-2}i + (1/2)ij(p^{m-2})^2p^{m-2} - (1/2)ijp^{m-2}p^{m-2})a + G' = G'$. So $\exp G/G' \leq p^{m-2} < \exp G$. Therefore, Theorem 25 does not apply in this case.

Theorem 41. *Let G be the group given before Lemma* 40*. Then* $M_E(G) \cong Z_{p^{m-1}}$.

Proof. Let $f \in M_E(G)$, and let $r_2b + r_1a \in G$. By part (viii) of the previous lemma, the map $\varphi(jb + ia) = jb + i(r_2b + r_1a)$ is an endomorphism of G with $\varphi(a) = r_2b + r_1a$.

Since $|a| = p^{m-1}$, then $\langle a\rangle$ is a maximal subgroup of G, and we can construct the endomorphism $\psi : G \to G/\langle a\rangle \to \langle b\rangle$. By Lemma 3 $f(\langle a\rangle) \subseteq \langle a\rangle$ and so $f(a) = ka$ for some integer k. Hence $f(r_2b+r_1a) = f\varphi(a) = \varphi f(a) = \varphi(ka) = k\varphi(a) = k(r_2b + r_1a)$. Since $r_2b + r_1a \in G$ is arbitrary, we have that $f = k \cdot id$ and $M_E(G) = \langle id\rangle \cong Z_{p^{m-1}}$. □

Although the group in Section 5 gives rise to a local centralizer nearring, this nearring does not have a very complicated structure. If G is any of the groups in this paper for which $M_E(G)$ is local, one might ask if there are other subsemigroups S of $\operatorname{End} G$ for which $M_S(G)$ is also local or local and nonabelian. Certainly in the former case a minimal such subsemigroup S' must exist. Perhaps $M_{S'}(G)$ will have a more interesting structure than $M_E(G)$.

REFERENCES

1. W. Burnside, *Theory of groups of finite order*, 2nd ed., Dover, New York, 1955.

2. G.A. Cannon, *Centralizer near-rings determined by* End G, Ph.D. Dissertation, Texas A&M University, 1995.

3. ———, *Centralizer near-rings determined by* End G, in *Near-rings and near-fields* (Y. Fong et al., eds.), Kluwer Academic Publishing, Dordrecht, The Netherlands, 1995, 89–111.

4. G.A. Cannon and L. Kabza, *Simplicity of the centralizer near-ring determined by* End G, Algebra Colloq. **5** (1998), 383–390.

5. J.R. Clay, *Nearrings*: *Geneses and applications*, Oxford Science Publ., Oxford, 1992.

6. P. Fuchs, C.J. Maxson, M.R. Pettet and K.C. Smith, *Centralizer near-rings determined by fixed point free automorphism groups*, Proc. Royal Soc. Edinburgh Sect. A **107** (1987), 327–337.

7. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968.

8. C.J. Maxson, *On local near-rings*, Math. Z. **106** (1968), 197–205.

9. C.J. Maxson and K.C. Smith, *The centralizer of a set of group automorphisms*, Comm. Alg. **8** (1980), 211–230.

10. ———, *Centralizer near-rings determined by local rings*, Houston J. Math. **11** (1985), 355–366.

11. J.D.P. Meldrum, *Near-rings and their links with groups*, Pitman, London, 1985.

12. G. Pilz, *Near-rings*, 2nd ed., North Holland/American Elsevier, Amsterdam, 1983.

13. W.R. Scott, *Group theory*, Dover, New York, 1987.

14. D.L. Winter, *The automorphism group of an extraspecial p-group*, Rocky Mountain J. Math. **2** (1972), 159–168.

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