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EXTREMAL DOMAINS FOR THE GEOMETRIC REFORMULATION OF BRENNAN'S CONJECTURE

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ABSTRACT. In [**7**], Carleson and Makarov convert the conjecture of Brennan on the order of growth of integral means of the derivative of a univalent function to a purely geometric and equivalent conjecture about some conformal invariants related to extremal length. Here we show that the failure of the geometric version of the conjecture always implies the existence of an extremal domain which must be bounded by the trajectory of a rational quadratic differential. We use a second variation developed by Chang, Schiffer and Schober to rule out the possibility of any zeros of order higher than two in such a trajectory.

1. Introduction. In [**4**], Brennan conjectured that for any conformal map ϕ of a simply connected domain $\Omega \subset \mathbf{C}$ onto the open unit disk $\mathbf{D}$

$$\iint_\Omega |\phi'|^p \, dx \, dy \, < \, +\infty$$

for all p such that $(4/3) < p < 4$.

A technique of Carleson from [**6**] for estimating harmonic measure is used in [**4**] to show that the upper bound for p is $3+\tau$ for some universal $\tau > 0$. By simpler methods it is shown in the same paper that the conjectured upper bound of 4 is the correct one for the class of close-to-convex domains. It is also explained in [**4**] that the problem has its origin in questions of approximation theory from the work of Metzger [**12**], Havin and Maz'ja [**10**] and Brennan [**5**].

An equivalent statement of the conjecture is that, for any conformal map f of the unit disk $\mathbf{D}$ into $\mathbf{C}$, any $p \in (-\infty, -2]$ and any $\varepsilon > 0$,

$$\int_0^{2\pi} |f'(re^{it})|^p \, dt \, = \, O\left(\frac{1}{(1-r)^{|p|-1+\varepsilon}}\right), \quad r \to 1$$

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holds. The conjectured upper bound of -2 corresponds to the upper bound of 4 in Brennan's statement.

By a very different approach than Brennan's based on manipulation of differential inequalities for the function

$$u(r) = \int_0^{2\pi} |f'(re^{it})|^p \, dt$$

Pommerenke proved in [**14**] that

$$\int_0^{2\pi} |f'(re^{it})|^p \, dt = O\left(\frac{1}{(1-r)^{|p|-.399+\varepsilon}}\right), \quad r \to 1$$

for all $p \leq -1$ and for all $\varepsilon > 0$. This corresponds to an upper bound of $p = 3.399$ in Brennan's original statement. Pommerenke's result has been improved slightly by Bertilsson [**3**], who combined the idea from [**14**] with new estimates for the coefficients

$$(f'(z))^p = \sum_{n=0}^{\infty} c_{n,p} z^n$$

for univalent functions f in $\mathbf{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$. The coefficient estimates in [**3**] are found by using the Loewner theory in a way suggested by DeBranges's proof of the Bieberbach conjecture.

The geometric approach to the problem was taken up again in [**7**] where Carleson and Makarov proved the following theorem on harmonic measure in simply connected domains:

Theorem A. *An absolute constant K exists such that, for every simply connected domain Ω satisfying*

$$\infty \in \Omega, \quad \operatorname{diam} \partial\Omega = 1$$

and any numbers $\varepsilon > 0$ and $\rho > 0$, the maximal number of disjoint disks of radius ρ and harmonic measure, evaluated at ∞, greater than $\rho^{(1/2)+\varepsilon}$ is at most

$$(abs.\ constant)\rho^{-K\varepsilon}.$$

This theorem has the corollary that, for any univalent f in $\mathbf{D}$ and any $p \leq -(1/2)K$,

$$\int_0^{2\pi} |f'(re^{it})\|^p \, dt \,=\, O\left(\frac{1}{(1-r)^{|p|-1+\varepsilon}}\right), \quad r \to 1$$

for every $\varepsilon > 0$. Thus, Brennan's conjecture is equivalent to the conjecture that $K = 4$ is the best possible constant in Theorem A.

The relationship between harmonic measure and extremal distance is exploited in [**7**] to give a further reformulation of Brennan's conjecture in terms of "beta numbers," which are introduced for the purpose. We will now briefly describe this version of the conjecture assuming some familiarity with the various conformal invariants. Some good references for harmonic measures, extremal length and extremal distance are [**1, 15**] and [**9**].

Let Ω be a simply connected domain and $a, b \in \hat{\mathbf{C}}$ any pair of points. For $\varepsilon > 0$, let Γ_ε denote the family of curves in $\mathbf{C}$ which join $\{z : |z-a| < \varepsilon\}$ and $\{z : |z-b| < \varepsilon\}$ (or if $b = \infty$, $\{z : |z| > (1/\varepsilon)\}$). Let $C_\varepsilon(w)$ denote the boundary of $\{z : |z-w| < \varepsilon\}$. If I_a and I_b are arcs of $C_\varepsilon(a) \cap \Omega$ and $C_\varepsilon(b) \cap \Omega$, respectively, let

$$\tilde{\lambda}_\varepsilon(a,b) \,=\, \inf_{(I_a, I_b)} d_\Omega\,(I_a, I_b)$$

where d_Ω denotes extremal distance in Ω. Following [**7**], define

$$\beta_\varepsilon(a,b,\Omega) \,=\, \exp\{2\pi\,[\lambda(\Gamma_\varepsilon) - \tilde{\lambda}_\varepsilon(a,b)]\}$$

and

$$\beta(a,b,\Omega) \,=\, \lim_{\varepsilon \to 0} \beta_\varepsilon(a,b,\Omega)$$

where λ denotes extremal length. The limit in the definition exists by the serial rule for extremal lengths.

By an n-configuration $\{\Omega; a_0, a_1, \dots, a_n\}$ we will mean a simply connected domain Ω together with $n+1$ distinct points $a_0, a_1, \dots, a_n$ on $\partial\Omega$. Let $\mathcal{C}_n$ denote the set of all n-configurations.

Fixing one point a_0 and taking n distinct other points $a_1, \dots, a_n$ on $\partial\Omega$, we denote

$$\beta_j \,=\, \beta(a_j, a_0, \Omega).$$

In [**7**] it is shown that Brennan's conjecture is equivalent to the statement that

$$\sum_{j=1}^{n} \beta_j^2 \leq 1 \tag{1}$$

for all n and all configurations $\{\Omega; a_0, a_1, \dots, a_n\}$ and the inequality is proven to hold in the case $n = 2$. In fact, it is shown that if

$$\sum_{j=1}^{n} \beta_j^2 > 1 \tag{2}$$

for some configuration, then it would be possible to find a domain bounded by a polygonal tree with $n+1$ tips for which (2) holds (the beta numbers being evaluated at the tips) and this tree can be used as a "generator" in the recursive construction of the fractal boundary of a domain for which Brennan's conjecture fails (in the form given in Theorem A with $K = 4$). Conversely, if Brennan's conjecture fails for some domain Ω then it is possible to construct a polygonal "tree" approximating $\partial\Omega$ for which (2) holds, the β_j's being evaluated at the tips of the tree. It should be mentioned that Baranski, Volberg and Zudnik [**2**] have shown that Brennan's conjecture is satisfied for simply connected basins of attraction for polynomials $z^2 + c$. The fractal domains considered by Carleson and Makarov are the critical case for the conjecture and are of a different nature than such basins of attraction.

The case $n = 2$ for the beta number formulation of Brennan's conjecture is handled in [**7**] as follows. Supposing that $\beta_1^2 + \beta_2^2 > 1$ for some configuration, the existence of an extremal configuration for the sum $\beta_1^2 + \beta_2^2$ is taken as being clear after normalization. The Schiffer variation is then applied to show that the boundary of the extremal domain is the trajectory of a certain quadratic differential. A differential equation is obtained for an appropriate mapping function, and this equation can be integrated completely to give an explicit computation of the sum $\beta_1^2 + \beta_2^2$ which is thereby shown to be strictly less than one. The present paper is an attempt to extend this approach to $n > 2$ using both first and second variation arguments. Brennan's conjecture remains open, but we are able to obtain some information on the extremal domains. We will prove (for all n) that the failure of the Carleson-Makarov version of the conjecture implies the existence of extremal domains for the sum (1), that such an extremal domain must be bounded by the trajectory of a quadratic differential, and that such a boundary trajectory cannot contain a zero of the quadratic differential with order higher than two. This means that one way to establish Brennan's conjecture might be to rule out the possibility of zeros of order one or two in an extremal trajectory of the above type.

In Section 2 we will prove

Theorem 1. *If M is the first integer for which there is some M-configuration*

$$\{\Omega_0; a_0, a_1, \dots, a_M\}$$

such that

$$\sum_{j=1}^{M} (\beta(a_j, a_0, \Omega_0))^2 > 1,$$

then there is an M-configuration

$$\{\Omega^*; a_0^*, a_1^*, \dots, a_M^*\}$$

such that

$$\sum_{j=1}^{n} (\beta(a_j^*, a_0^*, \Omega^*))^2 = \sup_{\mathcal{C}_M} \sum_{j=1}^{M} (\beta(a_j, a_0, \Omega))^2.$$

As mentioned earlier, this is stated in [**7**] for the case $n = 2$. For $n > 2$, a little argument is required since the positions of four or more endpoints cannot be completely normalized. The well-known second variation developed by Chang, Schiffer and Schober, originally intended as an investigative tool for coefficient problems, will be adapted in Section 3 to prove the following

Theorem 2. *Any extremal domain whose existence is established in Theorem 1 would necessarily be bounded by the trajectory of a rational quadratic differential with second order poles at $\{a_0^*, a_1^*, \dots, a_M^*\}$, and such a trajectory cannot contain a zero of order greater than 2.*

During the proof of Theorem 2 the differential equation (23) is also derived.

2. The existence of extremal domains. Proof of Theorem 1.

In the proofs of Theorem 1.1 and Theorem 1.2 we will use the notation

$$R_\nu(a_j) \;\equiv\; \{z : e^{-(\nu+1)A} < |z - a_j| < e^{-\nu A}\}$$

for open annuli centered at a_j, and

$$R_\nu(\infty) \;\equiv\; \left\{ z : e^{-(\nu+1)A} < \frac{1}{|z|} < e^{-\nu A} \right\}$$

for open annuli centered at ∞. Also, we denote

$$X_\nu(a_j) \;\equiv\; \lambda_\nu(a_j) - \lambda,$$

where $\lambda = A/2\pi$ is the extremal distance between the boundaries of the annuli, and where

$$\lambda_\nu(a_j) \;=\; \inf\{d_\Omega\,(l_+, l_-) : l_\pm \text{ are arcs on } \partial_\pm \cap \Omega\}.$$

Here $\partial_\pm$ are the inner and outer boundaries of $R_\nu(a_j)$ and d_Ω is extremal distance in Ω.

We will also use the following lemmas. They are from [**7**]. The proofs are included here for the sake of completeness.

Lemma 1. *Let $R = \{z : 1 < |z| < e^A\}$, and let E be a connected, compact subset of $\overline{R}$ such that $\partial R \cup E$ is connected and $R \setminus E$ is simply connected. Suppose also that neither $\{|z| = 1\} \setminus E$ nor $\{|z| = e^A\} \setminus E$ is empty. Let θ be the smallest angle of any sector containing $R \cap E$ ($\theta = 2\pi$ if E cannot be contained in a sector), and let I_0, I_A be arcs of $\{|z| = 1\} \setminus E$ and $\{|z| = e^A\} \setminus E$, respectively. Then there is a numerical constant $A_0 > 0$ such that if $A \geq A_0$,*

$$d_{R\setminus E}(I_0, I_A) \;\geq\; A + \frac{c\theta^3}{A}$$

for some numerical constant $c > 0$. The estimate holds for any A if θ is sufficiently small (depending on A).

Proof. Observe that the paths connecting the two arcs must pass outside the sector containing $R \cap E$. By a conformal mapping, it is therefore sufficient to estimate the extremal length of the family of paths Γ' in the rectangle $(0, A) \times (-\pi, \pi)$ which join the two vertical sides and which pass outside of $[0, A] \times (-\theta/2, \theta/2)$. Paths are allowed to continue through one horizontal edge to the other and no path can traverse the strip $\{|y| \le \theta/2\}$. Consider the metric

$$\rho(x, y) = \begin{cases} \dfrac{1}{A} & 0 \le x \le A,\ \theta/4 < |y| < \pi, \\ \dfrac{1}{\sqrt{A^2 + k\theta^2}} & 0 \le x \le A,\ 0 \le |y| \le \theta/4. \end{cases}$$

Of all paths in Γ' which intersect the strip $\{|y| \le (\theta/4)\}$, a path with minimal ρ-length must have the form $\overline{p_1p_2} \cup \overline{p_2p_3}$ where $p_1 = (0, (\theta/4))$, $p_2 = (d, (\theta/4))$ for some $0 \le d \le A$ and $p_3 = (A, (\theta/2))$, or be a reflection of such a path in $x = (A/2)$ or in $y = 0$. The minimal ρ-length of any path which intersects the strip $|y| \le (\theta/4)$ is then found by minimizing the function

$$l(d) = \frac{d}{\sqrt{A^2 + k\theta^2}} + \frac{1}{A}\sqrt{(A - d)^2 + \left(\frac{\theta}{4}\right)^2}.$$

By elementary methods, the minimizing value of d is determined to be

$$d = A\left(1 - \frac{1}{4\sqrt{k}}\right),$$

for which we have

$$l(d) = 1 + \left(\frac{\sqrt{k}}{4} - \frac{k}{2}\right)\frac{\theta^2}{A^2} + \cdots.$$

Choosing $k = 1/8$ ensures that all curves in the family have ρ-length greater than one, and we have

$$\iint \rho^2 \, dx \, dy = \frac{2\pi}{A} - \frac{\theta^3}{16A^3} + \cdots,$$

from which the lemma follows by taking reciprocals.

Lemma 2. *Let Ω be a simply connected domain in $\hat{\mathbf{C}}$ and $\xi \in \partial\Omega$. Let R be an annulus centered at ξ such that $\partial\Omega$ intersects both boundary circles of R. Let C_+ denote the outer and C_- the inner boundary circle of R, and let λ denote the extremal distance between C_+ and C_- in R. If I_+ is any arc of $C_+ \cap \Omega$ and I_- any arc of $C_- \cap \Omega$ then we have*

$$d_\Omega(I_+, I_-) - \lambda \geq \frac{c\theta^3}{\lambda}$$

where c is a numerical constant and θ is the smallest angle of a sector of R containing $R \cap \partial\Omega$.

Proof. Given the arcs I_+ and I_- we may chose two arcs I_+^* and I_-^* such that I_+^* and I_-^* both separate I_+ and I_- and so that I_+^* and I_-^* are not separated in Ω by any other arcs of $\Omega \cap C_+$ or $\Omega \cap C_-$. Let $\mathcal{U}$ denote the simply connected subdomain of R determined by Ω and the arcs I_+^*, I_-^*. Let θ_* denote the smallest angle of a sector containing $R \setminus \mathcal{U}$. By elementary properties of extremal distance and the previous lemma, we have

$$d_\Omega(I_+, I_-) \geq d_R(I_+^*, I_-^*) \geq \lambda + \frac{c\theta_*^3}{\lambda} \geq \lambda + \frac{c\theta^3}{\lambda}.$$

Proof of Theorem 1.1. Let M be the first integer such that some M-configuration $\{\Omega_0; a_0, a_1, \dots, a_M\}$ exists for which

$$\sum_{j=1}^{M} (\beta(a_j, a_0, \Omega_0))^2 > 1.$$

Let $\{\Omega^{(k)}; a_0, a_1^{(k)}, \dots, a_M^{(k)}\}$ be a sequence of M-configurations such that

$$\sum_{j=1}^{M} (\beta(a_j^{(k)}, a_0, \Omega^{(k)}))^2 \longrightarrow \sup_{\mathcal{C}_M} \sum_{j=1}^{M} (\beta(a_j, a_0, \Omega))^2 \equiv S.$$

Taking subsequences, we may assume that for each j, $a_j^{(k)} \to a_j$ and $\beta(a_j^{(k)}, a_0, \Omega^{(k)}) \to \beta_j > \delta > 0$. We may also assume that the sets

$\partial\Omega^{(k)}$ converge in the Hausdorff metric on the sphere to the boundary of a simply connected limit domain.

Suppose first that the points of $\{a_0, a_1, \dots, a_n\}$ are all distinct. Given $\delta > 0$ we want to show that

$$\sum_{j=1}^{M} \beta_j^2 \;>\; s - \delta.$$

Choose $\varepsilon' > 0$ such that for each $\varepsilon < \varepsilon'$,

$$\sum_{j=1}^{M} \beta_j^2 \;>\; \sum_{j=1}^{M} \beta_{j,\varepsilon}^2 - \frac{\delta}{2}.$$

By Lemma 2, if ε' is sufficiently small and k sufficiently large, the boundaries of the domains $\Omega^{(k)}$ and of the limit domain all pass through a narrow corridor of the annuli with radii ε and $e^{A}\varepsilon$ centered at the a_j. So the numbers $\beta_{j,\varepsilon}$ and $\beta_{j,\varepsilon}^{(k)}$ are computed using quadrilaterals whose end edges are the large circular arcs of the circles $C_\varepsilon(a_j)$. Now the theorem on convergence of the module from [**11**, p. 27] shows that

$$\sum_{j=1}^{M} \beta_{j,\varepsilon}^2 \;\geq\; \sum_{j=1}^{M} (\beta_{j,\varepsilon}^{(k)})^2 - \frac{\delta}{2}$$

for all sufficiently large k, and therefore

$$\sum_{j=1}^{M} \beta_j^2 \;\geq\; \sum_{j=1}^{M} (\beta_{j,\varepsilon}^{(k)})^2 - \delta \;\geq\; \sum_{j=1}^{M} (\beta_j^{(k)})^2 - \delta$$

and it follows that

$$\sum_{j=1}^{M} \beta_j^2 \;>\; S - \delta.$$

We finish the proof by ruling out the possibility that $\{a_0, a_1, \dots, a_n\}$ contains fewer than $n+1$ distinct points. By Moebius invariance, we may assume that $a_1^{(k)} = 0$, $a_2^{(k)} = 1$, $a_0 = \infty$ for every k.

If a sequence $a_j^{(k)}$ were to approach $a_0 = \infty$, the points $a_j^{(k)}$ and the parts of the boundary ending at the $a_j^{(k)}$ would have to pass through annular sectors of arbitrarily narrow angles containing the parts of the boundaries ending at a_0. We would then have

$$\lim_{k\to\infty}\sum_{\nu=1}^{\infty}X_\nu(a_j^{(k)}) \;=\; +\infty$$

and therefore we would have $\beta_{(j,k)} \to 0$.

We may suppose then that $a_{n-l}^{(k)},\ldots,a_M^{(k)}$ all have a common finite limit point a. If there were more than one such common limit point, we would repeat the following argument.

Choose a circle C_R of radius R centered at a. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j=M-l}^{M} & e^{(4\pi[\lambda_\varepsilon-\tilde{\lambda}_\varepsilon(a_j^k,a_0,\Omega_k)])}\\ &\le \lim_{\varepsilon\to 0} e^{(4\pi[\lambda(C_\varepsilon(a_0),C_R)-\tilde{\lambda}(C_\varepsilon(a_0),C_R,\Omega_k)])}\\ &\quad\cdot\sum_{j=M-l}^{M} e^{(4\pi[\lambda(C_R,C_\varepsilon(a_j^k))-\tilde{\lambda}(C_R,C_\varepsilon(a_j^k),\Omega_k)])}\end{aligned}$$

by the series rule.

Since M is the first integer for which

$$\sup_{\mathcal{C}_M}\sum_{j=1}^{M}\beta_j^2 \;>\; 1,$$

a rescaling shows that there is a function $\eta(\varepsilon)=o(1)$ as $\varepsilon\to 0$ so that

$$\begin{aligned}\lim_{\varepsilon\to 0}\sum_{j=M-l}^{M} & e^{(4\pi[\lambda_\varepsilon-\tilde{\lambda}_\varepsilon(a,a_0,\Omega_k)])}\\ &\le \lim_{\varepsilon\to 0} e^{(4\pi[\lambda(C_\varepsilon(a_0),C_R)-\tilde{\lambda}(C_\varepsilon(a_0),C_R,\Omega_k)])}(1+\eta(\varepsilon)).\end{aligned}\tag{3}$$

Let $\eta_2>0$. If R is sufficiently small, then we may choose k sufficiently large so that

$$\left|\lim_{\varepsilon\to 0} e^{(4\pi[\lambda(C_\varepsilon(a_0),C_R)-\tilde{\lambda}(C_\varepsilon(a_0),C_R,\Omega_k)])}-\beta^2(a,a_0,\Omega)\right| \;<\; \eta_2.$$

Then we would have

$$\beta^2(a, a_0, \Omega) + \sum_{j=1}^{M-l-1} \beta_j^2(a_j, a_0, \Omega) \; > \; 1.$$

This contradicts the definition of M so the theorem is proved.

3. A description of extremal domains. Proof of Theorem 2. To prove Theorem 2 we adapt the second variational argument devised by Chang, Schiffer and Schober to the beta numbers problem.

The existence of extremals for the problem has been shown in Section 1 and a standard application of the Schiffer variation, exactly as in [**7**], shows that extremal domains are bounded by the trajectories of a quadratic differential.

We will assume that $b = \infty$ and use the fact, also from [**7**], that in this situation,

$$\beta_j(a_j, b, \Omega) \; = \; \frac{2}{|f''(x_j)|} \tag{4}$$

where f is a conformal mapping to the upper half plane $\mathbf{H}$ onto Ω, normalized so that $f(z) \sim z^2$ at ∞ and where $x_j \in \mathbf{R}$ is the pre-image of the tip $a_j \in \partial\Omega$. Our immediate goal is to determine the form of a first variation for $f(z)$. We shall then use it to obtain a quadratic differential equation satisfied by f in $\mathbf{H}$.

In some fixed neighborhood of the boundary of the extremal domain Ω, let $v_\nu(w)$, $\nu = 1, 2, \ldots$, be defined, analytic and uniformly bounded. We vary $\partial\Omega$ by

$$w^* \; = \; w + \varepsilon\, v_1(w) + \varepsilon^2\, v_2(w) + \cdots .$$

Specific choices of the functions v_1 and v_2 will be made respectively in the first and second variation arguments which follow.

For small $\varepsilon > 0$ we obtain a union of analytic arcs $\partial\Omega^*$ which is the boundary of the new simply connected domain Ω^*. Let $f^*(z) = f^*(z, \varepsilon)$ be univalent and map $\mathbf{H}$ onto Ω^*. We may assume that $f^*(\infty) = \infty$, and we still have

$$\operatorname{Arg} f^*(t) \longrightarrow 0, \quad t \to \pm\infty,\ t \in \mathbf{R}.$$

Composing with a real dilation we may assume that $f^*(z) = z^2 + c'z + \dots$ as $z \to \infty$. Let $\tau_\delta(z) = z + \delta$. We suppose for now that the mappings $f^*(z, \varepsilon)$ all have the same fixed normalization, and we will be free to change $f^*(z, \varepsilon)$ by precomposition with $\tau_\delta(\varepsilon)$ where $\delta(\varepsilon)$ is an analytic function of ε in a neighborhood of $\varepsilon = 0$ and is real for real ε.

We have the Taylor series for $f^* : \mathbf{H} \to \Omega^*$ in ε

$$f^*(z) \;=\; f(z) + \varepsilon\, f_1(z) + \varepsilon^2\, f_2(z) + \cdots \tag{5}$$

and we define

$$\begin{aligned} z^*(z) \;&\equiv\; (f^*)^{-1}[f(z) + \varepsilon\, v_1(f(z)) + \varepsilon^2\, v_2(f(z)) + \cdots] \\ &=\; z + \varepsilon\, \psi_1(z) + \varepsilon^2\, \psi_2(z) + \cdots, \end{aligned} \tag{6}$$

where the ψ_ν are analytic in a neighborhood of $\mathbf{R}$ and real on $\mathbf{R}$, the values of $z^*(z)$ being determined by reflection below the real line.

We can insert (3) and (4) into the Taylor development of $f^*(z^*)$ in ε as follows

$$\begin{aligned} f^*(z^*(z)) \;&=\; f^*(z) + \varepsilon\, f^{*\prime}(z)\, \frac{\partial z^*}{\partial \varepsilon}\bigg|_{\varepsilon=0} \\ &\quad + \frac{1}{2}\,\varepsilon^2 \left[f^{*\prime}(z)\, \frac{\partial^2 z}{\partial \varepsilon^2}\bigg|_{\varepsilon=0} + f^{*\prime\prime}(z) \left(\frac{\partial z^*}{\partial \varepsilon}\right)^2 \bigg|_{\varepsilon=0} \right] \\ &\quad + o(\varepsilon^2) \\ &=\; f(z) + \varepsilon[f_1(z) + f'(z)\psi_1(z)] \\ &\quad + \varepsilon^2 \left[f_2(z) + f_1'(z)\psi_1(z) + f'(z)\psi_2(z) + f''(z)\, \frac{(\psi_1(z))^2}{2} \right] \\ &\quad + o(\varepsilon^2). \end{aligned} \tag{7}$$

Comparing coefficients of powers of ε in (4) and (5), we have

$$v_1(f(z)) \;=\; f_1(z) + f'(z)\psi_1(z) \tag{8}$$

and

$$v_2(f(z)) = \left[f_2(z) + f_1'(z)\psi_1(z) + f'(z)\psi_2(z) + f''(z)\frac{(\psi_1(z))^2}{2} \right]. \tag{9}$$

Since the ψ_ν are real on $\mathbf{R}$, (8) shows that

$$\operatorname{Im}\left[\frac{v_1(f(z))}{f'(z)}\right] = \operatorname{Im}\left[\frac{f_1(z)}{f'(z)}\right] \tag{10}$$

on $\mathbf{R}$.

We know that f_1/f' is analytic in $\mathbf{H}$. Following the accumulated experience of Schiffer and his collaborators, and in particular [**8**], we make the useful and admissible choice

$$v_1(f(z)) = \frac{x\,f(z)}{[f(z)-f(\xi)]f(\xi)} \tag{11}$$

for some $x \in \mathbf{C}$ and some $\xi \in \mathbf{H}$. Then we have

$$\begin{aligned}\frac{f_1(z)}{f'(z)} = {} & \frac{xf(z)}{f'(z)[f(z)-f(\xi)]f(\xi)} \\ & - \left[\frac{x}{[f'(\xi)]^2(z-\xi)} + \frac{\bar{x}}{[\overline{f'(\xi)}]^2(z-\bar{\xi})}\right] + C\end{aligned} \tag{12}$$

where C is some real constant. Equation (12) holds because $\psi_1(z)$ is analytic in a neighborhood of $\mathbf{R}$ and because its left and right sides are analytic in $\mathbf{H}$, bounded as $z \to \infty$, and have the same imaginary parts on $\mathbf{R}$. Replacing $f^*(z,\varepsilon)$ by $f^* \circ \tau_{C_\varepsilon} = f^{**}(z,\varepsilon)$ if necessary, we may assume that $C=0$. It follows from (8) that

$$\psi_1(z) = \left[\frac{x}{[f'(\xi)]^2\,(z-\xi)} + \frac{\bar{x}}{[\overline{f'(\xi)}]^2(z-\bar{\xi})}\right] \tag{13}$$

and

$$f_1(z) = \frac{x\,f(z)}{[f(z)-f(\xi)]f(\xi)} - \left[\frac{x\,f'(z)}{[f'(\xi)]^2(z-\xi)} + \frac{\bar{x}\,f'(z)}{[\overline{f'(\xi)}]^2(z-\bar{\xi})}\right]. \tag{14}$$

Note that, by the local structure of trajectories, f and f^* have derivatives of all orders at x_j and at $x*_j$, and recall that in the present situation $\beta_j(a_j,b,\Omega) = 2/|f''(x_j)|$. We now want to use the fact that

$$\sum_{j=1}^{n} \frac{1}{|f''(x_j)|^2} \geq \sum_{j=1}^{n} \frac{1}{|f^{*\prime\prime}(x_j^*)|^2}, \tag{15}$$

which follows from (4), to obtain a quadratic differential equation (see (23) below) which is satisfied by an extremal function. Here

$$\begin{aligned} x_j^* &= f^{*-1}(f(x_j) + \varepsilon\, v_1(f(x_j)) + \varepsilon^2\, v_2(f(x_j)) + \cdots) \\ &= x_j + \varepsilon\, \psi_1(x_j) + \varepsilon^2\, \psi_2(x_j) + \cdots \end{aligned} \tag{16}$$

and

$$\begin{aligned} f^{*\prime\prime}(x_j^*) &= f''(x_j + \varepsilon\, \psi_1(x_j) + \varepsilon^2\, \psi_2(x_j) + \cdots) \\ &\quad + \varepsilon\, f_1''(x_j + \varepsilon\, \psi_1(x_j) + \cdots) \\ &\quad + \varepsilon^2\, f_2''(x_j + \cdots) + o(\varepsilon^2) \\ &= f''(x_j) \\ &\quad + \varepsilon[\psi_1(x_j) f^{(3)}(x_j) + f_1''(x_j)] \\ &\quad + \varepsilon^2 \bigg[f_2''(x_j) + \psi_1(x_j) f_1^{(3)}(x_j) + \frac{(\psi(x_j))^2}{2} f^{(4)}(x_j) \\ &\qquad\qquad + f^{(3)}(x_j)\psi_2(x_j) \bigg] + o(\varepsilon^2). \end{aligned} \tag{17}$$

By correctly choosing v_2, we will arrange below that $\psi_2 \equiv 0$. Therefore, we define

$$V_1^{(j)} = \frac{\psi_1(x_j) f^{(3)}\,(x_j) + f_1''(x_j)}{f''(x_j)} \tag{18}$$

and

$$V_2^{(j)} = \frac{f_2''(x_j) + \psi_1(x_j) f^{(3)}{}_1(x_j) + ([\psi_1(x_j)]^2/2) f^{(4)}\,(x_j)}{f''(x_j)} \tag{19}$$

so that

$$\begin{aligned} \frac{1}{|f^{*\prime\prime}(x_j^*)|^2} &= \frac{1}{|f''(x_j)|^2} \big(1 - 2\,\varepsilon(\mathrm{Re}\,[V_1^{(j)}]) - \varepsilon^2(|V_1^{(j)}|^2 \\ &\qquad + 2\,\mathrm{Re}\,[V_2^{(j)}] - 4(\mathrm{Re}\,[V_1^{(j)}])^2) + o(\varepsilon^2)\big). \end{aligned} \tag{20}$$

By (15), it follows that

$$\mathrm{Re} \sum_{j=1}^{n} \frac{V_1^{(j)}}{|f''(x_j)|^2} \geq 0 \tag{21}$$

which is, after a computation,

$$\text{(22)}\quad \mathrm{Re}\left[\sum_{j=1}^{n} \frac{x}{|f''(x_j)|^2}\left(\frac{4}{[f'(\xi)]^2(x_j-\xi)^2} - \frac{1}{(f(x_i)-f(\xi))^2}\right)\right] \geq 0.$$

The inequality (22) must hold for all $x \in \mathbf{C}$, arbitrarily chosen in (11), and all $\xi \in \mathbf{H}$, so the first variation vanishes and we have

$$\text{(23)}\quad [f'(\xi)]^2 \sum_{j=1}^{n} \frac{1}{|f''(x_j)|^2(f(x_j)-f(\xi))^2} = \sum_{j=1}^{n} \frac{4}{|f''(x_j)|^2(x_j-\xi)^2}.$$

For the case $n = 2$, this equation is derived in [**7**] and used to prove that $\beta_1^2 + \beta_2^2 \leq 1$.

Now we compute a second variation, and we shall use it to complete the proof of Theorem 2. As mentioned earlier, with a proper choice of v_2, we can assume that $\psi_2 \equiv 0$. In fact, we have

$$\text{(24)}\quad v_2(f(z)) = f_2(z) + f_1'(z)\psi_1(z) + f''(z)\,\frac{(\psi_1(z))^2}{2} + f'(z)\psi_2(z)$$

and we will choose v_2 so that $v_2(f(z))$ has the same singularity at $z = \xi$ in $\mathbf{H}$ as

$$\text{(25)}\qquad f_1'(z)\psi_1(z) + f''(z)\,\frac{(\psi_1(z))^2}{2}.$$

Because $f_2(z)$ is at most $O(z)$ as $z \to \infty$, ψ_2 will be bounded and analytic in $\mathbf{H}$ and real on $\mathbf{R}$, therefore a real constant, say C'. Replacing $f^*(z,\varepsilon)$ by $f^* \circ \tau_{C'\varepsilon^2}$ if necessary, we may assume $C' = 0$.

After a computation we have

$$\text{(26)}\quad v_2(f(z)) = f_2(z) + x^2 A(z) + |x|^2 B(z) + \bar{x}^2 C(z) + f'(z)\psi_2(z),$$

where
(27)
$$A(z) = \frac{f'(z)}{[f'(\xi)]^4(z-\xi)^3} - \frac{f'(z)}{[f'(\xi)]^2(z-\xi)(f(z)-f(\xi))^2} - \frac{f''(z)}{2[f'(\xi)]^4(z-\xi)^2},$$

$$B(z) = \frac{f'(z)}{|f'(\xi)|^4(z-\xi)(z-\bar{\xi})^2} - \frac{f'(z)}{[\overline{f'(\xi)}]^2(z-\bar{\xi})(f(z)-f(\xi))^2} + \frac{f'(z) - f''(z)(z-\xi)}{|f'(\xi)|^4(z-\bar{\xi})(z-\xi)^2} \tag{28}$$

and

$$C(z) = \frac{f'(z)}{[\overline{f'(\xi)}]^4(z-\bar{\xi})^3} - \frac{f''(z)}{2[\overline{f'(\xi)}]^4(z-\bar{\xi})^2}. \tag{29}$$

We see that $C(z)$ is analytic in $\mathbf{H}$ and, after computation, that $A(z)$ has the following principal part at ξ:

$$\frac{f''(\xi)}{2[f'(\xi)]^4(z-\xi)^2} - \frac{1}{[f'(\xi)]^2}\left(\frac{f^{(3)}(\xi)}{6[f'(\xi)]^2} - \frac{(f''(\xi))^2}{4(f'(\xi))^3}\right)\frac{1}{(z-\xi)} \tag{30}$$

and that the principal part of $B(z)$ at ξ is

$$\frac{f'(\xi)}{|f'(\xi)|^4(z-\xi)(\xi-\bar{\xi})^2}. \tag{31}$$

We can therefore achieve the desired matching of singularities by choosing

$$\begin{aligned} v_2(f(z)) = x^2\Bigg[&\frac{f''(\xi)}{2[f'(\xi)]^2}\left(\frac{f(z)}{f(\xi)(f(z)-f(\xi))}\right)^2 \\ &+ \frac{1}{[f'(\xi)]^2}\left(M(\xi) - \frac{f''(\xi)}{f(\xi)}\right)\left(\frac{f(z)}{f(\xi)(f(z)-f(\xi))}\right)\Bigg] \\ &+ |x|^2\left[\frac{1}{[\overline{f'(\xi)}]^2(\xi-\bar{\xi})^2}\left(\frac{f(z)}{f(\xi)(f(z)-f(\xi))}\right)\right], \end{aligned} \tag{32}$$

where

$$M(\xi) \equiv \frac{1}{2}\left[\left(\frac{f''(z)}{f'(z)}\right)^2 - \frac{1}{3}\{f,\xi\}\right] \tag{33}$$

and

$$\{f,\xi\} = \frac{f^{(3)}(\xi)}{f'(\xi)} - \frac{3}{2}\left(\frac{f''(\xi)}{f'(\xi)}\right)^2 \tag{34}$$

denotes the Schwarzian derivative of f at ξ.

With this choice of v_2, we have

$$f_2(z) = v_2(f(z)) - f_1'(z)\,\psi_1(z) - f''(z)\,\frac{(\psi_1(z))^2}{2}. \tag{35}$$

From (13) and (14) it follows that

$$\frac{f_1'(x_j)}{f''(x_j)} = -\,\psi_1(x_j). \tag{36}$$

A computation with (19) now gives

$$V_2^{(j)} = v_2'(f(x_j)) - 2(V_1^{(j)}\,\psi_1'(x_j)) - (\psi_1'(x_j))^2. \tag{37}$$

And, since

$$\begin{aligned} V_1^{(j)} &= \mathrm{Re}\left[\frac{4x}{[f'(\xi)]^2\,(x_j-\xi)^2}\right] - \frac{x}{(f(x_j)-f(\xi))^2} \\ &= -\,2\psi_1'(x_j) - \frac{x}{(f(x_j)-f(\xi))^2} \end{aligned} \tag{38}$$

from (21) and (22), we have, using (11),

$$\begin{aligned} V_2^{(j)} = v_2'(f(x_j)) + x^2\Bigg(&\frac{3}{(f'(\xi))^4\,(x_j-\xi)^4} \\ &- \frac{2}{(f(x_j)-f(\xi))^2\,(f'(\xi))^2\,(x_j-\xi)^2}\Bigg) \\ + |x|^2\Bigg(&\frac{6}{|f'(\xi)|^4\,|x_j-\xi|^4} - \frac{2}{(f(x_j)-f(\xi))^2\,(\overline{f'(\xi)})^2\,(x_j-\bar{\xi})^2}\Bigg) \\ + \bar{x}^2\Bigg(&\frac{3}{(\overline{f'(\xi)})^4\,(x_j-\bar{\xi})^4}\Bigg). \end{aligned}$$

By (32), we have

$$\begin{aligned} v_2'(f(x_j)) = x^2\Bigg(&-\frac{f''(\xi)\,f(x_j)}{(f'(\xi))^2\,f(\xi)\,(f(x_j)-f(\xi))^3} - \frac{1}{(f'(\xi))^2} \\ &\cdot\left(M(\xi) - \frac{f''(\xi)}{f(\xi)}\right)\frac{1}{(f(x_j)-f(\xi))^2}\Bigg) \\ &- |x|^2\left(\frac{1}{(\overline{f'(\xi)})^2\,(\xi-\bar{\xi})^2\,(f(x_j)-f(\xi))^2}\right). \end{aligned}$$

The second variational inequality obtained from (20) by supposing we have a maximum is

$$\sum_{j=1}^{n} \frac{|V_1^{(j)}|^2 + 2\,\mathrm{Re}\,[V_2^{(j)}] - 4(\mathrm{Re}\,[V_1^{(j)}])^2}{|f''(x_j)|^2} \;\geq\; 0 \tag{40}$$

and we have after some algebra

$$|V_1^{(j)}|^2 + 2\,\mathrm{Re}\,V_2^{(j)} - 4(\mathrm{Re}\,[V_1^{(j)}])^2 \;=\; a_j x^2 + b_j |x|^2 + c_j \bar{x}^2 \tag{41}$$

where

$$\begin{aligned} a_j \;=\; & -\frac{6}{(f'(\xi))^4\,(x_j-\xi)^4} + \frac{2}{(f'(\xi))^2\,(x_j-\xi)^2\,(f(x_j)-f(\xi))^2} \\ & - \frac{1}{(f(x_j)-f(\xi))^4} - \frac{2f''(\xi)\,f(x_j)}{(f'(\xi))^2\,f(\xi)\,(f(x_j)-f(\xi))^3} \\ & - \frac{2}{(f'(\xi))^2}\left(M(\xi) - \frac{f''(\xi)}{f(\xi)}\right)\frac{1}{(f(x_j)-f(\xi))^2} \end{aligned} \tag{42}$$

$$\begin{aligned} b_j \;=\; & -\frac{12}{|f'(\xi)|^4\,|x_j-\xi|^4} - \frac{1}{|f(x_j)-f(\xi)|^4} \\ & - \frac{2}{(\overline{f'(\xi)})^2\,(\xi-\bar{\xi})^2\,(f(x_j)-f(\xi))^2} \\ & + 12\,\mathrm{Re}\left(\frac{1}{(\overline{f'(\xi)})^2\,(x_j-\bar{\xi})^2\,(f(x_j)-f(\xi))^2}\right) \\ & - 4\left(\frac{1}{(\overline{f'(\xi)})^2\,(x_j-\bar{\xi})^2\,(f(x_j)-f(\xi))^2}\right) \end{aligned}$$

and

$$\begin{aligned} c_j \;=\; & -\frac{6}{(\overline{f'(\xi)})^4\,(x_j-\bar{\xi})^4} - \frac{1}{(\overline{f(x_j)} - \overline{f(\xi)})^4} \\ & + \frac{6}{(\overline{f'(\xi)})^2\,(x_j-\bar{\xi})^2\,(\overline{f(x_j)}-\overline{f(\xi)})^2}. \end{aligned} \tag{43}$$

The inequality (40) has the form

$$\mathrm{Re}\,(ax^2 + b|x|^2 + c\bar{x}^2) \;\geq\; 0, \tag{44}$$

where

$$a = \sum_{j=1}^{n} \frac{a_j}{|f''(x_j)|^2}, \qquad b = \sum_{j=1}^{n} \frac{b_j}{|f''(x_j)|^2}$$

and

$$c = \sum_{j=1}^{n} \frac{c_j}{|f''(x_j)|^2},$$

and it holds for all $x \in \mathbf{C}$. Letting $x^2 = -|a+\bar{c}|/(a+\bar{c})$ gives

$$-\operatorname{Re} b \le -|a+\bar{c}|. \tag{45}$$

Note that, since (44) holds for all $x \in \mathbf{C}$, the inequality (45) also holds if $a = -\bar{c}$. Writing out (45) gives

$$\begin{aligned}
\operatorname{Re} \sum_{j=1}^{n} \frac{1}{|f''(x_j)|^2} \biggl(& \frac{12}{|f'(\xi)|^4 \, |x_j - \xi|^4} + \frac{1}{|f(x_j) - f(\xi)|^4} \\
& - \frac{8}{(\overline{f'(\xi)})^2 \, (x_j - \bar{\xi})^2 \, (f(x_j) - f(\xi))^2} \\
& + \frac{2}{(\overline{f'(\xi)})^2 \, (\xi - \bar{\xi})^2 \, (f(x_j) - f(\xi))^2} \biggr) \\
\le - \biggl| \sum_{j=1}^{n} \frac{1}{|f''(x_j)|^2} \biggl(& \frac{12}{(f'(\xi))^4 \, (x_j - \xi)^4} \\
& - \frac{8}{(f'(\xi))^2 \, (x_j - \xi)^2 \, (f(x_j) - f(\xi))^2} + \frac{2}{(f(x_j) - f(\xi))^4} \\
& + \frac{2 f''(\xi) \, f(x_j)}{(f'(\xi))^2 \, f(\xi) \, (f(x_j) - f(\xi))^3} + \frac{2}{(f'(\xi))^2} \\
& \cdot \left(M(\xi) - \frac{f''(\xi)}{f(\xi)} \right) \frac{1}{(f(x_j) - f(\xi))^2} \biggr) \biggr|.
\end{aligned} \tag{46}$$

This inequality must hold for all $\xi \in \mathbf{H}$.

Let η be a point on the real line such that $f(\eta)$ is a zero of order m of

$$\sum_{j=1}^{n} \frac{1}{|f''(x_j)|^2} \, \frac{1}{(f(x_j) - w)^2}.$$

By (23) such points η are exactly the preimages under f of the points of $\partial\Omega$ at which there are three or more analytic arcs meeting at a branching point.

Expand f as

$$f(\xi) = f(\eta) + \tau(\xi-\eta)^{\alpha} + o(\xi-\eta)^{\alpha}.$$

The theory of the local structure of trajectories implies that $\alpha = 2/(m+2)$. (See, for example, [**13**, Chapter 6].) Writing the asymptotic expansion of (46) as ξ makes a perpendicular approach to η, we see that if $m \geq 3$ then the only nonvanishing term on the lefthand side is the strictly positive second term. The only possibilities are therefore that $m = 1$ or $m = 2$, and this finishes the proof of the theorem. More information can be obtained by expanding the inequality near the endpoints $f(x_j)$ but the relationships obtained this way are complicated and do not seem to further clarify the geometric picture.

Note added in proof. In independent work but by the same method, Daniel Bertilsson has ruled out zeros of order two in extremal trajectories in his 1999 doctoral thesis for RIT Stockholm.

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