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CONTINUOUS MAPS ON IDEAL SPACES OF C^*-ALGEBRAS

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ABSTRACT. The set of ideals of a C^*-algebra can be given a natural topology, which restricts to the hull-kernel topology on the primitive ideals. Our primary interest is to study continuous maps on the space of all ideals, rather than on the subset of primitive ideals. We show how the properties of a map between primitive ideal spaces carry over to properties of the extended map between their ideal spaces.

As an application of these results we determine a number of properties of maps between ideal spaces of tensor products, both minimal and maximal. For example, for C^*-algebras A and B, the map $(\ker \pi, \ker \eta) \mapsto \ker(\pi \otimes \eta) : \operatorname{Id}(A) \times \operatorname{Id}(B) \to \operatorname{Id}(A \otimes B)$ is a homeomorphism onto its range. Finally, we apply these results to tensor products of continuous C^*-bundles.

Introduction. The set of ideals of a C^*-algebra can be given a natural topology associated to the partial ordering given by containment. This topology restricts to the usual hull-kernel topology on the subset of primitive ideals. It is clear that, given a continuous map between the ideal spaces of two C^*-algebras, it need not restrict to a map between primitive ideal spaces. On the other hand, given a continuous map between primitive ideal spaces, does it extend to a continuous map between ideal spaces? A less likely question to ask perhaps is, if we begin with a continuous map from the Cartesian product of two primitive ideal spaces into a primitive ideal space of a third C^*-algebra, will this property be retained when we attempt to extend to a map on the Cartesian product of the ideal spaces? We begin by answering these questions in the affirmative (Section 1).

Given C^*-algebras A and B, we apply these results of Section 1 to the study of three particular maps between the ideal spaces of A and $A \otimes B$, for both minimal and maximal tensor products. Every ideal I generates an ideal $\operatorname{Ext}(I)$ in $A \otimes B$. Every representation of $A \otimes B$

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restricts to a representation of A; this is well-defined on ideals, so for an ideal K in $A \otimes B$ we obtain the ideal $\mathrm{Res}\,(K)$ in A. There is a third map which takes $\ker \pi$ to $\ker(\pi \otimes \mathrm{id})$, where id is the universal representation of B. It turns out that this also gives a well-defined process on ideals, so for an ideal I in A, there is an ideal $\mathrm{Ind}\,(I)$ in $A \otimes B$.

The tensor product by a group C^*-algebra can be viewed as a crossed product by a trivial action. The maps Res, Ext and Ind have natural counterparts in the crossed product set-up, and the interplay between these three maps in that case gives an indication of what to expect of these maps for tensor products, [**11**, Section 3]. It is well-known from the theory of Morita equivalence that inducing representations in crossed products is continuous [**5**, Proposition 11], [**7**, Proposition 4.1]. In [**11**, Propositions 2.1, 2.7] we gave an elementary proof of the continuity of induction which works for crossed products by actions as well as coactions. However, the proof relies on group theoretic techniques and does not carry over to arbitrary tensor products. Part of what we show in Section 2 is that Ind is continuous for both minimal and maximal tensor products. To do so, we show that the map

$$(\ker \pi, \ker \eta) \longmapsto \ker(\pi \otimes \eta) : \ \mathrm{Id}\,(A) \times \mathrm{Id}\,(B) \longrightarrow \mathrm{Id}\,(A \otimes B)$$

is a homeomorphism onto its ranges, as is well known for primitive ideals [**6**, Theorem 5].

Archbold, Kaniuth, Schlichting and Somerset have studied continuous maps between ideal spaces of the Haagerup tensor products and showed, among other things, results analogous to some of ours (for example, [**2**, Theorem 1.5]).

Lastly, we use the results of Sections 1 and 2 to show that, given a continuous C^*-bundle $\mathcal{A}$ and a nuclear C^*-algebra B, there exists a continuous C^*-bundle $\mathcal{A} \otimes_{\max} B$ with section algebra $\Gamma_0(\mathcal{A}) \otimes_{\max} B$ and fibers $A_x \otimes_{\max} B$. The analogous result for exact C^*-algebras and minimal tensor products follows along similar lines (see Section 3).

1. Ideals and primitive ideals. Let X be a topological space and denote the collection of closed subsets of X by $\mathcal{P}(X)$ [**4**, Example 2.7]. This is a lattice with the partial order given by reverse containment.

Give $\mathcal{P}(X)$ the lower topology, which means it has subbasic open sets $O_F := \{G : F \not\supseteq G\}$, where F is a closed subset of X.

Lemma 1.1. *Let X and Y be topological spaces, $\alpha : X \to Y$, and define $\bar{\alpha} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\bar{\alpha}(F) = \overline{\alpha(F)}$. If α is continuous, then $\bar{\alpha}$ is continuous. If α is continuous and onto, then $\bar{\alpha}$ is onto. If α is a homeomorphism onto its range, then $\bar{\alpha}$ is a homeomorphism onto its range.*

Proof. Suppose α is continuous, and let S be a closed set in Y. We will show that $\bar{\alpha}^{-1}(O_S)$ is open in $\mathcal{P}(X)$ by showing that $\bar{\alpha}^{-1}(O_S) = O_{\alpha^{-1}(S)}$. Notice that, because α is continuous, $\alpha^{-1}(S)$ is closed in X, so $O_{\alpha^{-1}(S)}$ is a subbasic open set in $\mathcal{P}(X)$. Firstly, $\bar{\alpha}^{-1}(O_S) = \{C \in \mathcal{P}(X) : \bar{\alpha}(C) \in O_S\} = \{C : S \not\supseteq \bar{\alpha}(C)\}$. On the other hand, $O_{\alpha^{-1}(S)} = \{C : \alpha^{-1}(S) \not\supseteq C\}$, so it suffices to show that $\alpha^{-1}(S) \supseteq C$ if and only if $S \supseteq \bar{\alpha}(C)$. Suppose $\alpha^{-1}(S) \supseteq C$. Then $S \supseteq \alpha(\alpha^{-1}(S)) \supseteq \alpha(C)$. But S is closed so $S \supseteq \overline{\alpha(C)} = \bar{\alpha}(C)$. Conversely, suppose $S \supseteq \bar{\alpha}(C)$. Then $S \supseteq \alpha(C)$, and thus $\alpha^{-1}(S) \supseteq \alpha^{-1}(\alpha(C)) \supseteq C$.

Suppose α is continuous and onto, and let S be a closed subset of Y. Then $\bar{\alpha}(\alpha^{-1}(S)) = \overline{\alpha(\alpha^{-1}(S))} = \overline{S} = S$ and $\bar{\alpha}$ is onto.

Now suppose that α is a homeomorphism onto its range, and let F and G be closed sets in X such that $\overline{\alpha(F)} = \overline{\alpha(G)}$. To show that $\bar{\alpha}$ is injective, it is enough to show that $F \subseteq G$, so let $x \in F$. Then $\alpha(x) \in \overline{\alpha(G)}$, and thus there exists a net y_γ in $\alpha(G)$ converging to $\alpha(x)$. Since α is a homeomorphism onto its range, α^{-1} is continuous on the range of α, so $\alpha^{-1}(y_\gamma)$ converges to x. The injectivity of α implies that each $\alpha^{-1}(y_\gamma) \in \alpha^{-1}(\alpha(G)) = G$, and hence $x \in G$.

It remains to show that $\bar{\alpha}$ is open onto its range. Since $\bar{\alpha}$ is injective, it preserves intersections and it is enough to show that the image of a subbasic open set is open. Then $\bar{\alpha}(O_F) = \{\overline{\alpha(S)} : F \not\supseteq S\}$. On the other hand,

$$O_{\overline{\alpha(F)}} \cap \bar{\alpha}(\mathcal{P}(X)) = \{\overline{\alpha(S)} : \overline{\alpha(F)} \not\supseteq \overline{\alpha(S)}\}.$$

We need to know that $F \supseteq S$ if and only if $\overline{\alpha(F)} \supseteq \overline{\alpha(S)}$. The forward direction is clear, and the argument used to show injectivity gives the reverse direction. $\square$

Lemma 1.2. *Let X, Y and Z be topological spaces, $\alpha : X \times Y \to Z$, and define $\bar{\alpha} : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(Z)$ by $\bar{\alpha}(C, D) = \overline{\alpha(C \times D)}$. If α is continuous, then $\bar{\alpha}$ is continuous. If α is continuous and onto, then $\bar{\alpha}$ is onto. If α is a homeomorphism onto its range, then $\bar{\alpha}$ is a homeomorphism onto its range.*

Proof. Let $\overline{\alpha(C \times D)} \in O_S$. To show $\bar{\alpha}$ is continuous, it suffices to find closed sets $E \subseteq X$ and $F \subseteq Y$ such that $C \in O_E$, $D \in O_F$ and $\bar{\alpha}(C \times D) \in \bar{\alpha}(O_E \times O_F) \subseteq O_S$.

Since $S \not\supseteq \overline{\alpha(C \times D)}$ and S is closed, there exists $(c, d) \in C \times D$ such that $\alpha(c, d) \notin S$. Since α is continuous and $\operatorname{Comp} S$ is open, there exist closed sets $E \subseteq X$ and $F \subseteq Y$ such that $(c, d) \in \operatorname{Comp} E \times \operatorname{Comp} F \subseteq \alpha^{-1}(\operatorname{Comp} S)$.

Now $c \notin E$ and $d \notin F$, so $E \not\supseteq C$ and $F \not\supseteq D$, which means $(C, D) \in O_E \times O_F$ so that $\bar{\alpha}(C, D) \in \bar{\alpha}(O_E \times O_F)$.

It remains to check that $\bar{\alpha}(O_E \times O_F) \subseteq O_S$. Suppose that $E \not\supseteq E'$ and $F \not\supseteq F'$, so that $(E', F') \in O_E \times O_F$. That means $e' \in E'$ exists such that $e' \notin E$; similarly for an f'. Thus, $(e', f') \in \operatorname{Comp} E \times \operatorname{Comp} F \subseteq \alpha^{-1}(\operatorname{Comp} S)$ and hence $\alpha(\varepsilon', f') \in \operatorname{Comp} S$. We have found a point in $\alpha(E' \times F')$ which is not in S, so $S \not\supseteq \alpha(E' \times F')$ and $\overline{\alpha(E' \times F')} \in O_S$. Hence, $\bar{\alpha}$ is continuous.

The remaining assertions are proved as in Lemma 1.1. □

For C^*-algebra A, let $\operatorname{Id}(A)$ be the set of closed two-sided ideals of A with subbasic open sets $O_I := \{J \in \operatorname{Id}(A) : J \not\supseteq I\}$ for some $I \in \operatorname{Id}(A)$. The relative topology on the subspace of primitive ideals $\operatorname{Prim} A$ is the hull-kernel topology, because all the closed sets of $\operatorname{Prim} A$ are of the form $C_I := \{P \in \operatorname{Prim} A : P \supseteq I\}$ [**3**, 3.1.2]. All open sets of $\operatorname{Prim} A$ are of the form $U_I := \{P \in \operatorname{Prim} A : P \not\supseteq I\}$.

It is well known that the map $h : \operatorname{Id}(A) \to \mathcal{P}(\operatorname{Prim} A)$, defined by $h(I) = C_I$, is a homeomorphism. Indeed, it is injective because if $\{P : P \supseteq J\} = \{P' : P' \supseteq I\}$, then $I = J$. For a closed set S in $\operatorname{Prim} A$, $S = \{P' : P' \supseteq \cap_{I \in S} I\}$, so h is onto. Since $h^{-1}(O_{C_I}) = \{J : h(J) \in O_{C_I}\} = \{J : C_I \not\supseteq C_J\} = \{J : J \not\supseteq I\} = O_I$, h is continuous and a similar calculation shows $h(O_I) = O_{C_I}$, so h is also open.

Corollary 1.3. *Let A and B be C^*-algebras, $\alpha : \operatorname{Prim} A \to \operatorname{Prim} B$, and define $\tilde{\alpha} : \operatorname{Id}(A) \to \operatorname{Id}(B)$ by $\tilde{\alpha}(I) := \cap\{\alpha(P) : P \in \operatorname{Prim} A,\ P \supseteq I\}$. If α is continuous, then $\tilde{\alpha}$ is continuous. If α is continuous and onto, then $\tilde{\alpha}$ is onto. If α is a homeomorphism onto its range, then $\tilde{\alpha}$ is a homeomorphism onto its range.*

Corollary 1.4. *Let A, B and C be C^*-algebras, $\alpha : \operatorname{Prim} A \times \operatorname{Prim} B \to \operatorname{Prim} C$, and define $\tilde{\alpha} : \operatorname{Id}(A) \times \operatorname{Id}(B) \to \operatorname{Id}(C)$ by*

$$\tilde{\alpha}(I, J) := \cap\{\alpha(P, Q) : P \in \operatorname{Prim} A,\ Q \in \operatorname{Prim} B,\ P \supseteq I,\ Q \supseteq J\}.$$

If α is continuous, then $\tilde{\alpha}$ is continuous. If α is continuous and onto, then $\tilde{\alpha}$ is onto. If α is a homeomorphism onto its range, then $\tilde{\alpha}$ is a homeomorphism onto its range.

All nonzero representations will be nondegenerate, and the extension of a representation π of a C^*-algebra A to the multiplier algebra $M(A)$ will be denoted by $\bar{\pi}$. Let $AB := \overline{\operatorname{sp}}\{ab : a \in A,\ b \in b\}$ denote the closed span of the products of elements of C^*-algebras A and B. Let $\iota : A \to M(B)$ be a homomorphism and define

$$\operatorname{Res}_\iota : \ \operatorname{Id}(B) \longrightarrow \operatorname{Id}(A) \quad \text{by} \quad \operatorname{Res}_\iota(\ker \sigma) = \ker(\bar{\sigma} \circ \iota), \tag{1}$$

and

$$\operatorname{Ext}_\iota : \operatorname{Id}(A) \longrightarrow \operatorname{Id}(B) \quad \text{by} \quad \operatorname{Ext}_\iota(I) = B\iota(I)B. \tag{2}$$

In [**10**, Lemma 1.1] we showed that Res_ι is well defined, continuous and

$$J \in \operatorname{Res}_\iota(O_K) \Longleftrightarrow \operatorname{Ext}_\iota(J) \in O_K. \tag{3}$$

We also showed that Res_ι is open onto its range if and only if $\operatorname{Ext}_\iota|_{\operatorname{Res}_\iota(\operatorname{Id}(B))}$ is continuous. The proof states that this follows from the fact that

$$\operatorname{Ext}_\iota^{-1}(O_K) \cap \operatorname{Res}_\iota(\operatorname{Id}(B)) = \operatorname{Res}_\iota(O_K).$$

But sets of the form O_K are only subbasic open sets, so this alone is not quite enough. However, the following argument shows that Res_ι

preserves intersections of subbasic open sets, so this equation is in fact sufficient. It follows from the definitions that $\mathrm{Res}_\iota(O_K \cap O_L) \subseteq \mathrm{Res}_\iota(O_K) \cap \mathrm{Res}_\iota(O_L)$, so let $J \in \mathrm{Res}_\iota(O_K) \cap \mathrm{Res}_\iota(O_L)$. Equation 3 says that $\mathrm{Ext}_\iota(J)$ is in both O_K and O_L. That means $\mathrm{Ext}_\iota(J) \not\supseteq K$ and $\mathrm{Ext}_\iota(J) \not\supseteq L$. Since J is in the image of Res_ι, $\mathrm{Res}_\iota \mathrm{Ext}_\iota(J) = J$, so there exists an ideal M (namely $M = \mathrm{Ext}_\iota(J)$) such that $\mathrm{Res}_\iota(M) = J$ and $M \not\supseteq K$ and $M \not\supseteq L$. Hence $J \in \mathrm{Res}_\iota(O_K \cap O_L)$.

Corollary 1.5. *Let A be a C^*-algebra, I an ideal in A and $q : A \to A/I$ the quotient map. Then $\mathrm{Res}_q : \mathrm{Id}\,(A/I) \to \mathrm{Id}\,(A)$ is a homeomorphism onto its range.*

Proof. There is a map $\alpha : \mathrm{Prim}\,(A/I) \to \mathrm{Prim}\,(A)$, satisfying $\alpha(q(P)) = P$ (where P is a primitive ideal containing I), which is a homeomorphism onto its range [**3**, 3.2.1]. By Corollary 1.3, $\tilde{\alpha} : \mathrm{Id}\,(A/I) \to \mathrm{Id}\,(A)$ is a homeomorphism onto its range. It remains to check that $\tilde{\alpha}(q(J)) = \mathrm{Res}_q(q(J))$, which we do by showing they are both equal to J.

Since q is onto, $\mathrm{Ext}_q(J) = q(J)$ and then $\mathrm{Res}_q(q(J)) = J$ on the range of Res_q [**10**, Lemma 1.1]. To show $\tilde{\alpha}(q(J)) = J$ we need to see that $P \supseteq J$ if and only if $q(P) \supseteq q(J)$. The forward direction is clear, and the reverse direction depends on the fact that if $q(j) \in q(P)$ and P contains I, then $j \in P$. $\square$

2. Tensor products. For the applications in Section 3, we need to show that $\mathrm{Ind} : \mathrm{Id}\,(A) \to \mathrm{Id}\,(A \otimes_{\min} B)$, where $\mathrm{Ind}\,(\ker\sigma) = \ker\,(\sigma \otimes \mathrm{id})$ is continuous. However, more is true. In fact, the map $j : \mathrm{Id}\,(A) \times \mathrm{Id}\,(B) \to \mathrm{Id}\,(A \otimes_{\min} B)$ satisfying $j(\ker\sigma, \ker\tau) = \ker\,(\sigma \otimes \tau)$ is a homeomorphism onto its range, which is dense in $\mathrm{Id}\,(A \otimes_{\min} B)$.

As we will see, it is not that hard to show that j is injective and open onto its range, which is dense. The difficulty is the continuity. But, at the level of states, continuity is easy. More precisely, the map $S(A) \times S(B) \to S(A \otimes_{\min} B)$, satisfying $f \times g \mapsto f \otimes g$, is continuous. So one might try to lift this continuity from the states to ideals using the canonical map $\theta : S(A) \to \mathrm{Id}\,(A)$.

This idea certainly works for pure states and primitive ideals. This is how Guichardet proved that j, restricted to primitive ideals, is a homeomorphism onto its range, which is dense in $\mathrm{Prim}\,(A \otimes_{\min} B)$ [**6**, Theorem 5]. More precisely, there is a commutative diagram:

$$
\begin{tikzcd}
P(A) \times P(B) \arrow[r, "\Phi"] \arrow[d, "\theta_A \times \theta_B"'] & P(A \otimes_{\min} B) \arrow[d, "\theta_{A \otimes B}"] \\
\mathrm{Prim}\,(A) \times \mathrm{Prim}\,(B) \arrow[r, "j"'] & \mathrm{Prim}\,(A \otimes_{\min} B)
\end{tikzcd} \tag{4}
$$

The maps θ_A, θ_B and $\theta_A \times \theta_B$ are continuous, open surjections [**3**, p. 44], and a diagram chase shows that j is continuous on $\mathrm{Prim}\,(A) \times \mathrm{Prim}\,(B)$ (cf. [**13**, pp. 256–257]). One important ingredient in this recipe is the surjectiveness of the map from pure states to primitive ideals and, unfortunately, this is not always true from states to ideals. Primitive ideals are kernels of irreducible representations, and irreducible representations are always cyclic, and therefore are unitarily equivalent to the GNS representation obtained from the cyclic vector. To get around this difficulty we will first prove the continuity on primitive ideals and extend this to continuity on all ideals using Lemma 1.4.

Let A and B be C^*-algebras, and denote the algebraic tensor product by $A \odot B$. In general there will be more than one C^*-norm on $A \odot B$ [**9**, p. 190]. For a C^*-norm γ on $A \odot B$, we denote the completion of $A \odot B$ with respect to γ by $A \otimes_\gamma B$.

Proposition 2.1. *Let A and B be C^*-algebras and γ a C^*-norm on $A \odot B$. Define $j_\gamma : \mathrm{Id}\,(A) \times \mathrm{Id}\,(B) \to \mathrm{Id}\,(A \otimes_\gamma B)$ by $j_\gamma(\ker \pi, \ker \eta) = \ker\,(\pi \otimes \eta)$. Then j_γ is a homeomorphism onto its range, the restriction $j_\gamma : \mathrm{Prim}\,(A) \times \mathrm{Prim}\,(B) \to \mathrm{Prim}\,(A \otimes_\gamma B)$ is a homeomorphism onto its range. Also, $j_{\min}(\mathrm{Id}\,(A) \times \mathrm{Id}\,(B))$ is dense in $\mathrm{Id}\,(A \otimes_{\min} B)$ and $j_{\min}(\mathrm{Prim}\,(A) \times \mathrm{Prim}\,(B))$ is dense in $\mathrm{Prim}\,(A \otimes_{\min} B)$.*

Proof. We begin by showing that $j := j_{\min}$ is a homeomorphism onto its range. But first we need to see that it is well-defined on ideals. Let q_I be the quotient map from A to A/I and similarly for q_J. Then $\pi = \pi' \circ q_I$ and $\eta = \eta' \circ q_J$ for faithful nondegenerate representations π' and σ' of A/I and B/J, respectively. Note that $\pi' \otimes \sigma'$ is a faithful representation of $(A/I) \otimes_{\min} (B/J)$. It follows from [**9**, 6.5.1] that $q_I \otimes q_J$ is a well-defined bounded homomorphism from $A \otimes_{\min} B$ onto $(A/I) \otimes_{\min} (B/J)$. Moreover, $\pi \otimes \eta = (\pi' \otimes \eta') \circ (q_I \otimes q_J)$. Thus,

$\ker(\pi \otimes \eta) = \ker(q_I \otimes q_J)$ and it depends only on I and J. Thus, j is well-defined.

Define $k : \mathrm{Id}(A \otimes_{\min} B) \to \mathrm{Id}(A) \times \mathrm{Id}(B)$ by $k(\ker \pi) = (\ker(\bar{\pi} \circ i_A), \ker(\bar{\pi} \circ i_B))$, where i_A and i_B are the canonical embeddings of A and B into the multiplier algebra $M(A \otimes_{\min} B)$. Since $k \circ j = \mathrm{id}$ on $\mathrm{Id}(A) \times \mathrm{Id}(B)$, j is injective. Furthermore, the map $\ker \pi \mapsto \ker(\bar{\pi} \circ i_A)$, is just Res_{i_A}, and so is continuous, and thus k is continuous. It follows that $j : \mathrm{Id}(A) \times \mathrm{Id}(B) \to \mathrm{Id}(A \otimes_{\min} B)$ is open onto its range.

The range of j is dense in $\mathrm{Id}(A \otimes_{\min} B)$ because every nonempty subbasic open set contains the zero ideal, and the zero ideal is in the range of j [**9**, Theorem 6.5.1].

Define $\Phi : S(A) \times S(B) \to S(A \otimes_{\min} B)$ by $\Phi(f, g) = f \otimes g$ [**9**, Theorem 6.4.6]. This map is continuous for the weak* topologies and restricts to a map $P(A) \times P(B) \to P(A \otimes \min_B)$ [**13**, p. 256]. By [**3**, p. 44] we have continuous, open surjections θ_A, θ_B and $\theta_{A \otimes B}$, diagram 4 commutes and hence $j : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \mathrm{Prim}(A \otimes_{\min} B)$ is continuous. The restriction of an injective open map is open, so $j : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \mathrm{Prim}(A \otimes_{\min} B)$ is open onto its range. By Lemma 1.4, the map $\tilde{j} : \mathrm{Id}(A) \times \mathrm{Id}(B) \to \mathrm{Id}(A \otimes_{\min} B)$ is continuous. We need to show that

$$\ker(\sigma \otimes \tau) = \cap\{\ker(\pi \otimes \eta) : \ker \pi \in \mathrm{Prim}\, A, \ker \eta \in \mathrm{Prim}\, B, \\ \ker \pi \supseteq \ker \sigma, \ker \eta \supseteq \ker \tau\}.$$

This will follow from
(5)

$$\|x\|_{A \otimes_{\min} B} = \sup\{\|\pi \otimes \eta(x)\| : \ker \pi \in \mathrm{Prim}(A), \ker \eta \in \mathrm{Prim}(B)\},$$

[**6**, Theorem 2] for the following reasons. There is a bijection from $\mathrm{Prim}(A/I)$ onto the subset of $\mathrm{Prim}(A)$ for primitive ideals containing I [**3**, 2.11.5]. Thus, every irreducible representation $\tilde{\pi}$ of $A/\ker \sigma$ is of the form $\pi \circ q_\sigma$ where $q_\sigma : A \to A/\ker \sigma$ is the quotient map. Similarly for $A/\ker \tau$. Thus,

$$\begin{aligned} \|\sigma \otimes \tau(x)\| &= \|q_\sigma \otimes q_\tau(x)\|_{(A/\ker \sigma) \otimes \min(B/\ker \tau)} \\ &= \sup\{\|\tilde{\pi} \otimes \eta(q_\sigma \otimes q_\tau(x)\| : \ker \tilde{\pi} \in \mathrm{Prim}(A/\ker \sigma), \\ &\qquad \ker \tilde{\eta} \in \mathrm{Prim}(B/\ker \tau)\} \\ &= \sup\{\|\pi \otimes \eta(x)\| : \ker \pi \supseteq \ker \sigma, \ker \eta \supseteq \ker \tau)\}. \end{aligned}$$

[**9**, Theorem 6.4.2] says that $\|x\|_{A\otimes_{\min}B} = \sup\{\|\pi_f \otimes \pi_g(x)\| : f \in S(A),\, g \in S(B)\}$. The argument given there is based on the fact that for any C^*-algebra, there exists a faithful representation which is a direct sum of representations obtained from the GNS construction. But the direct sum of irreducible representations obtained from the GNS construction (that is all of them) is also faithful. So Equation (5) follows by the same argument. And it follows from Equation (5) that the range of $j|_{\mathrm{Prim}\,(A)\times\mathrm{Prim}\,(B)}$ is dense in $\mathrm{Prim}\,(A \otimes_{\min} B)$.

Let $\sigma : A \to B(H_\sigma)$ and $\eta : A \to B(H_\eta)$ be representations, and let q be the canonical quotient map $A\otimes_\gamma B \to A\otimes_{\min} B$. The representation $\sigma\otimes_\gamma \eta$ of $A\otimes_\gamma B$ is defined by $\sigma\otimes_\gamma\eta := (\sigma\otimes\eta)\circ q$ so that $j_\gamma = \mathrm{Res}_q \circ j$. Thus j_γ is injective and continuous (Lemma 1.5). Since Res_q is open and injective, it restricts to an open map on the range of j, and thus j_γ is open onto its range. $\square$

A C^*-algebra B is *nuclear* if $A \otimes_{\max} B = A \otimes_{\min} B$ for every C^*-algebra A [**9**, p. 193]. Proposition 2.2 is an analogue of the crossed product results [**11**, Section 3].

Proposition 2.2. *Let B be a C^*-algebra and $i_A : A \to M(A\otimes_{\max} B)$ the canonical embedding for any C^*-algebra A. Define*

$$\mathrm{Res}_A : \mathrm{Id}\,(A\otimes_{\max} B) \to \mathrm{Id}\,(A) \quad \textit{by} \quad \mathrm{Res}_A(\ker\pi) = \ker(\bar\pi\circ i_A),$$
$$\mathrm{Ext}_A : \mathrm{Id}\,(A) \to \mathrm{Id}\,(A\otimes_{\max} B) \quad \textit{by} \quad \mathrm{Ext}_A(I) = I\otimes_{\max} B,$$

and

$$\mathrm{Ind}_A : \mathrm{Id}\,(A) \to \mathrm{Id}\,(A\otimes_{\max} B) \quad \textit{by} \quad \mathrm{Ind}_A(\ker\pi) = \ker(\pi\otimes\mathrm{id}),$$

where id *is the faithful universal representation of B. Then*

(i) $\mathrm{Res}_A\mathrm{Ind}_A(I) = I = \mathrm{Res}_A\mathrm{Ext}_A(I)$; $\mathrm{Ext}_A(I) \subseteq \mathrm{Ind}_A(I)$;

(ii) Ind_A *is a homeomorphism onto its range*;

(iii) *$Ext_A = \mathrm{Ind}_A$ for all A, if and only if B is nuclear*; *and*

(iv) *if B is nuclear, then Ext_A is continuous for all A.*

Proof. By definition $\mathrm{Res}_A = \mathrm{Res}_{i_A}$, Equation (1). The ideal $I\otimes_{\max} B$ is identified with the ideal in $A\otimes_{\max} B$ generated by I [**13**, Proposition B.30], thus $\mathrm{Ext}_A = \mathrm{Ext}_{i_A}$.

(ii) To see that Ind_A is well defined, injective and continuous on ideals, note that Ind_A is obtained by composing $j_{\max}$ (Proposition 2.1) with the map $p : \mathrm{Id}\,(A) \to \mathrm{Id}\,(A) \times \mathrm{Id}\,(B)$ such that $p(I) = (I, 0)$, which is a homeomorphism onto its range. Since $j_{\max}$ is injective, it restricts to an open map on the range of p, and thus Ind_A is open onto its range.

(i) It follows from the formulas for Ind_A and Res_A that $\mathrm{Res}_A \mathrm{Ind}_A = \mathrm{id}$. Thus Res_A is onto and hence $\mathrm{Res}_A \mathrm{Ext}\,_A(I) = I$ [**10**, Proof of Lemma 1.1]. The inclusion $\mathrm{Ext}\,_A(I) \subseteq \mathrm{Ind}_A(I)$ is clear by looking at elementary tensors.

(iii) Let π be a representation with $\ker \pi = I$ and $\tilde{\pi} : (A/I) \to B(H)$ the faithful representation associated to π. The following diagram commutes because it does so on elementary tensors.

$$\begin{tikzcd} A \otimes_{\max} B \arrow[r, "\phi"] \arrow[d, "\pi\otimes\mathrm{id}"'] & (A/I) \otimes_{\max} B \arrow[dl, "\tilde{\pi}\otimes\mathrm{id}"] \\ B(H \otimes H_B). & \end{tikzcd}$$

The sequence $0 \to I \otimes_{\max} B \to A \otimes_{\max} B \xrightarrow{\phi} (A/I) \otimes_{\max} B \to 0$ is short exact since the tensor product is maximal, so $\ker \phi = \mathrm{Ext}\,_A(I)$. If B is nuclear, then $(A/I) \otimes_{\max} B \cong (A/I) \otimes_{\min} B$. Hence, $\tilde{\pi} \otimes \mathrm{id}$ is faithful and $\mathrm{Ext}\,_A(I) = \mathrm{Ind}_A(I)$.

Conversely, suppose $\mathrm{Ext}\,_A = \mathrm{Ind}_A$ for all C^*-algebras A. In particular, then, $\mathrm{Ext}\,_A(0) = \mathrm{Ind}_A(0)$, so $\pi \otimes \mathrm{id}$ is faithful for all faithful π representations of A.

$$\begin{tikzcd} A \otimes_{\max} B \arrow[r] \arrow[d, "\pi\otimes\mathrm{id}"'] & A \otimes_{\min} B \arrow[dl] \\ B(H \otimes H_B). & \end{tikzcd}$$

Since $\pi \otimes \mathrm{id}$ factors through $A \otimes_{\min} B$, $A \otimes_{\max} B \cong A \otimes_{\min} B$ for all A, and B is nuclear.

(iv) It follows immediately from parts (ii) and (iii) that if B is nuclear then $\mathrm{Ext}\,_A$ is a homeomorphism onto its range for all A. $\square$

A C^*-algebra B is *exact* if the following sequence is short exact for all C^*-algebras A and ideals I of A [**13**, B.33]: $0 \to I \otimes_{\min} B \to A \otimes_{\min} B \to (A/I) \otimes_{\min} B \to 0$.

Proposition 2.3. *Let B be a C^*-algebra and $i_A : A \to M(A \otimes_{\min} B)$ the canonical embedding for any C^*-algebra A. Define*

$$\begin{aligned} &\mathrm{Res}_A : \mathrm{Id}\,(A \otimes_{\min} B) \longrightarrow \mathrm{Id}\,(A) \quad \textit{by} \quad \mathrm{Res}_A(\ker \pi) = \ker(\bar{\pi} \circ i_A), \\ &\mathrm{Ext}_A : \mathrm{Id}\,(A) \longrightarrow \mathrm{Id}\,(A \otimes_{\min} B) \quad \textit{by} \quad \mathrm{Ext}_A(I) = I \otimes_{\min} B, \end{aligned}$$

and

$$\mathrm{Ind}_A : \mathrm{Id}\,(A) \longrightarrow \mathrm{Id}\,(A \otimes_{\min} B) \quad \textit{by} \quad \mathrm{Ind}_A(\ker \pi) = \ker(\pi \otimes \mathrm{id}),$$

where id *is the faithful universal representation of B. Then*

(i) $\mathrm{Res}_A \mathrm{Ind}_A(I) = I = \mathrm{Res}_A \mathrm{Ext}_A(I)$; $\mathrm{Ext}_A(I) \subseteq \mathrm{Ind}_A(I)$;

(ii) Ind_A *is a homeomorphism onto its range*;

(iii) $\mathrm{Ext}_A = \mathrm{Ind}_A$ *for all A if and only if B is exact*; *and*

(iv) *if B is exact, then* Ext_A *is continuous for all A.*

Proof. The proofs for parts (i), (ii) and (iv) work the same as in Proposition 2.2.

(iii) Let π be a representation with $\ker \pi = I$ and $\tilde{\pi} : (A/I) \to B(H)$ the faithful representation associated to π. We have the following commutative diagram.

$$\begin{tikzcd} A \otimes_{\min} B \arrow[r, "\phi"] \arrow[d, "\pi\otimes\mathrm{id}"'] & (A/I) \otimes_{\min} B \arrow[dl, "\tilde{\pi}\otimes\mathrm{id}"] \\ B(H \otimes H_B). & \end{tikzcd}$$

Since the tensor product is minimal, $\tilde{\pi} \otimes \mathrm{id}$ is faithful and so $\ker \phi = \mathrm{Ind}_A(I)$. The sequence $0 \to I \otimes_{\min} B \to A \otimes_{\min} B \xrightarrow{\phi} (A/I) \otimes_{\min} B \to 0$, is short exact for all C^*-algebras A and ideals I in A if and only if B is exact. Since $\ker \phi = \mathrm{Ind}_A(I)$, this happens if and only if $\mathrm{Ext}_A(I) = \mathrm{Ind}_A(I)$. $\square$

3. C^***-bundles.** A C^*-algebra A is a $C_0(X)$-*algebra* if there exists a nondegenerate injection ι of $C_0(X)$ into $ZM(A)$, the center of the multiplier algebra of A [**10**, Section 2]. An upper semi-continuous (continuous) C^*-bundle $\mathcal{A}$ can be thought of as a family of C^*-algebras A_x parametrized by a locally compact Hausdorff space X, together with an algebra $\Gamma_0(\mathcal{A})$ of upper semi-continuous (continuous) sections (maps f on X such that $f(x) \in A_x$ for all $x \in X$, with pointwise operations) [**10**, Section 1].

Let A be a $C_0(X)$-algebra with $i : C_0(X) \to ZM(A)$, and let B be a C^*-algebra with canonical embedding $i_A : A \to M(A \otimes B)$ (for both minimal and maximal tensor products). Then we can define $\iota : C_0(X) \to ZM(A \otimes B)$ by $\iota = \overline{i}_A \circ i$, so that $A \otimes B$ is again a $C_0(X)$-algebra.

The section algebra $\Gamma_0(\mathcal{A})$ of an upper semi-continuous C^*-bundle $\mathcal{A}$ over X is a $C_0(X)$-algebra because there is an embedding $i : C_0(X) \to ZM(\Gamma_0(\mathcal{A}))$ given by $i(g)f = gf$. Conversely, in [**10**, Theorem 2.3], we showed that every $C_0(X)$-algebra A gives rise to an upper semi-continuous C^*-bundle $\mathcal{C}$ over X. Specifically, there is an upper semi-continuous C^*-bundle $\mathcal{C}$ over X with section algebra $\Gamma_0(\mathcal{C}) \cong A$ and fibers $C_x \cong A/\operatorname{Ext}_i(I_x)$ where $I_x := \{g \in C_0(X) : g(x) = 0\}$.

This theorem can be used to show that if $\mathcal{A}$ is an upper semi-continuous C^*-bundle over X and B is a C^*-algebra, then there exists an upper semi-continuous C^*-bundle $\mathcal{C}$ over X such that $C_x \cong A_x \otimes_{\max} B$ and $\Gamma_0(\mathcal{C}) \cong \Gamma_0(\mathcal{A}) \otimes_{\max} B$. This is because $\operatorname{Ext}_i(I_x) = \{f \in \Gamma_0(\mathcal{A}) : f(x) = 0\}$ [**10**, Section 3], and

$$\begin{aligned} C_x &\cong (\Gamma_0(\mathcal{A}) \otimes_{\max} B)/(\operatorname{Ext}_i(I_x) \otimes_{\max} B) \\ &\cong (\Gamma_0(\mathcal{A})/\operatorname{Ext}_i(I_x)) \otimes_{\max} B \cong A_x \otimes_{\max} B. \end{aligned}$$

The more difficult step is to show that if $\mathcal{A}$ is a continuous C^*-bundle, then $\mathcal{C}$ is a continuous C^*-bundle. This requires the assumption that B is a nuclear C^*-algebra.

Lemma 3.1. *Let D be a $C_0(X)$-algebra. Then* Res_ι *restricts to a continuous map from* $\operatorname{Prim} D$ *into X with dense range such that*

$$P \supseteq \operatorname{Ext}_\iota(I_x) \Longleftrightarrow \operatorname{Res}_\iota(P) = x, \tag{6}$$

and $\mathrm{Res}_\iota : \mathrm{Prim}\, D \to X$ *is open if and only if* $\mathrm{Ext}_\iota : X \to Id(D)$ *is continuous.*

Proof. We showed in [**10**, Proposition 2.1] that Res_ι restricts to a continuous map from $\mathrm{Prim}\, D$ into X with dense range satisfying Equation (6). To verify the last claim, we will use the following:

$$x \in \mathrm{Res}_\iota(U_K) \iff \mathrm{Ext}_\iota(I_x) \in O_K. \tag{7}$$

Suppose that $x \in \mathrm{Res}_\iota(U_K)$. That means there exists a primitive ideal Q such that $Q \not\supseteq K$ and $\mathrm{Res}_\iota(Q) = x$. In other words, there exists a Q such that $Q \not\supseteq K$ and $Q \supseteq \mathrm{Ext}_\iota(I_x)$. That is, $\mathrm{Ext}_\iota(I_x) \not\supseteq K$, which means that $\mathrm{Ext}_\iota(I_x) \in O_K$.

Equation (7) says that $\mathrm{Res}\,(U_K) = \mathrm{Ext}^{-1}(O_K) \cap X$, which suffices because every open set in $\mathrm{Prim}\, D$ is of the form U_K. $\square$

Remark. Because $\mathrm{Res}\,\mathrm{Ext}\,\mathrm{Res} = \mathrm{Res}$, Ext_ι is injective on the range on Res_ι. This also shows that $\mathrm{Ext}_\iota : \mathrm{Res}_\iota(\mathrm{Prim}\, D) \to \mathrm{Id}\,(D)$ is open onto its range, because Res_ι is its continuous inverse. These facts, together with Lemma 3.1 and [**10**, Theorem 2.3] extend part of [**1**, Theorem 3.1]. Let $\mathcal{A}$ be an upper semi-continuous C^*-bundle $\mathcal{A}$ over X, and let X' be the subset of X for which $\mathrm{Ext}_i(I_x)$ is proper. Then $\mathcal{A}$ is continuous if and only if $\mathrm{Ext}_i|_{X'}$ is a homeomorphism onto its range.

Suppose $\mathcal{A}$ is a continuous C^*-bundle and B is nuclear. Since $\mathcal{A}$ is continuous, $\mathrm{Res}_i : \mathrm{Prim}\,(\Gamma_0(\mathcal{A})) \to X$ is open [**10**, Theorem 2.3], and thus $\mathrm{Ext}_i : X \to \mathrm{Id}\,(\Gamma_0(\mathcal{A}))$ is continuous, Lemma 3.1. The nuclearity of B implies that $\mathrm{Ext}_A : \mathrm{Id}\,(\Gamma_0(\mathcal{A})) \to \mathrm{Id}\,(\Gamma_0(\mathcal{A}) \otimes_{\max} B)$ is continuous, Proposition 2.2. Since $\mathrm{Ext}_\iota = \mathrm{Ext}_A \circ \mathrm{Ext}_i$, $\mathrm{Ext}_\iota : X \to \mathrm{Id}\,(\Gamma_0(\mathcal{A}) \otimes_{\max} B)$ is continuous. By Lemma 3.1 and [**10**, Theorem 2.3] again, $\mathrm{Res}_\iota : \mathrm{Prim}\,(\Gamma_0(\mathcal{A}) \otimes_{\max} B) \to X$ is open and $\mathcal{C}$ is a continuous C^*-bundle.

Kirchberg and Wassermann show that nuclearity is actually a necessary and sufficient condition. More precisely, they construct a continuous C^*-bundle $\mathcal{A}$ such that if B is not a nuclear C^*-algebra, then $\mathcal{A} \otimes_{\max} B$ is not a continuous C^*-bundle [**8**, p. 684].

In the case of minimal tensor products, there exists an upper semi-continuous C^*-bundle $\mathcal{C}$ over X such that $C_x \cong (\Gamma_0(\mathcal{A}) \otimes_{\min}$

$B)/(\mathrm{Ext}\,(I_x) \otimes_{\min} B)$ and $\Gamma_0(\mathcal{C}) \cong \Gamma_0(\mathcal{A}) \otimes_{\min} B$. If B is exact, then $\mathcal{C}$ is continuous and

$$C_x \cong (\Gamma_0(\mathcal{A})/\mathrm{Ext}\,(I_x)) \otimes_{\min} B \cong A_x \otimes_{\min} B.$$

Kirchberg and Wassermann's work does not show that exactness is a necessary and sufficient condition in this case because we are considering different bundles: they work with the lower semi-continuous C^*-bundle whose fibers are $A_x \otimes_{\min} B$ [**8**].

We have studied analogous results for full crossed products by action and coaction [**10**, Theorem 5.1, Corollary 5.3], [**12**, Theorem 4.3], but the argument given in the proof of [**10**, Corollary 5.3] is inaccurate. However, the proof given above for tensor products works in exactly the same manner for crossed products. There is just one point which should be made clear.

In the tensor product case, with the appropriate assumptions, $\mathrm{Ext}_A = \mathrm{Ind}_A$ on all of $\mathrm{Id}\,(A)$. However, for full crossed products, even when G is amenable, the set on which $\mathrm{Ext}_A = \mathrm{Ind}_A$ is *not* all of $\mathrm{Id}\,(A)$, so that Ext_A is not necessarily continuous on all of $\mathrm{Id}\,(A)$. So what we have to check is that the subset of $\mathrm{Id}\,(A)$ on which $\mathrm{Ext}_A = \mathrm{Ind}_A$ contains the image of Ext_i, so that the composition $\mathrm{Ext}_i = \mathrm{Ext}_A \circ \mathrm{Ext}_i$ is continuous.

For the action case, $\mathrm{Ext}_A = \mathrm{Ind}_A$ on the α-invariant ideals, when G is amenable [**10**, Lemma 5.2(iii)]. Given a C^*-bundle $\mathcal{A}$ over X, to construct a bundle with section algebra isomorphic to $\Gamma_0(\mathcal{A}) \times_\alpha G$, the action G must leave the embedding of $C_0(X)$ into $\Gamma_0(\mathcal{A})$ fixed, and it follows easily from this condition that the ideal $\mathrm{Ext}_i(I_x)$ is α-invariant [**10**, Proof of Theorem 5.1] (cf. [**14**, Definition 3.1]).

For the coaction case [**12**, Theorem 4.3], we have to be even more careful because invariant ideals come in more than one flavor. There are three sets of ideals in $\mathrm{Id}\,(A)$ worthy of attention. First the δ-invariant ideals are exactly the image of Res_A, that is, those ideals I for whom there exists a covariant representation (π, μ) such that $I = \ker \pi$ [**11**, Sections 1, 3]. Second, the nondegenerately δ-invariant ideals I satisfy a condition which ensure that the coaction on A restricts to a coaction on I [**12**, Section 2]. Thirdly, there are those ideals I for which there is a well-defined coaction on the quotient A/I. Fortunately, this third set contains the first two, and it is the third set on which

$\operatorname{Ext}_A = \operatorname{Ind}_A$. We verified in [**12**, Proposition 3.1] that the ideals $\operatorname{Ext}_i(I_x)$ are nondegenerately δ-invariant, and so the image of Ext_i is contained in the third set, and $\operatorname{Ext}_\iota = \operatorname{Ext}_A \circ \operatorname{Ext}_i$ is continuous.

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