ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 4, Winter 2000

# MODULES OVER DOMAINS LARGE IN A COMPLETE DISCRETE VALUATION RING

W. MAY AND P. ZANARDO

ABSTRACT. We consider a class of domains $R$ containing a maximal ideal $\mathfrak{N}$ such that $R$ is not complete with respect to the $\mathfrak{N}$-adic topology, but $T = R_{\mathfrak{N}}$ is a complete DVR. Such domains are called $T$-large because of the way to construct them. We characterize a $T$-large domain $R$ to be of the form $R = T \cap V$, where $V$ is a mildly restricted valuation domain of $Q$, the field of fractions of $T$. We show that the completion $\hat{V}$ of $V$ has infinite rank as a $V$-module. We investigate finite rank torsion-free modules $M$ over a $T$-large domain $R$ which are Hausdorff in the $\mathfrak{N}$-adic topology. Making use of known results on $V$-modules, we obtain the following results: there exist indecomposable torsion-free Hausdorff $R$-modules of any fixed rank $n$; every cotorsion-free Hausdorff $R$-algebra of rank $n$ is the endomorphism algebra of a torsion-free module of rank $3n$; the Krull-Schmidt theorem fails, that is, there exist finite rank torsion-free Hausdorff $R$-modules which admit non-isomorphic decompositions into indecomposable summands.

**Introduction.** In his 1962 book [**6**], Nagata exhibited the first example of a noncomplete discrete valuation ring $R$ such that $[\hat{Q} : Q] < \infty$, where $Q, \hat{Q}$ are the field of fractions of $R$ and its completion $\hat{R}$, respectively. The DVR's satisfying this property were called Nagata valuation domains in [**9**].

Recently the second author [**9**] and Arnold and Dugas [**1**] investigated torsion-free modules of finite rank over Nagata valuation domains $R$. In particular, in [**9**] it was proved that if $[\hat{Q} : Q] = 2$, then every finite rank torsion-free indecomposable $R$-module has rank $\leq 2$; in [**1**] it is shown that $[\hat{Q} : Q] = 3$ implies that every finite rank torsion-free indecomposable $R$-module has rank $\leq 3$, while if $[\hat{Q} : Q] \geq 4$, then there exist finite rank torsion-free indecomposable $R$-modules of arbitrarily large rank. It is worth noting that the Krull-Schmidt theorem holds for finite rank torsion-free modules over Nagata valuation domains since

Received by the editors on December 18, 1996, and in revised form on October 18, 1999.
1991 AMS *Mathematics Subject Classification.* 13C05, 13B35.

they all are henselian rings, and therefore we can apply Lemma 14 of [**8**].

In fact Vámos in [**8**] had previously studied nonhenselian valuation domains $R$ such that $[\tilde{Q} : Q] < \infty$, where $\tilde{Q}$ denotes the field of fractions of a maximal immediate extension $\tilde{R}$ of $R$. Inter alia, he proved that every finite rank torsion-free indecomposable $R$-module has rank $\leq 2$ ([**8**, Theorem 10]), and that the Krull-Schmidt theorem holds for finite rank torsion-free $R$-modules. Further investigations on finite rank torsion-free modules over nonhenselian valuation domains, especially concerning the failure of the Krull-Schmidt theorem, have been made by Goldsmith and the first author in [**3**] and by the authors in [**5**].

In the present paper we consider the limit situation of a certain noncomplete domain $R$, whose completion $\hat{R}$ is a DVR, and $R, \hat{R}$ have the same field of fractions, i.e. $[\hat{Q} : Q] = 1$. Note that $R$ cannot be a valuation domain in this case.

Our starting point is the following result by the second author and Zannier ([**10**, Theorem 7]): let $T$ be a domain, complete with respect to the $\mathfrak{M}$-adic topology where $\mathfrak{M}$ is a maximal ideal of $T$; then there exists a subring $R$ of $T$ satisfying the following: 1) $\mathfrak{N} = \mathfrak{M} \cap R$ is a maximal ideal of $R$, 2) $R$ is not local, 3) $T = R_{\mathfrak{N}}$, 4) $T$ is the completion of $R$ in its $\mathfrak{N}$-adic topology.

In the first section we recall the way to construct such an $R$ for any given $T$ (cf. [**10**]); because of this construction we shall say that $R$ is a *$T$-large domain*. We are interested in the case where $T$ is a complete DVR; in this case we characterize a $T$-large domain $R$ to be of the form $R = T \cap V$ where $V$ is a mildly restricted valuation domain of $Q$, the field of fractions of $T$ (Proposition 2). Also, a $T$-large domain $R$ has exactly two maximal ideals, namely $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$ and $\mathfrak{N} = \mathfrak{M} \cap R$ where $\overline{\mathfrak{P}}$ and $\mathfrak{M}$ are the maximal ideals of $V$ and $T$, respectively; nevertheless, it can have any admissible Krull dimension (Proposition 3). We also show that the valuation domain $V$ cannot be complete in the topology of the valuation; in fact if $\hat{V}$ denotes the completion of $V$, then $\operatorname{rank}_V \hat{V} = \infty$; as a consequence we also have that $\operatorname{rank}_V \tilde{V} = \infty$ where $\tilde{V}$ is any maximal immediate extension of $V$ (Proposition 4).

The second section is devoted to the study of finite rank torsion-free modules $M$ over a $T$-large domain $R = T \cap V$. We confine ourselves to

$R$-modules $M$ which are Hausdorff in the $\mathfrak{N}$-adic topology and carry over results on $V$-modules to these modules. In Proposition 7 we show that there exist indecomposable torsion-free $R$-modules of any fixed rank $n$ (which is different from the above case of those Nagata valuation domains such that $[\hat{Q} : Q] = 2$ or 3; see [**9**] and [**1**]).

In Theorem 8 we make use of Theorem 1 of [**3**] to prove a "Corner type" result: every cotorsion-free Hausdorff $R$-algebra $A$ of finite rank $n$ is the $R$-endomorphism algebra of a torsion-free Hausdorff $R$-module $M$ of rank $3n$. Finally in Theorem 10 we adapt the arguments developed in Theorem 2 of [**3**], showing that the Krull-Schmidt theorem fails for finite rank torsion-free $R$-modules: more precisely, for every $n > 0$ there exists a Hausdorff $R$-module $N$ of rank $> n$ which admits two non-isomorphic direct decompositions into indecomposable summands. This result provides another dramatic difference from the cases of both Nagata valuation domains ([**9**] and [**1**]) and the nonhenselian valuation domains studied by Vámos in [**8**].

**Section 1.** For general facts about valuation domains and their modules, we refer to the books by Schilling [**7**] and by Fuchs and Salce [**2**].

We recall the construction and the properties of the domains $R$ in which we are interested, as described in the proof of Theorem 7 in [**10**].

Let $T$ be a local domain, not a field, with maximal ideal $\mathfrak{M}$. We assume that $T$ is Hausdorff and complete in the $\mathfrak{M}$-adic topology. We choose an $x \in T \setminus \mathfrak{M}$ in such a way that $x \equiv 1 \bmod \mathfrak{M}$ and the family of subrings $\mathcal{F} = \{B \subset T : x \in B \text{ and } 1/x \notin B\}$ is nonempty. It is worth recalling a possible choice of $x$. If $\chi(T) = 0$ and $\chi(T/\mathfrak{M}) = p > 0$ (the eterocharacteristic case), we let $x$ be any prime number such that $x \equiv 1 \bmod (p)$; then $x \in \mathbf{Z}$, $1/x \notin \mathbf{Z}$, so that $\mathbf{Z} \in \mathcal{F}$. If $\chi(T) = \chi(T/\mathfrak{M})$ (the equicharacteristic case), then $T$ contains a field $L$ and any nonzero $z \in \mathfrak{M}$ is transcendental over $L$; in this case we set $x = 1 + z$; then $L[x] \in \mathcal{F}$, since $1/x \notin L[x]$, $x$ being transcendental over $L$. Let now $R \subset T$ be a maximal element of $\mathcal{F}$. Note that $R$ is a proper subring of $T$ since $1/x \in T$, $T$ being local. Let $\mathfrak{P}$ be a maximal ideal of $R$ containing the nonunit $x$ of $R$; let $\mathfrak{N} = \mathfrak{M} \cap R$. Then $R$ satisfies the following properties (see [**10**]):

A. $R$ is integrally closed in $T$.

B. If $z \in T \setminus \mathfrak{M}$, then either $z \in R$ or $1/z \in R$.

C. $\mathfrak{P}$ and $\mathfrak{N}$ are distinct and are the only maximal ideals of $R$; in particular, any $r \in R \setminus (\mathfrak{P} \cup \mathfrak{M})$ is a unit of $R$.

D. $\pi_{\mathfrak{M}}(R) = T/\mathfrak{M}$ where $\pi_{\mathfrak{M}} : T \to T/\mathfrak{M}$ is the canonical map.

E. $T = R_{\mathfrak{N}}$.

F. For all $n \in \mathfrak{N}$, we have $\mathfrak{N}^n = R \cap (\mathfrak{M}^n)$, that is, the $\mathfrak{N}$-adic topology of $R$ coincides with the topology induced on $R$ by the $\mathfrak{M}$-adic topology of $T$; $R$ is a dense subset of $T$, with respect to the $\mathfrak{M}$-adic topology; $T$ is the completion of $R$ in its $\mathfrak{N}$-adic topology.

A subring $R$ of $T$ constructed in the way described above will be called $T$-*large*.

*Remark.* It is important to observe that the apparently technical hypothesis $x \equiv 1 \bmod \mathfrak{M}$ cannot be weakened in the construction of the $T$-large domain $R$. Actually, it is not enough to choose $x \notin \mathfrak{M}$ and take a maximal element of the family $\mathcal{F} = \{S \subset T : x \in S \text{ and } 1/x \notin S\}$ to get a $T$-large domain. Let us in fact consider the following example: $T = \mathbf{Q}[[t]]$ (formal power series in the indeterminate $t$) and $R = \mathbf{Z}_p + t\mathbf{Q}[[t]]$ (formal power series with constant term in $\mathbf{Z}_p$, the integers localized at the prime $p$). It is known that $R$ is a valuation domain, hence certainly not a $T$-large domain; on the other hand, setting $x = p$, $R$ is a maximal element of the family $\mathcal{G} = \{S \subset T : x \in S, 1/x \notin S\}$. In fact if $f \in T \setminus R$, then the constant term of $f$ has to lie in $\mathbf{Q} \setminus \mathbf{Z}_p$ from which it readily follows that $1/p \in R[f]$. But $R[1/p] = T$; thus we get that $R$ is a maximal subring of $T$, whence obviously maximal in $\mathcal{G}$. Note that here $x = p \notin t\mathbf{Q}[[t]] = \mathfrak{M}$, but $p \not\equiv 1 \bmod \mathfrak{M}$.

From now on we shall consider $T$ to be a valuation domain, complete in the $\mathfrak{M}$-adic topology. This implies that $T$ is a DVR, so that $\mathfrak{M}$ is principal; we shall denote by $\pi$ a fixed generator of $\mathfrak{M}$; $R$, $\mathfrak{P}$, $\mathfrak{N}$, $x$ will maintain the same meaning as above. Finally, we shall denote by $Q$ the common field of fractions of $T$ and $R$.

We want to give a description of the $T$-large domain $R$ in terms of intersection of two valuation domains, which is more useful for our purposes.

Let us first show some further properties of $R$ which may be used in the sequel.

G. $\pi$ can be chosen in $R$, more precisely in $\mathfrak{N}$ not in $\mathfrak{P}$; for such a choice we have $\mathfrak{N} = \pi R$.

*Proof.* Since $T = R_{\mathfrak{N}}$ we can write $\pi$ in the form $\pi = \sigma/t$, where $\sigma, t \in R$ and $t \notin \mathfrak{N} = R \cap \mathfrak{M}$. If now $\sigma \notin \mathfrak{P}$, we replace $\pi$ with $\sigma$. If $\sigma \in \mathfrak{P}$, let us pick an element $\alpha \in R$ such that $\alpha \in \mathfrak{N} \setminus \mathfrak{P}$; $\alpha$ does exist in view of (C). If $\alpha$ is a generator of $\pi T$ we replace $\pi$ with $\alpha$. Otherwise, let $\beta = \alpha + \sigma$; then $\pi \in \sigma T$ and $\pi \notin \alpha T$ imply $\pi \in \beta T$; moreover, $\alpha \notin \mathfrak{P}$ and $\sigma \in \mathfrak{P}$ imply $\beta \notin \mathfrak{P}$. Thus $\pi T = \beta T$ and $\beta \in \mathfrak{N} \setminus \mathfrak{P}$ which shows the first part of our statement. Let us now pick any $z \in \mathfrak{N} \subseteq \pi T$ where $\pi \in \mathfrak{N} \setminus \mathfrak{P}$. We have $z = \pi^k \lambda$, where $k > 0$ is a suitable integer and $\lambda \in T \setminus \pi T$. In view of (B) either $\lambda \in R$ or $1/\lambda \in R$. In the first case we have $z \in \pi R$, whence $\mathfrak{N} \subseteq \pi R$ as desired. If on the contrary $\lambda \notin R$, then by (C), $1/\lambda \in \mathfrak{P}$ since it is not a unit of $R$ and $1/\lambda \notin \mathfrak{N} \subseteq \pi T$. We then get $\pi^k = z/\lambda \in \mathfrak{P}$, whence $\pi \in \mathfrak{P}$, impossible. The desired conclusion follows.

H. $\mathfrak{P} = \mathrm{Rad}\,(x)$.

*Proof.* Let $b \in \mathfrak{P}$; then in view of (G), $b = \pi^h c$ where $c \in \mathfrak{P} \setminus \pi R$. Then $1/c \in T \setminus R$ so that $R \subsetneqq R[1/c] \subset T$. The maximality of $R$ implies that $1/x \in R[1/c]$ so that

$$1/x = a_0 + a_1 1/c + \cdots + a_r 1/c^r, \quad a_i \in R. \tag{1}$$

By (1) we readily get $c^r \in xR$ so that $b^r \in xR$ too. Since $b \in \mathfrak{P}$ was arbitrary, we conclude that $\mathfrak{P} \subseteq \mathrm{Rad}\,(x)$. The converse inclusion is trivial.

**Proposition 1.** *In the above notation, $V = R_{\mathfrak{P}}$ is a valuation domain and $R = V \cap T$.*

*Proof.* In view of (G) any element $z$ of $R \setminus \mathfrak{P}$ can be written in the form $z = \pi^k u$ where $k$ is a positive integer and $u$ is a unit of $R$. It follows that any element $w$ of $V = R_{\mathfrak{P}}$ can be written in the form $w = \pi^h \alpha$ where $h$ is an integer, possibly negative, and $\alpha \in R \setminus \pi R$. If $w \in \mathfrak{P}V$, then necessarily $\alpha \in \mathfrak{P}$. Let then $\pi^h\alpha, \pi^m\beta$ be arbitrary elements of $\mathfrak{P}V$ where $\alpha, \beta \in \mathfrak{P} \setminus \pi R$. Since $\alpha/\beta \in T \setminus \pi T$, by (B) we see that either $\alpha/\beta \in R$ or $\beta/\alpha \in R$; let us assume that $\alpha/\beta \in R$. Then $\pi^{h-m}(\alpha/\beta) \in R_{\mathfrak{P}} = V$. It follows that, given two elements of $\mathfrak{P}V$, necessarily one divides the other and this is enough to ensure that $V$ is a valuation domain. If now $z \in V \cap T$, we have $z = \pi^t \gamma$ with $t \in \mathbf{Z}$, $\gamma \in R \setminus \pi R$, since $z \in V$; since $z \in T$, too, we must have $t \geq 0$, whence $z \in R$. This shows that $R \supseteq V \cap T$ so that $R = V \cap T$ as desired.

From the above proposition we deduce further properties of $R$.

I. The ideal $\mathfrak{P}_1 = \cap_n x^n R$ is prime in $R$.

*Proof.* Since $V$ is a valuation domain, it is known that $\cap_n x^n V$ is a prime ideal of $V$. It is then enough to show that $\mathfrak{P}_1 = (\cap_n x^n V) \cap R$; we will prove that $x^n R = x^n V \cap R$. Let $x^n v = r$ where $v \in V$ and $r \in R$: we have to show that $v \in R$. In fact, if $v \in V \setminus R$, we can write $v = \alpha/\pi^h$ where $\alpha \in R \setminus \pi R$, $h > 0$; it follows that $x^n \alpha \in \pi R$, impossible, since $x, \alpha \notin \pi R$.

J. $R$ is a maximal subring of $T$ if and only if $\mathfrak{P}_1 = \cap_n x^n R = \{0\}$.

*Proof.* ($\Rightarrow$). Let us suppose that $\mathfrak{P}_1 \neq \{0\}$; we shall show that $R[1/x] \neq T$ so that $R$ is not a maximal subring. Let us choose $b \in \mathfrak{P}_1$, $b \neq 0$. We can write $b = \pi^h c$ with $c \notin \pi R$. Now $c \in \mathfrak{P}_1$ since $\pi \notin \mathfrak{P}_1 \subset \mathfrak{P}$ and $\mathfrak{P}_1$ is a prime ideal. Then $1/c \in T$ and we will verify that $1/c \notin R[1/x]$ from which our assertion will follow. In fact from $1/c \in R[1/x]$, it readily follows that $x^m \in cR$ for a suitable positive integer $m$, so that $x^m \in \mathfrak{P}_1 = \cap_n x^n R$, which is a plain contradiction.

($\Leftarrow$). By the properties of $R$, for every $v \in T \setminus R$ we have $1/x \in R[v]$; therefore, to show that $R$ is a maximal subring of $T$, it is enough to check that $R[1/x] = T$. Since $T$ is a local ring it is plainly sufficient to show that every unit $u$ of $T$ not in $R$ lies in $R[1/x]$. In view of (B) we must have $1/u \in \mathfrak{P} \setminus \mathfrak{N}$. By contradiction, let us assume that $u \notin R[1/x]$; in particular, for no integer $k$ we have $x^k \in (1/u)R$ which shows that $x \notin \mathrm{Rad}\,(1/u)$. Therefore $\mathfrak{P} \supsetneqq \mathrm{Rad}\,(1/u)$ and, on the other hand, $\mathrm{Rad}\,(1/u) \not\subseteq \mathfrak{N}$ since $1/u \notin \mathfrak{N}$. Thus a prime ideal of $R$ must exist, say $\mathfrak{J}$ containing $1/u$, properly contained in $\mathfrak{P}$ and not containing $x$. To conclude we show that necessarily $\mathfrak{J} \subseteq \cap_n x^n R = \mathfrak{P}_1$ so that $\mathfrak{P}_1 \neq \{0\}$, which is our required contradiction. In fact let $b \in \mathfrak{J}$; as usual let us write $b = \pi^h c$ with $c \notin \pi R$; $c \in \mathfrak{J}$ since $\pi \notin \mathfrak{P} \supseteq \mathfrak{J}$, and it suffices to show that $c \in \mathfrak{P}_1$. Assuming that $c \in x^k R$, let us verify that then $c \in x^{k+1}R$ too. We have $c = x^k d$ where $d \in \mathfrak{J}$ since $x \notin \mathfrak{J}$. Now both $x$ and $d$ lie in $\mathfrak{P} \setminus \mathfrak{N}$ so that as a consequence of (B), one divides the other in $R$; but $x \notin dR \subseteq \mathfrak{J}$ and so $d \in xR$, whence $c \in x^{k+1}R$. The desired conclusion follows.

We shall see in the following that it is possible that $\mathfrak{P}_1 = \{0\}$ so that $R$ can be a maximal subring of $T$. However, as we have seen in the above remark, not every maximal subring of a complete DVR $T$ is a $T$-large domain.

The following result completes the description of $T$-large domains in terms of intersections of valuation domains.

**Proposition 2.** *Let $V$ be a valuation domain of $Q$ with maximal ideal $\overline{\mathfrak{P}}$. Then $V \cap T = R$ is a $T$-large domain if and only if $V$ is not contained in $T$ and $\overline{\mathfrak{P}}$ is a radical ideal.*

*Proof.* ($\Rightarrow$). If $V$ is contained in $T$, then $V \cap T = V = R$ is local and so it cannot be a $T$-large domain. Let us now assume that $V \not\subseteq T$; of course then $T \not\subseteq V$ since $T$ is a DVR; therefore we can invoke Theorem 11.11 of [**6**, p. 38]. $R$ has exactly two distinct maximal ideals, $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$ and $\mathfrak{N} = \mathfrak{M} \cap R$, and we have $R_{\mathfrak{P}} = V$, $R_{\mathfrak{M}} = T$. If $\overline{\mathfrak{P}}$ is not a radical ideal, then $\overline{\mathfrak{P}}$ is the union of a strictly ascending chain of prime ideals of $V$; therefore, also $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$ is the union of a strictly ascending chain of prime ideals of $R$ since $V$ is a localization of $R$. Then $\mathfrak{P}$ cannot be a radical ideal in $R$, whence $R$ does not satisfy property (H) and so it is not a $T$-large domain.

($\Leftarrow$). Since $V \not\subseteq T$, again by Theorem 11.11 of [**6**], we have that $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$ and $\mathfrak{N} = \mathfrak{M} \cap R$ are the only maximal ideals of $R$ and they do not coincide. Let us now choose $y \in V$ such that $\mathrm{Rad}_V(y) = \overline{\mathfrak{P}}$. We may assume that $y \in R$; in fact if $y \notin T$, then $1/y \in \mathfrak{M} \setminus V$, whence $1+1/y$ is a unit of $T$ not in $V$, whence $(1+1/y)^{-1} = y/(1+y) \in V \cap T = R$. Moreover, $\mathrm{Rad}_V(y/(1+y)) = \mathrm{Rad}_V(y) = \overline{\mathfrak{P}}$ since $1+y$ is a unit of $V$. We may also assume that $y \in R \setminus \mathfrak{N}$. If $y = \pi^k y'$ with $y' \in R \setminus \mathfrak{N}$, we have $\mathrm{Rad}_V(y') = \mathrm{Rad}_V(y)$ since $\pi$ is a unit of $V$. Now if $y \in R \setminus \mathfrak{N}$ and $\mathrm{Rad}_V(y) = \overline{\mathfrak{P}}$, it is easily seen that $\mathrm{Rad}_R(y) = \overline{\mathfrak{P}} \cap R = \mathfrak{P}$. In fact for any $w \in \mathfrak{P}$ there exists a $k > 0$ such that $w^k \in yV$, whence $w^k/y \in V \cap T$ (since $y$ is a unit in $T$) and therefore $w^k \in yR$ which shows that $\mathfrak{P} \subseteq \mathrm{Rad}_R(y)$. Finally, let us pick $z \in \mathfrak{N} \setminus \mathfrak{P}$; then $y+z$ is a unit in $R$ since $y+z \notin \mathfrak{N} \cup \mathfrak{P}$. Let us set $x = y/(y+z) \in R$. Then $\mathrm{Rad}_R(x) = \mathrm{Rad}_R(y) = \mathfrak{P}$ and $x \equiv 1 \bmod \mathfrak{N}$. To conclude that $R$ is a $T$-large domain, it suffices to show that $R$ is maximal with respect to the property of not containing $x$. Let $u \in T \setminus R$; possibly substituting $u$ by $1+u$, we may assume that $u$ is a unit of $T$. We must show that $1/x \in R[u]$. Since $u \in T \setminus \mathfrak{M}$ and $u \notin R$, then $u \notin V$ so that $1/u \in \overline{\mathfrak{P}}$. It follows that $1/u \in \mathfrak{P} = \mathrm{Rad}_R(x)$, whence $(1/u)^k \in xR$ for a suitable $k$, and so $1/x \in u^k R \subseteq R[u]$ as desired.

Using the above characterization, we are able to show that a $T$-large domain $R$ can have any admissible Krull dimension. We need first the following well-known lemma on valuation domains.

**Lemma.** *Let $L$ be a field, $\{x_\alpha : \alpha < \gamma\}$ a set of indeterminates over $L$ indexed by the ordinal $\gamma$; then there exists a valuation domain $V$ of the field $L(x_\alpha : \alpha < \gamma)$ of Krull dimension $\gamma$; $\mathfrak{P}_0 = x_0 V$ is the maximal ideal of $V$, and the set $\{\mathfrak{P}_\alpha : \alpha < \gamma\}$ of nonzero prime ideals of $V$ is well-ordered by the opposite inclusion.*

*Proof.* We consider the group $G = \oplus_{\alpha<\gamma} \mathbf{Z}_\alpha$ where $\mathbf{Z}_\alpha \cong \mathbf{Z}$ for all $\alpha$. We endow $G$ with the anti-lexicographic order, i.e. the vector $(c_\alpha)_{\alpha<\gamma} \in G$ $c_\alpha \in \mathbf{Z}$, is positive if and only if the element of its support with the largest index is positive. For $\alpha < \gamma$ let $e_\alpha$ be the element of $G$ whose coordinates are 1 at the $\alpha$-th place and 0 otherwise. We define a valuation $v : L(x_\alpha : \alpha < \gamma) \to G \cup \{\infty\}$ by extending the assignments $x_\alpha \mapsto e_\alpha$. Then the valuation domain $V$ corresponding to $v$ satisfies our requirements.

**Proposition 3.** *Let $L$ be the prime subfield of $Q$, the field of fractions of $T$, and let $\lambda$ be the transcendence degree of $Q$ over $L$. Then for any cardinal $\gamma \leq \lambda$, there exists a valuation domain $V$ of $Q$ such that $V \cap T = R$ is a $T$-large domain with Krull dimension equal to $\gamma$.*

*Proof.* Let us choose a well-ordered set $\{x_\alpha : \alpha < \lambda\}$ of elements of $Q$ which constitutes a basis of transcendence of $Q$ over $L$; it is not restrictive to choose $x_0 \in T$ and $x_0 \equiv 1 \bmod \mathfrak{M}$. Making use of the lemma for a fixed $\gamma \leq \lambda$, let us construct a valuation domain $W$ of the field $L' = L(x_\alpha : \alpha < \gamma)$ of Krull dimension $\gamma$, where the set $\{\mathfrak{P}_\alpha : \alpha < \gamma\}$ of nonzero prime ideals of $W$ is well-ordered by the opposite inclusion. Let us extend the valuation on $L'$ to the field $L'' = L'(x_\alpha : \gamma \leq \alpha < \lambda)$ in such a way that the value group remains the same (see [**7**]). Now $Q$ is an algebraic extension of $L''$ and we can extend the valuation on $L''$ to a valuation on $Q$. Let $V$ be the valuation domain of $Q$ with respect to this last valuation. These extensions of valuations do not affect the lattice structure of prime ideals, and so $V$ has Krull dimension $\gamma$ and its maximal ideal, $\overline{\mathfrak{P}}$, (though no longer principal) is not the union of a strictly ascending chain of prime ideals, and therefore $\overline{\mathfrak{P}}$ is a radical ideal of $V$. Moreover, $V \not\subseteq T$ by our choice of $x_0 \in \overline{\mathfrak{P}}$. By Proposition 2 it follows that $R = V \cap T$ is a $T$-large domain and $R_{\mathfrak{P}} = V$ (where $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$) implies that the Krull dimension of $R$ is $\gamma$ too.

Let us note that in this case $\lambda$ is always infinite and the Krull dimension of $V$ cannot exceed the transcendence degree of $Q$ over $L$, and so $R$ can have any admissible Krull dimension.

The following result will be crucial in our investigation of Hausdorff finite rank torsion-free $R$-modules in the next section.

**Proposition 4.** *If $V$ is a valuation domain of $Q$, the field of fractions of the complete* DVR *$T$, and $V \not\subseteq T$, then $V$ is never complete in the topology of the valuation. If $\hat{V}$ denotes the completion of $V$, we have* $\mathrm{rank}_V \hat{V} = \infty$. *As a consequence we also have* $\mathrm{rank}_V \tilde{V} = \infty$ *where $\tilde{V}$*

*is any maximal immediate extension of* $V$.

*Proof*. We set $R = T \cap V$, $\pi T$ is the maximal ideal of $T$, $\overline{\mathfrak{P}}$ is the maximal ideal of $V$; $Q$ denotes the common field of fractions of $R$, $T$, $V$; $\pi T \cap R$ and $\mathfrak{P} = \overline{\mathfrak{P}} \cap R$ are the only maximal ideals of $R$ in view of Theorem 11.11 of [**6**]. We may assume that $\pi \in R$ and $R \cap \pi T = \pi R$, as one can see with the same proof as for property (G) above. Let us choose $\lambda \in R$ such that $\lambda\pi \equiv 1 \bmod \mathfrak{P}$. For every prime number $q$ greater than the $\pi$-adic value of $\lambda\pi$, the polynomial $f(Y) = Y^q - \lambda\pi$ has no roots in $T$, and thus also none in $Q$ since $T$ is integrally closed. By field theory it follows that $f$ is irreducible in $Q[Y]$. Let us also assume that $q$ is *different from the characteristic of* $V/\overline{\mathfrak{P}}$. Let us now prove the following

**Claim.** *Let* $\hat{V}$ *be the completion of* $V$; *then the polynomial* $f(Y)$ *has a root in* $\hat{V}$.

Since $\pi R$ is a maximal ideal of $R$, we can find a subset $F$ of $V$ such that $\{rV : r \in F\}$ is a basis of neighborhoods of zero for the $V$-topology of $V$, satisfying the following:

$$r \in R \cap \overline{\mathfrak{P}}; \quad r \equiv 1 \mod \pi R, \quad \forall\, r \in F.$$

For all $r \in F$, let us consider the polynomials

$$f_r = Y^q + rY^{q-1} - \lambda\pi \in R[Y] \subset T[Y].$$

Now $T$ is a complete DVR, hence henselian (see, e.g., [**7**]). Since $f_r \equiv Y(Y^{q-1} + 1) \bmod \pi T$, we may apply Hensel's lemma to find a root $\xi_r \in Q$ of $f_r$. Let us note that $\xi_r \in T \cap V = R$ since $T$ and $V$ are integrally closed. We have $\xi_r^q = -r\xi_r^{q-1} + \lambda\pi \equiv \lambda\pi \equiv 1 \bmod \overline{\mathfrak{P}}$ so that $\xi_r$ is a unit of $V$ for all $r \in F$. Moreover, $\xi_r + \overline{\mathfrak{P}}$ is a root of the polynomial $Z^q - 1 \in (V/\overline{\mathfrak{P}})[Z]$. Since $F$ is an infinite set, there must exist a subset $F'$ of $F$ such that $\{rV : r \in F'\}$ is a basis of neighborhoods of zero and $\xi_r \equiv \xi_s \bmod \overline{\mathfrak{P}}$ for all $r, s \in F'$. Let us now remark that $\xi_r \equiv \xi_s \bmod \overline{\mathfrak{P}}$ implies that

$$\eta_{rs} = \xi_r^{q-1} + \xi_r^{q-2}\xi_s + \cdots + \xi_r\xi_s^{q-2} + \xi_s^{q-1} \equiv q\xi_r^{q-1} \not\equiv 0 \mod \overline{\mathfrak{P}},$$

since $\xi_r$ and $q$ are units of $V$ (recall that $q$ is different from the characteristic of $V/\overline{\mathfrak{P}}$).

We want to verify that $\{\xi_r : r \in F'\}$ is a Cauchy net in $V$. In fact let us pick $r, s \in F'$ where $r$ divides $s$ in $V$; we have $\xi_r^q - \xi_s^q = s\xi_s^{q-1} - r\xi_r^{q-1} \in rV$ from which $(\xi_r - \xi_s)\eta_{rs} \in rV$. Since $\eta_{rs}$ is a unit in $V$ as observed above, we have $\xi_r - \xi_s \in rV$; it follows that $\{\xi_r : r \in F'\}$ is a Cauchy net as desired.

Let us now choose $\xi \in \hat{V}$ such that $\xi - \xi_r \in r\hat{V}$ for all $r \in F'$. Then

$$\xi^q \equiv \xi_r^q = -r\xi_r^{q-1} + \lambda\pi \equiv \lambda\pi \mod r\hat{V}, \quad \forall\, r \in F';$$

since $\cap_{r\in F'} r\hat{V} = \{0\}$, we conclude that $\xi^q - \lambda\pi = 0$ as desired.

Since the prime element $q$ may be chosen arbitrarily large, and $f$ is irreducible in $Q[Y]$, from the above claim we deduce that $[\hat{Q} : Q] = \infty$; equivalently, $\operatorname{rank}_V \hat{V} = \infty$. Finally, since $\hat{V}$ embeds into $\hat{V}$, we also have $\operatorname{rank}_V \tilde{V} = \infty$.

*Remark.* In his book [**4**] Matlis describes the theory of $D$-rings, i.e. those domains $R$ such that every torsion-free $R$-module of finite rank is a direct sum of modules of rank 1. One main result is that an integrally closed domain $R$ is a $D$-ring if and only if $R$ is the intersection of at most two maximal valuation rings of its field of fractions. Examples are given of $D$-rings $R$ which are not valuation domains; of course the two maximal valuation domains $V_1, V_2$ such that $R = V_1 \cap V_2$ are both nondiscrete according to our Proposition 4. As a consequence we can state that our $T$-large domains are not $D$-rings, but we shall see much more in the next section.

**Section 2.** In this section we shall examine torsion-free modules, of finite rank $M$ over a $T$-large domain $R$, which are Hausdorff in the $\mathfrak{N}$-adic topology.

**Proposition 5.** *Let $M$ be a finite rank torsion-free $R$-module, Hausdorff in the $\mathfrak{N}$-adic topology. Then the $\mathfrak{N}$-adic completion of $M$ coincides with the localization $M_{\mathfrak{N}}$, which is a direct sum of as many copies of $T$ as the rank of $M$.*

*Proof.* Since $R_{\mathfrak{N}} = T$, then $M_{\mathfrak{N}}$ is a $T$-module; since $T$ is a complete

DVR, $M_{\mathfrak{N}}$ has to be a direct sum of as many copies of $T$ as the $T$-rank of $M_{\mathfrak{N}}$ which is equal to the $R$-rank of $M$. It also follows that the $R$-module $M_{\mathfrak{N}}$ is complete in the $\mathfrak{N}$-adic topology since $T$ is the completion of $R$ in the $\mathfrak{N}$-adic topology. To conclude that $M_{\mathfrak{N}}$ is the completion of $M$, it suffices to show that the $\mathfrak{N}$-topology on $M$ coincides with the topology induced on $M$ by the $\mathfrak{N}$-topology of $M_{\mathfrak{N}}$ and that $M$ is dense in $M_{\mathfrak{N}}$. This amounts to prove that for all $n \in \mathbf{N}$, $\pi^n M = (\pi^n M_{\mathfrak{N}}) \cap M$ and that for all $t \in M_{\mathfrak{N}}$ and $n \in \mathbf{N}$, $(t + \pi^n M_{\mathfrak{N}}) \cap M \neq \varnothing$. These two facts can be proved with so straightforward a generalization of the arguments in Lemmas 5 and 6 of [**10**] that we have thought it appropriate to omit the verifications.

An immediate consequence of the above proposition is the following

**Corollary 6.** *A finite rank torsion-free $R$-module $M$ is Hausdorff if and only if it is contained in a finite direct sum of copies of $T$.*

In the next results, we shall carry over known results on modules over the nonmaximal valuation domain $V$ to Hausdorff modules over $R = V \cap T$.

**Proposition 7.** *For any fixed $n > 0$, there exists an indecomposable torsion-free $R$-module $M$ of rank $n$ which is Hausdorff in the $\mathfrak{N}$-adic topology.*

*Proof.* In view of Proposition 4, we know that $\mathrm{rank}_V(\tilde{V}) = \infty$. Let us then choose $a_1, \dots, a_n \in \tilde{V}$, linearly independent over $Q$. Let us consider the $V$-module

$$N = (Qa_1 + \cdots + Qa_n) \cap \tilde{V};$$

$N$ is an indecomposable $V$-module in view of Theorem 3(a) of [**8**]. Let us now consider the following $R$-module

$$M = N \cap (Ta_1 + \cdots + Ta_n).$$

$M$ has rank $n$ and it is Hausdorff, being contained in $Ta_1 + \cdots + Ta_n$. Let us verify that $M$ is indecomposable. It is clear that any possible nontrivial direct decomposition of the $R$-module $M$ gives rise to a nontrivial direct decomposition of the $V$-module $M_{\mathfrak{P}}$ ($\mathfrak{P}$ is the maximal ideal of $V$). It is then enough to show that $M_{\mathfrak{P}} = N$. Obviously, $N \supseteq M_{\mathfrak{P}}$ since $N$ is a $V$-module. Let us now choose $\eta \in N$; since $\eta \in Qa_1 + \cdots + Qa_n$, there exist $h \geq 0$ and $t_1, \ldots, t_n \in T$ such that

$$\eta = (1/\pi)^h (t_1 a_1 + \cdots + t_n a_n).$$

It follows that $\pi^h \eta \in N \cap (Ta_1 + \cdots + Ta_n) = M$ whence $\eta \in M_{\mathfrak{P}}$ since $\pi$ is a unit of $V$. This yields $N \subseteq M_{\mathfrak{P}}$ and the desired conclusion follows.

Our next Theorem 8 will be based on Theorem 1 in [**3**]; we state it in the following, less general form, which is exactly what we need.

**Theorem** [**3**]. *Let $V$ be a valuation domain such that* $\operatorname{rank}_V \hat{V} = \infty$. *Let $A_V$ be a reduced torsion-free $V$-algebra of finite rank $n$; let $\hat{A}_V$ be the completion of $A_V$ in the $V$-topology. Then there exist $\alpha \in \hat{V}$, $\delta \in \hat{A}_V$ such that*:

(i) $1_{A_V}, \alpha, \delta$ *are independent over $A_V$*;

(ii) *the $V$-submodule $N = A_V + A_V\alpha + A_V\delta$ of $\hat{A}_V$ has rank* $3n$;

(iii) $\operatorname{End}_V(N_*) = A_V$, *where $N_*$ is the purification of $N$ in $\hat{A}_V$.*

We recall that an $R$-module $M$ is said to be *cotorsion-free* if $M$ is reduced, torsion-free and does not contain isomorphic copies of $\hat{R} = T$. We will need the simple fact that $M$ is cotorsion-free if and only if $M_{\mathfrak{P}}$ is reduced. In fact if $M$ contains a copy of $T$, then $M_{\mathfrak{P}}$ contains a copy of $T_{\mathfrak{P}} = Q$. Conversely let us suppose that $M_{\mathfrak{P}} \supseteq Q\xi$ where without loss of generality, $\xi \in M$. In order to show that $M \supseteq T\xi$, it is enough to prove that $\xi/y \in M$ for any $y \in R \setminus \pi R$, since $T = R_{\mathfrak{N}}$. Now $\xi/y$ belongs to $Q\xi \subseteq M_{\mathfrak{P}}$ whence we can write $\xi/y = a/\pi^k$, where $a \in M$. To reach the desired conclusion we have to show that $a \in \pi^k M$. If not we may assume that $a \notin \pi M$ and $k > 0$. We have $ya = \pi^k \xi \in \pi^k M$; let $z \in R$ be such that $yz \equiv 1 \bmod \pi R$, $1 = yz + \lambda\pi$, say; then $a = zya + \lambda\pi a \in \pi M$ against our assumption.

**Theorem 8.** *Let $R = V \cap T$ be a $T$-large domain. Every cotorsion-free Hausdorff $R$-algebra $A$ of finite rank $n$ is the $R$-endomorphism* *algebra of a torsion-free Hausdorff $R$-module $M$ of rank $3n$.*

*Proof.* Since $A$ is Hausdorff, $A_{\mathfrak{N}}$ is the $\mathfrak{N}$-adic completion of $A$; we can write $A_{\mathfrak{N}} = Tz_1 \oplus \cdots \oplus Tz_n$ where the $z_i$ lie in $A$ and $z_1 = 1_A$. Let us now consider the $V$-algebra $A_{\mathfrak{P}}$; let us note that $A_{\mathfrak{P}}$ is reduced since $A$ is cotorsion-free. We are thus in the position to apply the preceding theorem to $A_{\mathfrak{P}}$ since $\operatorname{rank}_V \hat{V} = \infty$ holds in view of Proposition 4. Let $N = A_{\mathfrak{P}} + A_{\mathfrak{P}}\alpha + A_{\mathfrak{P}}\delta \subseteq \hat{A}_{\mathfrak{P}}$ be the $V$-module such that $\operatorname{End}_V(N_*) = A_{\mathfrak{P}}$. Let us note that the elements $z_1, \dots, z_n$, $z_1\alpha, \dots, z_n\alpha$, $z_1\delta, \dots, z_n\delta$ of $N$ are linearly independent over $Q$ since $1_A, \alpha, \delta$ are independent over $A$; moreover, $A \subseteq Qz_1 \oplus \cdots \oplus Qz_n$ implies that

$$N_* \subseteq Qz_1 \oplus \cdots Qz_n \oplus Qz_1\alpha \oplus \cdots \oplus Qz_n\alpha \oplus Qz_1\delta \oplus \cdots \oplus Qz_n\delta.$$

Let us consider the $R$-module

$$C = Tz_1 \oplus \cdots Tz_n \oplus Tz_1\alpha \oplus \cdots \oplus Tz_n\alpha \oplus Tz_1\delta \oplus \cdots \oplus Tz_n\delta,$$

and let us set $M = N_* \cap C$. It is clear that the $R$-module $M$ has rank $3n$ since it contains $Rz_1 \oplus \cdots \oplus Rz_n\delta$; we will show that $\operatorname{End}_R(M) = A$ whence our statement follows. Let us first show that $M_{\mathfrak{P}} = N_*$. It is clear that $N_* \supseteq M_{\mathfrak{P}}$ since $N_*$ is a $V$-module. Let now $\eta$ be any element of $N_*$; then $\eta \in Qz_1 \oplus \cdots \oplus Qz_n\delta$ so that $\eta = \sum a_i z_i + \sum b_i z_i\alpha + \sum c_i z_i\delta$, where $a_i, b_i, c_i \in Q$. Since any $d \in Q$ is of the form $d = v/\pi^k$ with $v \in T$ and $k \geq 0$, it is clear that we can write $\eta = (1/\pi)^h\theta$, where $h \geq 0$ is a suitable integer and $\theta \in C$. It follows that $\theta = \pi^h\eta \in N_* \cap C = M$ whence $\eta \in M_{\mathfrak{P}}$, since $\pi \notin \mathfrak{P}$; we conclude that $N_* \subseteq M_{\mathfrak{P}}$ too. From $N_* = M_{\mathfrak{P}}$ we deduce that every $R$-endomorphism of $M$ extends uniquely to an $R$-endomorphism of $N_*$. Moreover, $\operatorname{End}_R(N_*) = \operatorname{End}_V(N_*) = A_{\mathfrak{P}}$ whence $\operatorname{End}_R(M) \subseteq A_{\mathfrak{P}}$. We also have $A \subseteq \operatorname{End}_R(M)$ since $M$ is an $A$-module; in fact $N_*$ is by definition an $A$-module and to see that $C$ is an $A$-module, it is enough to observe that $A \subseteq A_{\mathfrak{N}} = Tz_1 \oplus \cdots \oplus Tz_n$. To end the proof, it suffices to check that no $\rho \in A_{\mathfrak{P}} \backslash A$ can be an endomorphism of $M$. In fact since $A$ is Hausdorff, for such a $\rho$ we can write $\rho = a/\pi^k$ where $a \in A$, $k > 0$ and $a/\pi \notin A$. From $\pi A_{\mathfrak{N}} \cap A = \pi A$ (see the proof of Proposition 5), it follows that $a/\pi \notin A_{\mathfrak{N}}$ too. Then $\rho z_1 = \rho 1_A = \rho \notin A_{\mathfrak{N}}$. Since $C = A_{\mathfrak{N}} \oplus A_{\mathfrak{N}}\alpha \oplus A_{\mathfrak{N}}\delta$ and $1_A, \alpha, \delta$ are independent over $A$, we conclude that $\rho z_1 \notin C$ whence $\rho z_1 \notin M$ and $\rho \notin \operatorname{End}_R(M)$.

The assumption that the $R$-algebra $A$ is cotorsion-free cannot be eliminated, due to the following example.

**Example 9.** The finite rank Hausdorff $R$-algebra $A = T \times T$ is not the endomorphism algebra of a Hausdorff torsion-free $R$-module of finite rank. This fact can be verified exactly as in the proof of the corollary of Theorem 1 in [**3**].

Our last result shows the remarkable fact that the Krull-Schmidt theorem fails for finite rank torsion-free modules over a $T$-large domain $R$; its proof is an adaptation of the argument developed in Theorem 2 of [**3**].

**Theorem 10.** *Let $R = V \cap T$ be a $T$-large domain. Then for every $n \in \mathbf{N}$ there exists an $R$-module $N$ of rank $> n$ which admits two non-isomorphic direct decompositions into indecomposable summands.*

*Proof.* Let $\lambda \in R$ be such that $\lambda\pi \equiv 1 \bmod \mathfrak{P}$. We know that for almost all prime numbers $q$, the polynomial $f = Y^q - \lambda\pi$ is irreducible in $Q[Y]$ whence, a fortiori in $R[Y]$, since it is monic. Let us also assume that $q \notin \mathfrak{P} \cup \mathfrak{N}$ whence $q$ is a unit of $R$, and let $f_1 = Y - 1$, $f_2 = Y^{q-1} + \cdots + Y + 1$. Then from $q \in (f_1, f_2)R[Y]$ it follows that $1 \in (f_1, f_2)R[Y]$. Let $g_2$ be a monic irreducible factor of $f_2$ in $R[Y]$; let us set $g_1 = f_1$ which is irreducible in $R[Y]$ too. We conclude that we can write $1 = g_1h_1 + g_2h_2$ for suitable $h_1, h_2 \in R[Y]$. Let us also observe that from $\lambda\pi \equiv 1 \bmod \mathfrak{P}$ it follows that $\bar{g}_1\bar{g}_2$ divides $\bar{f}$ in $(R/\mathfrak{P})[Y]$, where $\bar{g}_1, \bar{g}_2, \bar{f}$ are the reductions of $g_1, g_2, f$ modulo $\mathfrak{P}$. Moreover, since $R$ is integrally closed and $g_1, g_2, f$ are all irreducible in $R[Y]$ and monic, they are also prime elements of $R[Y]$. Let us consider the $R$-algebra $A = R[Y]/(g_1g_2f)$; $A$ is cotorsion-free of finite rank since $g_1g_2f$ is monic. We are in the position to apply Theorem 8. Let $M$ be a finite rank torsion-free Hausdorff $R$-module such that $\mathrm{End}_R(M) = A$. As in [**3**] we can see that from $g_1, g_2, f$ prime in $R[Y]$, it follows that $A$ has no idempotent elements whence $M$ is indecomposable. Note that $\operatorname{rank} M > 3q$ since $\operatorname{rank} A > q$. Let $\phi : R[Y] \to A$ denote the canonical map and let us define submodules $M_i \subset M$ by $M_i = \phi(g_ih_i)(M)$, $i = 1, 2$. Define mappings $M \to M_1 \oplus M_2$ by $m \mapsto (\phi(g_1h_1)(m), \phi(g_2h_2)(m))$ and

$M_1 \oplus M_2 \to M$ by $(m_1, m_2) \mapsto m_1 + m_2$. The composition map is $\phi(g_1h_1 + g_2h_2) = \phi(1) = 1_A$ so that $N = M_1 \oplus M_2 = M \oplus K$ for a suitable $K$. Moreover, $\operatorname{rank} M > \operatorname{rank} M_i$, $i = 1, 2$ (see [**3**]). Thus when $M_1, M_2, K$ are expressed as direct sums of indecomposable modules, we get inequivalent decompositions. Finally, if we choose $q$ such that $3q > n$, we get $n < 3q < \operatorname{rank} M < \operatorname{rank} N$.

## REFERENCES

**1.** D. Arnold and M. Dugas, *Indecomposable modules over Nagata valuation domains*, Proc. Amer. Math. Soc. **122** (1994), 689–696.

**2.** L. Fuchs and L. Salce, *Modules over valuation domains*, Marcel Dekker, New York, 1985.

**3.** B. Goldsmith and W. May, *The Krull-Schmidt problem for modules over valuation domains*, J. Pure Appl. Algebra **140** (1999), 57–63.

**4.** E. Matlis, *Torsion-free modules*, Chicago University Press, Chicago, 1972.

**5.** W. May and P. Zanardo, *Nonhenselian valuation domains and the Krull-Schmidt property for torsion-free modules*, Forum Math. **10** (1998), 249–257.

**6.** M. Nagata, *Local rings*, Wiley, Interscience, New York, 1962.

**7.** O.F.G. Schilling, *The theory of valuations*, AMS Surveys **4** (1950), New York.

**8.** P. Vámos, *Decomposition problems for modules over valuation domains*, J. London Math. Soc. (2) **41** (1990), 10–26.

**9.** P. Zanardo, *Kurosch invariants for torsion-free modules over Nagata valuation domains*, J. Pure Appl. Algebra **82** (1992), 195–209.

**10.** P. Zanardo and U. Zannier, *Commutative domains large in their* $\mathfrak{M}$*-adic completions*, Rend. Sem. Mat. Univ. Padova **95** (1996), 1–9.

Department of Mathematics, University of Arizona, Tucson, AZ 85721, U.S.A.
*E-mail address:* MAY@MATH.ARIZONA.EDU

Dipartimento di Matematica Pura e Applicata, Via Belzoni 7, 35131 Padova, Italy
*E-mail address:* PZANARDO@PDMAT1.MATH.UNIPD.IT