

## A WALLMAN-SHANIN-TYPE COMPACTIFICATION FOR APPROACH SPACES

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**ABSTRACT.** In [11] a Čech-Stone-type compactification theory was developed for  $\mathbf{UAP}_2$ . In this paper we construct a Wallman-Shanin-type compactification theory for weakly symmetric  $T_1$  approach spaces which form a full subcategory of  $\mathbf{AP}$  properly containing  $\mathbf{UAP}_2$ . For a weakly symmetric  $T_1$  approach space, we also investigate the relation between the topological bicoreflection of its Wallman-type compactification and the classical Wallman compactification of its topological bicoreflection, and we show that our theory extends the classical topological Wallman compactification theory. It is shown in [14] that our present theory also extends the Čech-Stone-type theory from [11].

**1. Introduction.** In the ‘classical’ study of extensions of topological spaces, a significant role is played by compactification theories, in particular, by the Wallman-Shanin compactification theory since it applies to all  $T_1$  topological spaces. This approach, based on the use of so-called closed ultrafilters, was put forward by Wallman in his 1938 paper [16], where he defined his ‘ultrafilter space’ in the setting of distributive lattices and then applied the result to the lattice of all closed sets of a  $T_1$  topological space, obtaining the so-called Wallman compactification, which for normal spaces yields the well-studied Čech-Stone compactification. His ideas were subsequently generalized by Banaschewski [2] who defined what he called a “Wallman basis” to construct Hausdorff compactifications of Tychonoff topological spaces. See also Frink [4], who used what he called ‘normal basis’ to end up with Hausdorff compactifications for Tychonoff topological spaces, and by Steiner [15], using the concept of a ‘separating base’ to create more general  $T_1$  compactifications for  $T_1$  spaces. This last line of work is also followed in [12] and we refer hereto for more details, since we will restrict ourselves to listing basic definitions and facts concerning

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the separating base approach in the preliminaries. Also Shanin came up with ideas about working with separating bases and, therefore, the theory is often referred to as the “Wallman-Shanin’ compactification theory and, as can be seen from the work of Bentley and Naimpally [3], Gagrut and Naimpally [5] and Hušek [8], this theory still attracts attention in recent years.

It is also well-known that these Wallman-Shanin compactifications are most often nonmetrizable, so when we start from a metric space, somewhere during the formation of the Wallman-Shanin compactification, the canonical numerical information seems to be lost somewhere. It is precisely here that approach spaces as defined in Lowen [9] come into the picture, because they represent exactly that part of the metric information which can be preserved by topological constructions. Therefore the question whether we can build a numerified compactification theory for a quite large class of approach spaces, using some kind of ultrafilter concept and extending the topological theory, imposes itself. It is our aim in this paper to propose such a quantified, Wallman-Shanin-type compactification theory, allowing us, amongst others, to end up with a canonically numerical compactification, starting from metric or even some quasi-metric spaces.

**2. Preliminaries.** We first give an account of some definitions and facts about approach spaces which we will use in the sequel and we refer to [9], [10] for any further information, whereas all background material of categorical nature can be found in [1], [6], [7] and [13].

Approach spaces were introduced in Lowen [9] as a concept generalizing both topological and metric spaces at the same time, and representing exactly that part of the canonical metric information which can be retained when performing products or more general, initial liftings of metric objects, while maintaining compatibility with the underlying topologies. As shown in Lowen [10], approach spaces have many different equivalent characterizations, the following two of which we will discuss here: ‘distances’ and ‘regular function frames.’ We start with the definition of a distance. In what follows,  $X$  will be an arbitrary set and we will use the notations  $2^X$ , respectively  $2_0^X$ ,  $2^{(X)}$  and  $2_0^{(X)}$ , for the set of all, respectively all nonempty, all finite, all nonempty finite, subsets of  $X$ . For making calculations in  $[0, \infty]$ , we also adopt the convention that  $0 \cdot \infty = \infty \cdot 0 = 0$  and  $\infty - \infty = 0$ .

**Definition 2.1.** A function  $\delta : X \times 2^X \rightarrow [0, \infty]$  is called a ‘distance’ (on  $X$ ) if it satisfies the following properties:

- (D1)  $\forall x \in X : \delta(x, \{x\}) = 0,$
- (D2)  $\forall x \in X : \delta(x, \emptyset) = \infty,$
- (D3)  $\forall x \in X, \forall A, B \in 2^X : \delta(x, A \cup B) = \delta(x, A) \wedge \delta(x, B),$
- (D4)  $\forall x \in X, \forall A \in 2^X, \forall \varepsilon \in [0, \infty] : \delta(x, A) \leq \delta(x, A^{(\varepsilon)}) + \varepsilon,$

where for every  $A \in 2^X$  and every  $\varepsilon \in [0, \infty]$ ,

$$A^{(\varepsilon)} \doteq \{x \in X \mid \delta(x, A) \leq \varepsilon\}.$$

For every  $x \in X$  and every  $A \in 2^X$ ,  $\delta(x, A)$  should be seen as an indication of ‘how far  $x$  is away from being an adherence point of  $A$ ,’ as will become clearer later on. We now recall the definition of the concept ‘regular function frame.’

**Definition 2.2.** A set of functions  $\mathcal{R} \subset [0, \infty]^X$  is called a ‘regular function frame’ (on  $X$ ) if the following properties are satisfied:

- (R1)  $\forall \mathcal{S} \subset \mathcal{R} : \vee \mathcal{S} \in \mathcal{R},$
- (R2)  $\forall \mu, \nu \in \mathcal{R} : \mu \wedge \nu \in \mathcal{R},$
- (R3)  $\forall \mu \in \mathcal{R}, \forall \alpha \in [0, \infty] : \mu + \alpha \in \mathcal{R},$
- (R4)  $\forall \mu \in \mathcal{R}, \forall \alpha \in [0, \inf_{x \in X} \mu(x)] : \mu - \alpha \in \mathcal{R}.$

Note that it is an immediate consequence of (R1) and (R3) that a regular function frame contains all constant functions. As indicated above, it is shown in [9], [10] that there is a one-to-one correspondence between the distances on  $X$  and the regular function frames on  $X$  and we refer hereto for more information. A pair  $(X, \delta)$  where  $\delta$  is a distance on  $X$ , or equivalently, a pair  $(X, \mathcal{R})$  where  $\mathcal{R}$  is a regular function frame on  $X$ , is called an ‘approach space’ and, from now on, we will make no distinction between a distance and its associated regular function frame anymore. If confusion might arise, we will write, e.g.,  $\mathcal{R}_\delta$  for the frame on  $X$  associated with  $\delta$ . We only recall that if  $\mathcal{R}$  is a regular function frame on  $X$ , the corresponding distance  $\delta$  is given by the formula

$$\delta(x, A) \doteq \sup\{\mu(x) \mid \mu \in \mathcal{R}, \mu|_A = 0\} \quad x \in X, A \in 2^X.$$

This automatically implies that for every distance  $\delta$  on  $X$  and every  $A \in 2^X$ , the function

$$\delta(\cdot, A) : X \longrightarrow [0, \infty] : x \longrightarrow \delta(x, A)$$

belongs to the corresponding regular function frame. Concerning the other equivalent characterizations of approach spaces given in [9], [10], we only mention that approach spaces can be completely determined by means of so-called approach systems. An approach system on  $X$  is a collection  $\mathcal{A} \doteq (\mathcal{A}(x))_{x \in X}$  of order-theoretic ideals in  $[0, \infty]^X$ , subject to a set of axioms stated in [9], [10], and where for every  $x \in X$  the ideal  $\mathcal{A}(x)$  should be interpreted as a collection of ‘local distance functions with respect to  $x$ .’ We only recall that if  $(X, \delta)$  is an approach space with associated regular function frame  $\mathcal{R}$  and associated approach system  $\mathcal{A}$ , the following transition formulas hold:

$$\delta(x, A) = \sup_{\varphi \in \mathcal{A}(x)} \inf_{y \in A} \varphi(y), \quad x \in X, A \subset X$$

and

$$\mathcal{R} = \left\{ \mu \in [0, \infty]^X \mid \forall x \in X : \mu(x) = \sup_{\varphi \in \mathcal{A}(x)} \inf_{y \in X} (\mu + \varphi)(y) \right\}.$$

If  $(X, \delta), (X', \delta')$  are two approach spaces, then a function  $f : X \rightarrow X'$  is called a contraction if it satisfies

$$\forall x \in X, \forall A \in 2^X : \delta'(f(x), f(A)) \leq \delta(x, A),$$

or, equivalently, in terms of the associated regular function frames  $\mathcal{R}$  and  $\mathcal{R}'$ , if

$$\forall \mu' \in \mathcal{R}' : \mu' \circ f \in \mathcal{R}.$$

This admits yet another characterization of the regular function frame which can be found in [10]. If we define

$$\delta_{\mathbf{P}} : [0, \infty] \times 2^{[0, \infty]} \longrightarrow [0, \infty] : (x, A) \longrightarrow \begin{cases} (x - \sup A) \vee 0 & \text{if } A \neq \emptyset, \\ \infty & \text{if } A = \emptyset, \end{cases}$$

and  $\mathbf{P} \doteq ([0, \infty], \delta_{\mathbf{P}})$  we have the following:

**Proposition 2.3.** *If  $(X, \delta)$  is an approach space with associated regular function frame  $\mathcal{R}$  and if  $\mu \in [0, \infty]^X$ , then the following are equivalent:*

- (1)  $\mu \in \mathcal{R}$ ,

(2)  $\mu : (X, \delta) \rightarrow \mathbf{P}$  is a contraction.

It then can be proved (see [9], [10]) that approach spaces and contractions are the objects and morphisms of a topological construct, which is denoted as **AP**. The following proposition recalls how initial liftings are formed in **AP**, where we adopt the convention that, for every  $\mathcal{S} \subset [0, \infty]^X$ ,  $\mathcal{S}^\wedge$ , respectively  $\mathcal{S}^\vee$ , stands for the saturation of  $\mathcal{S}$  with respect to finite infima, respectively arbitrary suprema.

**Proposition 2.4.** *Let*

$$(f_j : X \longrightarrow (X_j, \mathcal{R}_j))_{j \in J}$$

*be a structured source in **AP**. Then the regular function frame corresponding to the unique initial lift of this source is given by*

$$\mathcal{R} \doteq (\{\mu_j \circ f_j \mid j \in J, \mu_j \in \mathcal{R}_j\}^\wedge)^\vee.$$

We now briefly discuss how both topological and metric spaces give rise to approach spaces. If  $(X, \mathcal{T})$  is a topological space,

$$\delta_{\mathcal{T}} : X \times 2^X \longrightarrow [0, \infty] : (x, A) \longrightarrow \begin{cases} 0 & \text{if } x \in \text{cl}_{\mathcal{T}}(A), \\ \infty & \text{if } x \notin \text{cl}_{\mathcal{T}}(A), \end{cases}$$

(where  $\text{cl}_{\mathcal{T}}$  denotes the  $\mathcal{T}$ -closure operator), defines a distance on  $X$ , the associated regular function frame of which is given by

$$\mathcal{R}_{\mathcal{T}} \doteq \{\mu \in [0, \infty]^X \mid \mu \text{ is lower semi-continuous with respect to } \mathcal{T}\}.$$

It is easy to see that this correspondence, assigning an approach space to each topological space is one-to-one and that a function between topological spaces is continuous if and only if it is a contraction between the associated approach spaces, yielding that the topological construct **TOP** (of topological spaces and continuous maps) can be concretely embedded as a full subconstruct of **AP**. If, on the other hand,  $(X, d)$  is an  $\infty pq$ -metric space,

$$\delta_d : X \times 2^X \longrightarrow [0, \infty] : (x, A) \longrightarrow \inf_{a \in A} d(x, a),$$

is a distance and since this correspondence is one-to-one and since a function between two  $\infty pq$ -metric spaces is nonexpansive if and only if it is a contraction between the corresponding approach spaces, this shows that the topological constructs  $pq\mathbf{MET}^\infty$  (of  $\infty pq$ -metric spaces and nonexpansive maps) and  $p\mathbf{MET}^\infty$  (of  $\infty p$ -metric spaces and nonexpansive maps) can be concretely embedded as full subconstructs of  $\mathbf{AP}$ .

These embeddings are also categorically well-behaved, in the sense that with respect to the embeddings given above,  $\mathbf{TOP}$  is a concretely bireflective and bicoreflective subconstruct of  $\mathbf{AP}$  and that both  $p\mathbf{MET}^\infty$  and  $pq\mathbf{MET}^\infty$  are concretely bicoreflective subconstructs of  $\mathbf{AP}$ . Of most interest to us are the topological and  $\infty p(q)$ -metric coreflections of a given approach space which should be interpreted as the topology and the  $\infty p(q)$ -metric ‘underlying’ the given approach space. If  $(X, \delta)$  is an approach space (with corresponding regular function frame  $\mathcal{R}$ ), the topological coreflection, which is denoted by  $\mathcal{T}_\delta$ , or  $\mathcal{T}_\mathcal{R}$ , is completely determined by the following closure operator:

$$\text{cl}_{\mathcal{T}_\delta}(A) \doteq \{x \in X \mid \delta(x, A) = 0\} \quad A \in 2^X$$

and the  $\infty pq$ -metric, respectively  $\infty p$ -metric, coreflection, which is denoted by  $d_\delta^{\infty pq}$  or  $d_\mathcal{R}^{\infty pq}$ , respectively  $d_\delta$  or  $d_\mathcal{R}$ , is given by

$$d_\delta^{\infty pq}(x, y) \doteq \delta(x, \{y\}) \quad x, y \in X,$$

(respectively

$$d_\delta(x, y) \doteq \delta(x, \{y\}) \vee \delta(y, \{x\}) \quad x, y \in X.)$$

We will call an approach space  $(X, \delta)$  a  $T_1$  space, respectively a compact space, if its topological coreflection is a  $T_1$ , respectively a compact, topological space whereas we will call it ‘symmetric’ if its  $\infty pq$ -metric and  $\infty p$ -metric coreflections coincide, i.e., if it satisfies  $\delta(x, \{y\}) = \delta(y, \{x\})$  for every  $x, y \in X$ . We will denote the full subconstruct of  $\mathbf{AP}$  formed by the  $T_1$  objects, respectively the symmetric  $T_1$  objects and the compact  $T_1$  objects, by  $\mathbf{AP}_1$ , respectively  $\mathbf{AP}_1^s$  and  $k\mathbf{AP}_1$ .

For any subset  $A$  of  $X$ ,  $\theta_A$  stands for the function on  $X$ , taking the value 0 on  $A$  and the value  $\infty$  on  $X \setminus A$ , whereas for any  $\alpha \in [0, \infty]$ ,

we will use the symbol  $\alpha$  to represent the constant function with value  $\alpha$  on  $X$ . If  $\varphi \in [0, \infty]^X$ , we write  $\varphi \gg 0$ , respectively  $\varphi \sim 0$ , to indicate that  $\inf_{x \in X} \varphi(x) > 0$ , respectively  $\inf_{x \in X} \varphi(x) = 0$  and finally, if  $\mathcal{S} \subset [0, \infty]^X$ , we put  $\mathcal{S}_0 \doteq \{\mu \in \mathcal{S} \mid \inf_{x \in X} \mu(x) = 0\}$ .

We also mention that, since

$$\mathcal{T}_{\delta_{\mathbf{p}}} = \{]a, \infty[ \mid a \in [0, \infty[ \cup \{\emptyset, [0, \infty]\},$$

it follows directly from Proposition 2.3 that, for any approach space  $(X, \mathcal{R})$ , all regular functions are lower semi-continuous with respect to  $\mathcal{T}_{\mathcal{R}}$ .

To conclude this section we recall the formulation of the classical Wallman-Shanin compactification theory, and we refer to [12] for more information.

**Definition 2.5.** If  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$  and  $\mathcal{B}$  is a closed base for  $(X, \mathcal{T})$ , then  $\mathcal{B}$  is called a separating base (or a  $T_1$  base) for  $(X, \mathcal{T})$  if it satisfies the following supplementary conditions:

- (1)  $\emptyset \in \mathcal{B}$ ,
- (2)  $\forall B, C \in \mathcal{B} : B \cup C, B \cap C \in \mathcal{B}$ ,
- (3)  $\forall x \in X, \forall B \in \mathcal{B} : x \notin B \Rightarrow \exists C \in \mathcal{B} : (x \in C \text{ and } B \cap C = \emptyset)$ .

**Definition 2.6.** If  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$  and  $\mathcal{B}$  is a separating base for  $(X, \mathcal{T})$ , then  $\mathcal{C} \in 2_0^{\mathcal{B}}$  is called a  $\mathcal{B}$ -filter if it fulfills the following properties:

- (CF1)  $\emptyset \notin \mathcal{C}$ ,
- (CF2)  $\forall B, C \in \mathcal{C} : B \cap C \in \mathcal{C}$ ,
- (CF3)  $\forall B \in \mathcal{B}, \forall C \in \mathcal{C} : C \subset B \Rightarrow B \in \mathcal{C}$ .

For  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$  and for a separating base  $\mathcal{B}$  for  $(X, \mathcal{T})$ , we define  $\sigma(X, \mathcal{B})$  to be the set of all maximal  $\mathcal{B}$ -filters. If, on the other hand, we put

$$\tilde{\mathcal{B}} \doteq \{\mathcal{C} \in \sigma(X, \mathcal{B}) \mid B \in \mathcal{C}\},$$

for all  $B \in \mathcal{B}$ , then  $\tilde{\mathcal{B}} \doteq \{\tilde{B} \mid B \in \mathcal{B}\}$  is a closed base for a topology  $\sigma(\mathcal{T}, \mathcal{B})$  on  $\sigma(X, \mathcal{B})$  which can be shown to be a compact  $T_1$  topology. Moreover,

$$w_{(X, \mathcal{B})}^* : (X, \mathcal{T}) \longrightarrow (\sigma(X, \mathcal{B}), \sigma(\mathcal{T}, \mathcal{B})) : x \longrightarrow \mathcal{C}_x,$$

where  $\mathcal{C}_x \doteq \{B \in \mathcal{B} \mid x \in B\}$  for each  $x \in X$ , is a dense embedding, whence a compactification of  $(X, \mathcal{T})$  which is called the  $\mathcal{B}$ -Wallman-Shanin compactification of  $(X, \mathcal{T})$ . In the special case that  $\mathcal{B}$  is taken to be the set of all  $\mathcal{T}$  closed sets, we call the corresponding Wallman-Shanin compactification simply the Wallman compactification and we write  $w(X)$ , respectively  $w(\mathcal{T})$  and  $w_X^*$  for  $\sigma(X, \mathcal{B})$ , respectively  $\sigma(\mathcal{T}, \mathcal{B})$  and  $w_{(X, \mathcal{B})}^*$ .

**3. Construction of the compactification.** First of all we have to find a suitable description of the points of our compactification. If we look at the axioms for regular function frames, it is very apparent that the frame of an approach space has properties which are closely related to the ones of the lattice of closed subsets of a topological space, when we interpret suprema as intersections and infima as unions. This seems very plausible if we look, e.g., at the indicator functions  $\theta_C$  of sets rather than at the sets themselves. Moreover, in this light, the hull operator looks very much like a closure operator, which is also made plausible by the observation that, for any approach space  $(X, \delta)$ , we have that

$$\forall A \in 2^X : h(\theta_A) = \delta(\cdot, A) \in \mathcal{R}$$

whence for every topological space  $(X, \mathcal{T})$ ,

$$\forall A \in 2^X : h_{\mathcal{T}}(\theta_A) = \theta_{\text{cl}_{\mathcal{T}}(A)}.$$

(We recall that, for an approach space  $(X, \delta)$ , the associated hull operator  $h : [0, \infty]^X \rightarrow [0, \infty]^X$  is given by

$$h(\mu) \doteq \bigvee \{\rho \in \mathcal{R} \mid \rho \leq \mu\}, \quad \mu \in [0, \infty]^X.$$

For  $(X, \mathcal{T}) \in |\mathbf{TOP}|$ , we write  $h_{\mathcal{T}}$  for the hull operator associated with  $\delta_{\mathcal{T}}$  and we refer to [10] for more information.) This motivates the choice that we will take maximal order-theoretic ideals of particular regular functions as the points of our compactification. We recall that we introduced the notations  $\rho \sim 0$ , respectively  $\rho \gg 0$ , to indicate that  $\inf_{x \in X} \rho(x) = 0$ , respectively  $\inf_{x \in X} \rho(x) > 0$  for  $\rho \in [0, \infty]^X$  and that

$$\mathcal{S}_0 \doteq \{\rho \in \mathcal{S} \mid \rho \sim 0\}$$

for  $\mathcal{S} \subset [0, \infty]^X$ . Saying that  $\rho \vee \nu \sim 0$ , respectively  $\rho \vee \nu \gg 0$  for  $\rho, \nu \in [0, \infty]^X$  should therefore be interpreted as stating that  $\rho$



and  $\nu$  ‘meet,’ respectively are ‘disjoint.’ Since translating functions by a constant does not affect their ‘shape,’ it is also reasonable to consider only ideals at zero level. We now start by defining *WS* bases as generalizations of separating bases.

**Definition 3.1.** If  $(X, \mathcal{R}) \in |\mathbf{AP}|$  and if  $\mathcal{S} \subset \mathcal{R}$ , we call  $\mathcal{S}$  a ‘regular base’ (of  $(X, \mathcal{R})$ ) if it satisfies the following conditions:

- (RB1)  $\forall \mu \in \mathcal{S}, \forall \alpha \in [0, \infty] : \mu + \alpha \in \mathcal{S}$ ,
- (RB2)  $\forall \mu \in \mathcal{S}, \forall \alpha \in [0, \inf_{x \in X} \mu(x)] : \mu - \alpha \in \mathcal{S}$ ,
- (RB3)  $\forall \mu \in \mathcal{R} : \exists \mathcal{S}' \in 2^{\mathcal{S}} : \mu = \vee \mathcal{S}'$ .

Note that it follows from (RB3) that  $0 \in \mathcal{S}$ , so together with (RB1) this implies that  $\forall \alpha \in [0, \infty] : \alpha \in \mathcal{S}$ .

**Definition 3.2.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1|$ , we call a regular base  $\mathcal{S}$  a ‘Wallman-Shanin base,’ or shortly a *WS* base, for  $(X, \mathcal{R})$ , if it satisfies the following supplementary conditions:

- (WS1)  $\forall \mathcal{S}' \in 2^{\mathcal{S}} : \bigvee \mathcal{S}', \bigwedge \mathcal{S}' \in \mathcal{S}$ ,
- (WS2)  $\forall \mu \in \mathcal{S}, \forall x \in X : \mu(x) > 0 \Rightarrow \exists \nu \in \mathcal{S} : (\nu(x) = 0 \text{ and } \mu \vee \nu \gg 0)$ .

*Remark 3.3.* If  $(X, \mathcal{R}) \in |\mathbf{AP}_1|$  and  $\mathcal{S}$  is a *WS* base for  $(X, \mathcal{R})$ , then

$$\forall \varphi \in \mathcal{S}, \forall \varepsilon \in [0, \infty] : (\varphi - \varepsilon) \vee 0 \in \mathcal{S}.$$

*Proof.* This is obvious since for each  $\varphi \in \mathcal{S}$  and every  $\varepsilon \in [0, \infty]$

$$(\varphi - \varepsilon) \vee 0 = \varphi \vee \varepsilon - \varepsilon. \quad \square$$

**Proposition 3.4.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^s|$ , then  $\mathcal{R}$  is a *WS* base for  $(X, \mathcal{R})$ .

*Proof.* It is obvious that  $\mathcal{R}$  is a regular base which also satisfies (WS1), so only (WS2) needs to be checked. Therefore, suppose that  $\mu \in \mathcal{R}$  and

$x \in X$  such that  $\mu(x) > 0$ . Then  $\delta(\cdot, \{x\}) \in \mathcal{R}$  with  $\delta(\cdot, \{x\})(x) = 0$  and since  $\delta$  is a symmetric distance, we have that

$$\begin{aligned} \inf_{y \in X} (\mu \vee \delta(\cdot, \{x\}))(y) &= \inf_{y \in X} (\mu(y) \vee \delta(x, \{y\})) \\ &= \inf_{y \in X} \sup_{\varphi \in \mathcal{A}(x)} (\mu \vee \varphi)(y) \\ &\geq \frac{1}{2} \left( \sup_{\varphi \in \mathcal{A}(x)} \inf_{y \in X} (\mu + \varphi)(y) \right) \\ &= \frac{1}{2} \mu(x) > 0, \end{aligned}$$

so we are done.  $\square$

**Proposition 3.5.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1|$ , the following assertions are equivalent:*

- (1)  $\exists \mathcal{S} \subset \mathcal{R} : \mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ ,
- (2)  $\forall \mu \in \mathcal{R}, \forall x \in X : \mu(x) > 0 \Rightarrow \exists \nu \in \mathcal{R} : (\nu(x) = 0 \text{ and } \mu \vee \nu \gg 0)$ ,
- (3)  $\mathcal{R}$  is a WS base for  $(X, \mathcal{R})$ .

*Proof.* The implication (3)  $\Rightarrow$  (1) is trivially fulfilled and the implication (2)  $\Rightarrow$  (3) is obviously true because, as mentioned in the proof of the previous proposition,  $\mathcal{R}$  is a regular base for  $(X, \mathcal{R})$  which satisfies (WS1). We therefore only have to verify the implication (1)  $\Rightarrow$  (2). So assume that  $\mathcal{S} \subset \mathcal{R}$  is a WS base for  $(X, \mathcal{R})$  and fix  $x \in X$  and  $\varphi \in \mathcal{R}$  with  $\varphi(x) > 0$ . Because  $\mathcal{S}$  is a regular base for  $(X, \mathcal{R})$ ,  $\emptyset \neq \mathcal{S}' \subset \mathcal{S}$  exists with  $\varphi = \bigvee_{\varphi' \in \mathcal{S}'} \varphi'$ , so we can find  $\varphi'_0 \in \mathcal{S}'$  with  $\varphi'_0(x) > 0$ . Since  $\mathcal{S}$  satisfies (WS2), this yields that a  $\mu \in \mathcal{S} \subset \mathcal{R}$  exists with  $\mu(x) = 0$  and  $\varphi'_0 \vee \mu \gg 0$ , whence  $\varphi \vee \mu \gg 0$  and therefore we are done.  $\square$

*Remark 3.6.* We now give an example of a  $q$ -metric  $T_1$ -approach space for which the regular function frame is a WS base. Define a function

$$d : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+ : (x, y) \longrightarrow \begin{cases} 1 & y < x, \\ 0 & x = y, \\ 2 & x < y. \end{cases}$$

Then one easily verifies that  $d$  is a  $q$ -metric on  $\mathbf{R}^+$  and that  $\mathcal{T}_d$  is the discrete topology, whence  $T_1$ , so that  $(\mathbf{R}^+, \mathcal{R}_d) \in |\mathbf{AP}_1|$ . Moreover, since for  $\varphi \in \mathcal{R}_d$  and  $x \in X$  with  $\varphi(x) > 0$ , clearly  $d(\cdot, x) = \delta_d(\cdot, \{x\}) \in \mathcal{R}_d$  with  $d(\cdot, x)(x) = 0$  and

$$\inf_{y \in X} (\varphi(y) \vee d(y, x)) \geq \varphi(x) \wedge 1 > 0,$$

it follows that  $\mathcal{R}_d$  is a WS base for  $(\mathbf{R}^+, \mathcal{R}_d)$ .

Apparently, the existence of a WS base seems to be a sort of weak symmetry condition, justifying the following definition.

**Definition 3.7.** We will call a  $T_1$  approach space  $(X, \mathcal{R})$  weakly symmetric if it satisfies one of the properties stated in Proposition 3.5. We write  $(k)\mathbf{AP}_1^{ws}$  for the full subcategory of  $\mathbf{AP}$ , consisting of all (compact) weakly symmetric  $T_1$  objects.

If we make no distinction between a topological space and the corresponding approach space, we have the following proper inclusions:

$$|\mathbf{TOP}_1| \subset |\mathbf{AP}_1^s| \subset |\mathbf{AP}_1^{ws}|.$$

We now come to giving a formal definition of the entities which will be the points in our compactification.

**Definition 3.8.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we call  $\Phi \in 2_0^{\mathcal{S}_0}$  a ‘zero ideal over  $\mathcal{S}$ ,’ if it satisfies the following conditions:

- (I1)  $\forall \varphi, \psi \in \Phi : \varphi \vee \psi \in \Phi$ ,
- (I2)  $\forall \varphi \in \Phi, \forall \psi \in \mathcal{S}_0 : \psi \leq \varphi \Rightarrow \psi \in \Phi$ .

We write  $\mathcal{I}_0(X, \mathcal{S})$  for the set of all zero ideals over  $\mathcal{S}$  and we denote the set of all maximal zero ideals over  $\mathcal{S}$  (with respect to inclusion) by  $\mathcal{W}(X, \mathcal{S})$ . If  $\Phi \in \mathcal{I}_0(X, \mathcal{S})$  and  $\Psi \subset \Phi$ , then we call  $\Psi$  a base for  $\Phi$  if and only if

$$\Phi = \{\mu \in \mathcal{S} \mid \exists \psi \in \Psi : \mu \leq \psi\}.$$

We call  $\Psi \subset \mathcal{S}$  a zero ideal base over  $\mathcal{S}$  if and only if

$$\{\mu \in \mathcal{S} \mid \exists \psi \in \Psi : \mu \leq \psi\} \in \mathcal{I}_0(X, \mathcal{S}).$$

**Proposition 3.9.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ ,  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$  and  $\Phi \subset \mathcal{S}$ , then  $\Phi$  is a zero ideal base over  $\mathcal{S}$  if and only if the following conditions are fulfilled:*

- (1)  $\emptyset \neq \Phi \subset \mathcal{S}_0$ ,
- (2)  $\forall \varphi, \psi \in \Phi : \exists \rho \in \Phi : \varphi \vee \psi \leq \rho$ .  $\square$

As we will see later on, the weak symmetry is exactly what we need in order to embed the space we start from nicely into the compactification.

**Proposition 3.10.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall x \in X : \Phi_x \doteq \{\varphi \in \mathcal{S} \mid \varphi(x) = 0\} \in \mathcal{W}(X, \mathcal{S}).$$

*Proof.* Fix  $x \in X$ . Clearly  $\Phi_x$  is a zero ideal over  $\mathcal{S}$ . Now assume that  $\Phi_x$  are not maximal. Then there would exist a zero ideal  $\Phi$  over  $\mathcal{S}$  such that  $\Phi_x \subsetneq \Phi$ , so we could find  $\varphi \in \Phi$  with  $\varphi(x) > 0$ . By (WS2),  $\psi \in \mathcal{S}$  would exist with  $\psi(x) = 0$  and  $\varphi \vee \psi \gg 0$ . This would imply that  $\psi \in \Phi_x \subset \Phi$ , whence it would follow that  $\varphi \vee \psi \in \Phi \subset \mathcal{S}_0$ , yielding a contradiction, and this completes the proof.  $\square$

**Proposition 3.11.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \Phi \in \mathcal{I}_0(X, \mathcal{S}) : \exists \Psi \in \mathcal{W}(X, \mathcal{S}) : \Phi \subset \Psi.$$

*Proof.* Fix  $\Phi \in \mathcal{I}_0(X, \mathcal{S})$  and put  $\mathcal{K} \doteq \{\Phi' \in \mathcal{I}_0(X, \mathcal{S}) \mid \Phi \subset \Phi'\}$ . Then  $\mathcal{K}$  is partially ordered with respect to inclusion. Now suppose that  $\emptyset \neq \mathcal{L} \subset \mathcal{K}$  such that  $\mathcal{L}$  is totally ordered with respect to inclusion. Then it easily follows that  $\Phi(\mathcal{L}) \doteq \cup \mathcal{L} \in \mathcal{I}_0(X, \mathcal{S})$  and that  $\Phi(\mathcal{L})$  is an upper bound for  $\mathcal{L}$  in  $(\mathcal{K}, \subset)$ . Applying Zorn's lemma yields the existence of a maximal element  $\Psi$  in  $(\mathcal{K}, \subset)$  and, since obviously  $\Psi \in \mathcal{W}(X, \mathcal{S})$ , we are done.  $\square$

**Proposition 3.12.** *Let  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , and let  $\mathcal{S}$  be a WS base for  $(X, \mathcal{R})$ . Then*

$$w_{(X, \mathcal{S})} : X \longrightarrow \mathcal{W}(X, \mathcal{S}) : x \longrightarrow \Phi_x$$

*is injective.*

*Proof.* Suppose that  $x, x' \in X$  for which  $\Phi_x = \Phi_{x'}$ . Then, since  $\mathcal{S}$  is a regular base for  $(X, \mathcal{R})$ , we have that

$$\begin{aligned} \delta(x', \{x\}) &= \sup\{\varphi(x') \mid \varphi \in \mathcal{R}, \varphi(x) = 0\} \\ &= \sup\left\{\left(\bigvee \mathcal{S}'\right)(x') \mid \emptyset \neq \mathcal{S}' \subset \mathcal{S}, \left(\bigvee \mathcal{S}'\right)(x) = 0\right\} \\ &= \sup\{\varphi(x') \mid \varphi \in \Phi_x = \Phi_{x'}\} = 0, \end{aligned}$$

yielding that  $x' \in \text{cl}_{\mathcal{T}_\delta}(\{x\})$  and because  $(X, \delta)$  is a  $T_1$  space,  $x = x'$ .  $\square$

We now prove some basic lemmas concerning zero ideals.

**Lemma 3.13.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$  we have that*

$$\begin{aligned} \forall \Phi \in \mathcal{I}_0(X, \mathcal{S}), \forall \mu \in \mathcal{S}_0 : \\ (\forall \varphi \in \Phi : \varphi \vee \mu \sim 0) \implies \exists \Psi \in \mathcal{I}_0(X, \mathcal{S}) : \Phi \cup \{\mu\} \subset \Psi. \end{aligned}$$

*Proof.* Take  $\Phi \in \mathcal{I}_0(X, \mathcal{S})$  and  $\mu \in \mathcal{S}_0$  such that

$$(1) \quad \forall \varphi \in \Phi : \varphi \vee \mu \sim 0$$

and put

$$\Psi \doteq \{\psi \in \mathcal{S} \mid \exists n \in \mathbf{N}_0 : \exists \psi_1, \dots, \psi_n \in \Phi \cup \{\mu\} : \psi \leq \bigvee_{j=1}^n \psi_j\}.$$

Obviously,  $\Phi \cup \{\mu\} \subset \Psi$ , and because  $\Phi \subset \mathcal{S}_0$  is closed under the formation of finite suprema,  $\mu \in \mathcal{S}_0$  and (1) holds, it follows that

$\emptyset \neq \Psi \subset \mathcal{S}_0$ . Since, on the other hand,  $\Psi$  trivially fulfills (I1) and (I2), we obtain that  $\Psi \in \mathcal{I}_0(X, \mathcal{S})$  which finishes the proof.  $\square$

**Proposition 3.14.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ ,  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$  and  $\Phi \in \mathcal{I}_0(X, \mathcal{S})$ , the following assertions are equivalent:*

- (1)  $\Phi \in \mathcal{W}(X, \mathcal{S})$ ,
- (2)  $\forall \mu \in \mathcal{S}_0 : (\forall \varphi \in \Phi : \varphi \vee \mu \sim 0) \Rightarrow \mu \in \Phi$ .

*Proof.* Suppose  $\Phi \in \mathcal{W}(X, \mathcal{S})$  and take  $\mu \in \mathcal{S}_0$  such that

$$\forall \varphi \in \Phi : \varphi \vee \mu \sim 0.$$

By the preceding lemma, a  $\Psi \in \mathcal{I}_0(X, \mathcal{S})$  exists such that  $\Phi \cup \{\mu\} \subset \Psi$  and, by maximality of  $\Phi$ , this implies that  $\mu \in \Psi = \Phi$ . Now assume that (2) holds and take  $\Psi \in \mathcal{I}_0(X, \mathcal{S})$  with  $\Phi \subset \Psi$ . Then obviously we have that

$$\forall \psi \in \Psi, \forall \varphi \in \Phi : \varphi \vee \psi \sim 0,$$

whence by (2),  $\Psi \subset \Phi$ , which finishes the proof.  $\square$

**Proposition 3.15.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$  we have that*

$$\forall \Phi \in \mathcal{W}(X, \mathcal{S}), \forall \varphi, \psi \in \mathcal{S}_0 : \varphi \wedge \psi \in \Phi \implies (\varphi \in \Phi \text{ or } \psi \in \Phi).$$

*Proof.* Suppose that  $\Phi \in \mathcal{W}(X, \mathcal{S})$  and  $\varphi, \psi \in \mathcal{S}_0$  for which  $\varphi \notin \Phi$  and  $\psi \notin \Phi$ . According to the previous proposition,  $\varphi_1, \varphi_2 \in \Phi$  exist such that  $\varphi_1 \vee \varphi \gg 0$  and  $\varphi_2 \vee \psi \gg 0$ . Since  $\varphi_1 \vee \varphi_2 \in \Phi$  and since

$$\begin{aligned} & \inf_{x \in X} ((\varphi \wedge \psi) \vee (\varphi_1 \vee \varphi_2))(x) \\ &= \inf_{x \in X} ((\varphi(x) \vee (\varphi_1(x) \vee \varphi_2(x))) \\ & \quad \wedge (\psi(x) \vee (\varphi_1(x) \vee \varphi_2(x)))) \\ & \geq \inf_{x \in X} ((\varphi(x) \vee \varphi_1(x)) \wedge (\psi(x) \vee \varphi_2(x))) \\ & \geq \left( \inf_{x \in X} (\varphi(x) \vee \varphi_1(x)) \right) \wedge \left( \inf_{x \in X} (\psi(x) \vee \varphi_2(x)) \right) > 0, \end{aligned}$$

it follows that  $\varphi \wedge \psi \notin \Phi$ , and this completes the proof.  $\square$

The following definition and propositions will help us to extend the structure to the compactification.

**Definition 3.16.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we define for every  $\rho \in \mathcal{S}$  the function

$$\hat{\rho} : \mathcal{W}(X, \mathcal{S}) \longrightarrow [0, \infty] : \Phi \longrightarrow \inf \{ \beta \in [0, \infty] \mid \exists \varphi \in \Phi : \rho \leq \varphi + \beta \}.$$

We also write

$$\hat{\mathcal{S}} \doteq \{ \hat{\rho} \mid \rho \in \mathcal{S} \}.$$

**Proposition 3.17.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ ,  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ ,  $\Phi \in \mathcal{W}(X, \mathcal{S})$ ,  $\rho \in \mathcal{S}$  and  $\alpha \in [0, \infty]$ , the following assertions are equivalent:*

- (1)  $\hat{\rho}(\Phi) \leq \alpha$ ,
- (2)  $(\rho - \alpha) \vee 0 \in \Phi$ ,
- (3)  $\inf_{x \in X} \rho(x) \leq \alpha$  and  $\forall \varepsilon \in ]\alpha, \infty]$ ,  $\forall \mu \in \mathcal{S}_0 : \mu|_{\{\rho \leq \varepsilon\}} = 0 \Rightarrow \mu \in \Phi$ ,
- (4)  $\inf_{x \in X} \rho(x) \leq \alpha$  and  $\forall \varepsilon \in ]\alpha, \infty]$ ,  $\forall \mu \in \mathcal{S}_0 : \mu|_{\{\rho \leq \varepsilon\}} = 0 \Rightarrow \hat{\mu}(\Phi) = 0$ .

*Proof.* First note that for  $\alpha = \infty$  the four statements are obviously true, so we may assume without loss of generality that  $\alpha < \infty$ . It is obvious that (2) implies (1) since we have that

$$\rho \leq (\rho - \alpha) \vee 0 + \alpha.$$

Conversely, to see that (1) implies (2), assume that  $\hat{\rho}(\Phi) \leq \alpha$ . Then, for every  $\varepsilon > 0$ ,  $\varphi_\varepsilon \in \Phi$  exists such that

$$\rho \leq \varphi_\varepsilon + \alpha + \varepsilon,$$

whence for every  $\varepsilon > 0$ ,

$$(\rho - \alpha) \vee 0 \leq \varphi_\varepsilon + \varepsilon.$$

Therefore, for every  $\varphi \in \Phi$  and every  $\varepsilon > 0$ , we have that

$$\begin{aligned} \inf_{x \in X} (((\rho - \alpha) \vee 0) \vee \varphi)(x) &\leq \inf_{x \in X} ((\varphi_\varepsilon + \varepsilon) \vee \varphi)(x) \\ &\leq \inf_{x \in X} (\varphi \vee \varphi_\varepsilon)(x) + \varepsilon = \varepsilon. \end{aligned}$$

Since this also implies that  $\inf_{x \in X} \rho(x) \leq \alpha$ , whence  $(\rho - \alpha) \vee 0 = \rho \vee \alpha - \alpha \in \mathcal{S}_0$ , it follows from Proposition 3.14 that  $(\rho - \alpha) \vee 0 \in \Phi$ . The equivalence of (3) and (4) is proved in exactly the same way. Now assume that (2) holds. It immediately follows from (2) that  $\inf_{x \in X} ((\rho - \alpha) \vee 0)(x) = 0$ , whence  $\inf_{x \in X} \rho(x) \leq \alpha$ . Fix  $\varepsilon \in ]\alpha, \infty]$  and  $\mu \in \mathcal{S}_0$  such that  $\mu|_{\{\rho \leq \varepsilon\}} = 0$ . If  $\varepsilon = \infty$ , it is obvious that  $\mu = 0 \in \Phi$ , so suppose that  $\varepsilon \in \mathbf{R}^+$  and fix  $\varepsilon' \in \mathbf{R}_0^+$  with  $\alpha + \varepsilon' < \varepsilon$ . Take  $\varphi \in \Phi$  arbitrary. It now follows from (2) that

$$\inf_{x \in X} (\varphi \vee ((\rho - \alpha) \vee 0))(x) = 0,$$

so for each  $\gamma \in ]0, \varepsilon']$ , we can pick  $x_\gamma \in X$  with  $\varphi(x_\gamma) \leq \gamma$  and  $\rho(x_\gamma) \leq \alpha + \gamma < \varepsilon$ , whence

$$\forall \gamma \in ]0, \varepsilon'] : (\varphi \vee \mu)(x_\gamma) \leq \gamma.$$

This shows that  $\varphi \vee \mu \sim 0$  and since  $\varphi \in \Phi$  was chosen arbitrarily, applying Proposition 3.14 yields that  $\mu \in \Phi$ . Finally, assume that (3) holds. Since  $\inf_{x \in X} \rho(x) \leq \alpha$ ,  $(\rho - \alpha) \vee 0 = \rho \vee \alpha - \alpha \in \mathcal{S}_0$ . On the other hand, we have that

$$\forall \varepsilon \in \mathbf{R}_0^+ : (\rho - \alpha) \vee 0 \leq (\rho - \alpha - \varepsilon) \vee 0 + \varepsilon$$

and since, for every  $\varepsilon \in \mathbf{R}_0^+$ ,  $(\rho - \alpha - \varepsilon) \vee 0 \in \mathcal{S}_0$  and  $((\rho - \alpha - \varepsilon) \vee 0)|_{\{\rho \leq \alpha + \varepsilon\}} = 0$ , it follows that

$$\forall \varepsilon \in \mathbf{R}_0^+ : (\rho - \alpha - \varepsilon) \vee 0 \in \Phi.$$

This implies that  $((\widehat{\rho - \alpha}) \vee 0)(\Phi) = 0$ , whence in the same way as indicated above,  $(\rho - \alpha) \vee 0 \in \Phi$ .  $\square$

**Corollary 3.18.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ ,  $\Phi \in \mathcal{W}(X, \mathcal{R})$ ,  $\rho \in \mathcal{R}$ ,  $\alpha \in [0, \infty]$  and if we put  $\mathcal{S} \doteq \mathcal{R}$  the following assertions are equivalent:*



- (1)  $\hat{\rho}(\Phi) \leq \alpha$ ,
- (2)  $(\rho - \alpha) \vee 0 \in \Phi$ ,
- (3)  $\forall \varepsilon \in ]\alpha, \infty] : \delta(\cdot, \{\rho \leq \varepsilon\}) \in \Phi$ ,
- (4)  $\forall \varepsilon \in ]\alpha, \infty] : \delta(\cdot, \widehat{\{\rho \leq \varepsilon\}})(\Phi) = 0$ .

*Proof.* Because of Proposition 3.5, this is an immediate consequence of the previous proposition if we show that (3) is equivalent to its counterpart in Proposition 3.17, since the equivalence of (3) and (4) is proved in the same way as the equivalence of their counterparts in Proposition 3.17. If  $\alpha = \infty$ , (3) and its counterpart in Proposition 3.17 trivially are equivalent, so we assume without loss of generality that  $\alpha \in \mathbf{R}^+$ . Suppose that  $\inf_{x \in X} \rho(x) \leq \alpha$  and that

$$\forall \varepsilon \in ]\alpha, \infty], \forall \mu \in \mathcal{R}_0 : \mu|_{\{\rho \leq \varepsilon\}} = 0 \implies \mu \in \Phi.$$

Fix  $\varepsilon \in ]\alpha, \infty]$ . Then  $\{\rho \leq \varepsilon\} \neq \emptyset$ , whence

$$\delta(\cdot, \{\rho \leq \varepsilon\}) \in \mathcal{R}_0$$

and since

$$\delta(\cdot, \{\rho \leq \varepsilon\})|_{\{\rho \leq \varepsilon\}} = 0,$$

we have that

$$\delta(\cdot, \{\rho \leq \varepsilon\}) \in \Phi.$$

Conversely, assume that (3) holds. Then from the fact that

$$\forall \varepsilon \in ]\alpha, \infty] : \delta(\cdot, \{\rho \leq \varepsilon\}) \in \Phi,$$

it follows that for every  $\varepsilon \in ]\alpha, \infty]$ ,  $\{\rho \leq \varepsilon\} \neq \emptyset$ , showing that  $\inf_{x \in X} \rho(x) \leq \alpha$ . Now fix  $\varepsilon \in ]\alpha, \infty]$  and  $\mu \in \mathcal{R}_0$  with  $\mu|_{\{\rho \leq \varepsilon\}} = 0$ . Then it follows from the transition formula (regular function frame  $\rightarrow$  distance) that

$$\mu \leq \delta(\cdot, \{\rho \leq \varepsilon\}),$$

whence  $\mu \in \Phi$ .  $\square$

**Corollary 3.19.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ ,  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ ,  $\rho \in \mathcal{S}$  and  $\Phi \in \mathcal{W}(X, \mathcal{S})$ , we have the following equalities:*

$$\begin{aligned} \hat{\rho}(\Phi) &= \inf \{ \beta \in [0, \infty] \mid (\rho - \beta) \vee 0 \in \Phi \} \\ &= \min \{ \beta \in [0, \infty] \mid (\rho - \beta) \vee 0 \in \Phi \} \\ &= \min \{ \beta \in [0, \infty] \mid \exists \varphi \in \Phi : \rho \leq \varphi + \beta \}. \end{aligned}$$

*Proof.* The first equality follows directly from the fact that

$$\forall \beta \in [0, \infty] : (\exists \varphi \in \Phi : \rho \leq \varphi + \beta) \iff (\rho - \beta) \vee 0 \in \Phi.$$

The second equality follows from Proposition 3.17, since Proposition 3.17 yields that  $(\rho - \hat{\rho}(\Phi)) \vee 0 \in \Phi$ , and the last equality again follows directly from (1).  $\square$

**Proposition 3.20.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \rho \in \mathcal{S} : \hat{\rho} \circ w_{(X, \mathcal{S})} = \rho.$$

*Proof.* This is clear since, for every  $x \in X$ , it follows from Corollary 3.19 that

$$\begin{aligned} \hat{\rho}(\Phi_x) &= \min\{\beta \in [0, \infty] \mid (\rho - \beta) \vee 0 \in \Phi_x\} \\ &= \min\{\beta \in [0, \infty] \mid \rho(x) \leq \beta\} \\ &= \rho(x). \quad \square \end{aligned}$$

**Proposition 3.21.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \mu, \rho \in \mathcal{S} : \widehat{\mu \vee \rho} = \hat{\mu} \vee \hat{\rho}.$$

*Proof.* Fix  $\mu, \rho \in \mathcal{S}$ . Since  $\mu, \rho \leq \mu \vee \rho$ , it is clear that  $\hat{\mu} \vee \hat{\rho} \leq \widehat{\mu \vee \rho}$ . To prove the converse inequality, pick  $\Phi \in \mathcal{W}(X, \mathcal{S})$ . If  $\alpha \in [0, \infty]$  with

$$\hat{\mu}(\Phi) \vee \hat{\rho}(\Phi) \leq \alpha$$

it follows from Proposition 3.17 that  $(\mu - \alpha) \vee 0, (\rho - \alpha) \vee 0 \in \Phi$ , whence

$$(\mu \vee \rho - \alpha) \vee 0 = ((\mu - \alpha) \vee 0) \vee ((\rho - \alpha) \vee 0) \in \Phi,$$

yielding that

$$\widehat{\mu \vee \rho}(\Phi) \leq \alpha. \quad \square$$

**Proposition 3.22.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \mu, \rho \in \mathcal{S} : \widehat{\mu \wedge \rho} = \hat{\mu} \wedge \hat{\rho}.$$

*Proof.* Take  $\mu, \rho \in \mathcal{S}$ . Then the inequality  $\hat{\mu} \wedge \hat{\rho} \geq \widehat{\mu \wedge \rho}$  is obvious since  $\mu, \rho \geq \mu \wedge \rho$ . Pick  $\Phi \in \mathcal{W}(X, \mathcal{S})$  with  $\hat{\mu}(\Phi) \wedge \hat{\rho}(\Phi) > 0$  and take  $\alpha \in [0, \infty]$  with

$$\hat{\mu}(\Phi) \wedge \hat{\rho}(\Phi) > \alpha.$$

Then it follows that  $(\mu - \alpha) \vee 0 \notin \Phi$  and that  $(\rho - \alpha) \vee 0 \notin \Phi$ , whence applying the technique used in the proof of Proposition 3.15 yields that

$$(\mu \wedge \rho - \alpha) \vee 0 = ((\mu - \alpha) \vee 0) \wedge ((\rho - \alpha) \vee 0) \notin \Phi,$$

showing that  $\widehat{\mu \wedge \rho}(\Phi) > \alpha$ .  $\square$

**Proposition 3.23.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \alpha \in [0, \infty] : \hat{\alpha} = \alpha.$$

*Proof.* Since for every  $\Phi \in \mathcal{W}(X, \mathcal{S})$ ,  $\Phi \subset \mathcal{S}_0$ , it is clear that

$$\forall \alpha \in [0, \infty], \forall \Phi \in \mathcal{W}(X, \mathcal{S}) :$$

$$\hat{\alpha}(\Phi) = \min\{\beta \in [0, \infty] \mid (\alpha - \beta) \vee 0 \in \Phi\} = \alpha. \quad \square$$

**Proposition 3.24.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \rho \in \mathcal{S}, \forall \alpha \in [0, \infty] : \widehat{\rho + \alpha} = \hat{\rho} + \alpha.$$

*Proof.* Take  $\rho \in \mathcal{S}$  and  $\alpha \in [0, \infty]$  arbitrary. For  $\alpha = \infty$ ,  $\rho + \alpha = \infty$ , so we are done by the previous proposition. Now suppose that  $\alpha < \infty$  and fix  $\Phi \in \mathcal{W}(X, \mathcal{S})$ . This implies that

$$\forall \varphi \in \Phi, \forall \beta \in [0, \infty] : \rho \leq \varphi + \beta \iff \rho + \alpha \leq \varphi + \alpha + \beta,$$

yielding that

$$\begin{aligned}\hat{\rho}(\Phi) + \alpha &= \min\{\beta \in [0, \infty] \mid \exists \varphi \in \Phi : \rho \leq \varphi + \beta\} + \alpha \\ &= \min\{\beta + \alpha \mid \beta \in [0, \infty] \text{ and } \exists \varphi \in \Phi : \rho \leq \varphi + \beta\} \\ &= \min\{\beta + \alpha \mid \beta \in [0, \infty] \text{ and } \exists \varphi \in \Phi : \rho + \alpha \leq \varphi + \alpha + \beta\} \\ &= \min\{\kappa \in [\alpha, \infty] \mid \exists \varphi \in \Phi : \rho + \alpha \leq \varphi + \kappa\}.\end{aligned}$$

If  $\kappa \in [0, \infty]$  such that

$$\exists \varphi \in \Phi : \rho + \alpha \leq \varphi + \kappa,$$

then we see that

$$\alpha \leq \inf_{x \in X} (\rho + \alpha)(x) \leq \inf_{x \in X} (\varphi + \kappa)(x) = \kappa.$$

This implies that

$$\hat{\rho}(\Phi) + \alpha = \min\{\kappa \in [0, \infty] \mid \exists \varphi \in \Phi : \rho + \alpha \leq \varphi + \kappa\} = \widehat{\rho + \alpha}(\Phi),$$

which finishes the proof.  $\square$

**Proposition 3.25.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that*

$$\forall \rho \in \mathcal{S}, \forall \alpha \in [0, \inf_{x \in X} \rho(x)] : \widehat{\rho - \alpha} = \hat{\rho} - \alpha.$$

*Proof.* Fix  $\rho \in \mathcal{S}$  and  $\alpha \in [0, \inf_{x \in X} \rho(x)]$ . If  $\alpha = \infty$ , it follows that  $\rho = \infty$ , so the result follows from Proposition 3.23. Now assume that  $\alpha \in \mathbf{R}^+$ . It now follows from (RB2) that  $\rho - \alpha \in \mathcal{S}$  and, since  $\rho = (\rho - \alpha) + \alpha$ , the previous proposition implies that

$$\hat{\rho} = \widehat{\rho - \alpha} + \alpha,$$

so we are done.  $\square$

Now we are ready to define a regular function frame on  $\mathcal{W}(X, \mathcal{S})$ .

**Definition 3.26.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we define

$$\mathcal{W}(\mathcal{R}, \mathcal{S}) \doteq \{\hat{\rho} \mid \rho \in \mathcal{S}\}^\vee.$$

**Proposition 3.27.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , then we have for every  $\rho \in \mathcal{S}$  that

$$\inf_{\Phi \in \mathcal{W}(X, \mathcal{S})} \hat{\rho}(\Phi) = \inf_{x \in X} \rho(x).$$

*Proof.* Fix  $\rho \in \mathcal{S}$ . If  $\alpha \in [0, \infty]$  such that  $\hat{\rho} \geq \alpha$  on  $\mathcal{W}(X, \mathcal{S})$ , it follows from Proposition 3.20 that  $\rho \geq \alpha$  on  $X$ . Conversely, we have for every  $\alpha \in [0, \infty]$  for which  $\rho \geq \alpha$  on  $X$  that  $\hat{\rho} \geq \hat{\alpha} = \alpha$  on  $\mathcal{W}(X, \mathcal{S})$ , so we are done.  $\square$

**Proposition 3.28.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that  $\mathcal{W}(\mathcal{R}, \mathcal{S})$  is a regular function frame on  $\mathcal{W}(X, \mathcal{S})$  having  $\hat{\mathcal{S}}$  as a regular base.

*Proof.* By definition, (R1) is fulfilled and, in order to verify that (R2) is fulfilled, fix  $\emptyset \neq \mathcal{S}', \mathcal{S}'' \subset \mathcal{S}$ . Using Proposition 3.22, we now obtain that for every  $\Phi \in \mathcal{W}(X, \mathcal{S})$

$$\begin{aligned} \left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho}(\Phi) \right) \wedge \left( \bigvee_{\mu \in \mathcal{S}''} \hat{\mu}(\Phi) \right) &= \bigvee_{\rho \in \mathcal{S}'} \left( \hat{\rho}(\Phi) \wedge \left( \bigvee_{\mu \in \mathcal{S}''} \hat{\mu}(\Phi) \right) \right) \\ &= \bigvee_{\rho \in \mathcal{S}'} \bigvee_{\mu \in \mathcal{S}''} (\hat{\rho}(\Phi) \wedge \hat{\mu}(\Phi)) \\ &= \bigvee_{\rho \in \mathcal{S}'} \bigvee_{\mu \in \mathcal{S}''} \widehat{\rho \wedge \mu}(\Phi), \end{aligned}$$

so that it follows from (WS1) that  $(\bigvee_{\rho \in \mathcal{S}'} \hat{\rho}) \wedge (\bigvee_{\mu \in \mathcal{S}''} \hat{\mu}) \in \mathcal{W}(\mathcal{R}, \mathcal{S})$ . Since, thanks to Proposition 3.24, for every  $\emptyset \neq \mathcal{S}' \subset \mathcal{S}$  and  $\alpha \in [0, \infty]$ , we have that

$$\left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho} \right) + \alpha = \bigvee_{\rho \in \mathcal{S}'} (\hat{\rho} + \alpha) = \bigvee_{\rho \in \mathcal{S}'} \widehat{\rho + \alpha},$$

it follows from (RB1) that (R3) holds. Finally, fix  $\emptyset \neq \mathcal{S}' \subset \mathcal{S}$  and

$$\alpha \in \left[ 0, \inf_{\Phi \in \mathcal{W}(X, \mathcal{S})} \left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho} \right) (\Phi) \right].$$

Then, applying Propositions 3.21, 3.23 and 3.25 yields that

$$\begin{aligned} \left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho} \right) - \alpha &= \left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho} \right) \vee \alpha - \alpha \\ &= \bigvee_{\rho \in \mathcal{S}'} (\hat{\rho} \vee \alpha - \alpha) \\ &= \bigvee_{\rho \in \mathcal{S}'} (\widehat{\rho \vee \alpha} - \alpha) \\ &= \bigvee_{\rho \in \mathcal{S}'} \widehat{\rho \vee \alpha} - \alpha. \end{aligned}$$

Hence, it follows from the fact that  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$  that (R4) is fulfilled. This proves that  $\mathcal{W}(\mathcal{R}, \mathcal{S})$  is a regular function frame on  $\mathcal{W}(X, \mathcal{S})$  and obviously, by Propositions 3.23, 3.24, 3.25 and 3.27,  $\{\hat{\rho} \mid \rho \in \mathcal{S}\}$  is a regular base for this frame.  $\square$

We will now prove step-by-step that

$$w_{(X, \mathcal{S})} : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

is a  $T_1$  compactification.

**Proposition 3.29.** *For every  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and every WS base  $\mathcal{S}$  for  $(X, \mathcal{R})$ ,*

$$w_{(X, \mathcal{S})} : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

*is initial, whence an embedding.*

*Proof.* The injectivity of  $w_{(X, \mathcal{S})}$  was verified in Proposition 3.12, whereas, as indicated in the first section, the initial regular function frame on  $X$  for

$$w_{(X, \mathcal{S})} : X \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

is given by

$$\begin{aligned} \left( \left( \bigvee_{\rho \in \mathcal{S}'} \hat{\rho} \right) \circ w_{(X, \mathcal{S})} \mid \mathcal{S}' \subset \mathcal{S} \right)^\wedge^\vee &= \left( \left( \bigvee_{\rho \in \mathcal{S}'} \rho \mid \mathcal{S}' \subset \mathcal{S} \right)^\wedge \right)^\vee \\ &= ((\mathcal{S}^\vee)^\wedge)^\vee = \mathcal{R}. \quad \square \end{aligned}$$

**Proposition 3.30.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that  $(\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S})) \in |\mathbf{AP}_1|$ .*

*Proof.* Take  $\Phi, \Psi \in \mathcal{W}(X, \mathcal{S})$  with  $\Phi \neq \Psi$ . It then follows from the maximality of  $\Phi$  and  $\Psi$  that  $\Phi \not\subset \Psi$  and  $\Psi \not\subset \Phi$ , whence we can find  $\varphi \in \Phi \setminus \Psi$  and  $\psi \in \Psi \setminus \Phi$ . Because  $\hat{\varphi}, \hat{\psi} \in \mathcal{W}(\mathcal{R}, \mathcal{S})$ , we have that  $\hat{\varphi}, \hat{\psi}$  are lower semi-continuous with respect to  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ , whence we obtain, using Proposition 3.17, that  $\{\hat{\varphi} > 0\}, \{\hat{\psi} > 0\} \in \mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$  with

$$\Phi \in \{\hat{\psi} > 0\} \not\preceq \Psi \quad \text{and} \quad \Psi \in \{\hat{\varphi} > 0\} \not\preceq \Phi,$$

so we are done.  $\square$

**Proposition 3.31.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , then  $\hat{\mathcal{S}}$  is a WS base for  $(\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$ .*

*Proof.* To begin with, note that it follows from Propositions 3.28, 3.21 and 3.22 that  $\hat{\mathcal{S}}$  is a regular base for  $(\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$  that satisfies (WS1). To verify (WS2), pick  $\Phi \in \mathcal{W}(X, \mathcal{S})$  and  $\mu \in \mathcal{S}$  and suppose that  $\hat{\mu}(\phi) > 0$ . Then  $\varphi \in \Phi$  exists such that

$$\mu \vee \varphi \gg 0.$$

(Namely, if  $\mu \sim 0$ , this follows from Proposition 3.14, whereas in the case that  $\mu \gg 0$ , we can pick  $\varphi \in \Phi$  arbitrarily). Applying Proposition 3.17, respectively Propositions 3.27 and 3.21, now yields that  $\hat{\varphi}(\Phi) = 0$  and that

$$\hat{\mu} \vee \hat{\varphi} \gg 0,$$

so we are done.  $\square$

The next proposition will be handy to prove compactness of  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$  and denseness of the embedding  $w_{(X, \mathcal{S})}$ .

**Proposition 3.32.** *For  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  a WS base for  $(X, \mathcal{R})$ , we have that*

$$\{\{\hat{\rho} = 0\} \mid \rho \in \mathcal{S}_0\} \cup \{\emptyset\}$$

is a base for the  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ -closed subsets of  $\mathcal{W}(X, \mathcal{S})$ .

*Proof.* First note that for every  $\rho \in \mathcal{S}_0$  the fact that  $\hat{\rho}$  is lower semi-continuous with respect to  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$  implies that  $\{\hat{\rho} = 0\}$  is  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ -closed. Conversely, let  $\mathfrak{A}$  be a nonempty,  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ -closed subset of  $\mathcal{W}(X, \mathcal{S})$ . Then it follows that

$$\mathfrak{A} = \{\delta_{\mathcal{W}(\mathcal{R}, \mathcal{S})}(\cdot, \mathfrak{A}) = 0\}.$$

Since, on the other hand,  $\delta_{\mathcal{W}(\mathcal{R}, \mathcal{S})}(\cdot, \mathfrak{A}) \in \mathcal{W}(\mathcal{R}, \mathcal{S})$  and since  $\{\hat{\rho} \mid \rho \in \mathcal{S}\}$  is a base for  $\mathcal{W}(\mathcal{R}, \mathcal{S})$ , we know that

$$\delta_{\mathcal{W}(\mathcal{R}, \mathcal{S})}(\cdot, \mathfrak{A}) = \bigvee_{\rho \in \mathcal{S}, \hat{\rho}|_{\mathfrak{A}} = 0} \hat{\rho}.$$

Moreover, since  $\mathfrak{A} \neq \emptyset$ , we can find  $\Phi \in \mathfrak{A}$  for which it follows from Proposition 3.17 that

$$\{\rho \mid \rho \in \mathcal{S}, \hat{\rho}|_{\mathfrak{A}} = 0\} \subset \Phi \subset \mathcal{S}_0$$

and because

$$\mathfrak{A} = \left\{ \bigvee_{\substack{\rho \in \mathcal{S} \\ \hat{\rho}|_{\mathfrak{A}} = 0}} \hat{\rho} = 0 \right\} = \bigcap_{\substack{\rho \in \mathcal{S} \\ \hat{\rho}|_{\mathfrak{A}} = 0}} \{\hat{\rho} = 0\},$$

we are done.  $\square$

**Proposition 3.33.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that  $w_{(X, \mathcal{S})}(X)$  is dense in  $\mathcal{W}(X, \mathcal{S})$  with respect to  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ .*



*Proof.* If  $X = \emptyset$  there is nothing to prove, so assume that  $X \neq \emptyset$ . Fix  $\emptyset \neq \mathfrak{A} \subset \mathcal{W}(X, \mathcal{S})$  open with respect to  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ . In case that  $\mathfrak{A} = \mathcal{W}(X, \mathcal{S})$  it is clear that  $\mathfrak{A} \cap w_{(X, \mathcal{S})}(X) \neq \emptyset$ , so we can assume that  $\mathfrak{A} \neq \mathcal{W}(X, \mathcal{S})$ . Then Proposition 3.32 implies that we can find  $S' \in 2_0^{S_0}$  with

$$\mathcal{W}(X, \mathcal{S}) \setminus \mathfrak{A} = \bigcap_{\rho \in S'} \{\hat{\rho} = 0\}$$

and such that a  $\mu \in S'$  exists which is not the constant zero function. If we take  $x \in X$  such that  $\mu(x) > 0$ , it is clear that  $\Phi_x \in w_{(X, \mathcal{S})}(X) \cap \mathfrak{A}$ , since  $\hat{\mu}(\Phi_x) = \mu(x) > 0$ , and this completes the proof.  $\square$

The main result is now proved.

**Proposition 3.34.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and if  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we have that  $(\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$  is compact, whence  $(\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S})) \in |k\mathbf{AP}_1|$ .*

*Proof.* We only need to verify compactness since the rest of the proposition was proven in Propositions 3.28 and 3.30. To do so, let  $(\mathfrak{A}_i)_{i \in I}$ , with  $I$  a nonempty set, be a family of  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ -closed subsets of  $\mathcal{W}(X, \mathcal{S})$  with the finite intersection property. By Proposition 3.32 we can find for every  $i \in I$  a family  $(\rho_k^i)_{k \in K_i}$  in  $S_0$  (with  $K_i$  a nonempty set) for which

$$\mathfrak{A}_i = \bigcap_{k \in K_i} \{\hat{\rho}_k^i = 0\}.$$

Then  $(\{\hat{\rho}_k^i = 0\})_{(i, k \in \cup_{j \in I} \{j\} \times K_j)}$  again is a family of  $\mathcal{T}_{\mathcal{W}(\mathcal{R}, \mathcal{S})}$ -closed subsets of  $\mathcal{W}(X, \mathcal{S})$  with the finite intersection property since, for every  $n \in \mathbf{N}_0$ ,  $i_1, \dots, i_n \in I$ ,  $k_1 \in K_{i_1}, \dots, k_n \in K_{i_n}$  we have that

$$\bigcap_{s=1}^n \{\widehat{\rho_{k_s}^{i_s}} = 0\} \supset \bigcap_{s=1}^n \mathfrak{A}_{i_s} \neq \emptyset.$$

Therefore, Proposition 3.17 implies that, for every  $n \in \mathbf{N}_0$ ,  $i_1, \dots, i_n \in I$ ,  $k_1 \in K_{i_1}, \dots, k_n \in K_{i_n}$ ,  $\Phi \in \mathcal{W}(X, \mathcal{S})$  exists with

$$\forall s \in \{1, \dots, n\} : \rho_{k_s}^{i_s} \in \Phi,$$

whence  $\bigvee_{s=1}^n \rho_{k_s}^{i_s} \in \Phi \subset \mathcal{S}_0$ . This yields that

$$\Psi \doteq \left\{ \varphi \in \mathcal{S} \mid \exists n \in \mathbf{N}_0 : \exists i_1, \dots, i_n \in I : \right. \\ \left. \exists k_1 \in K_{i_1} : \dots : \exists k_n \in K_{i_n} : \varphi \leq \bigvee_{s=1}^n \rho_{k_s}^{i_s} \right\}$$

is a zero ideal over  $\mathcal{S}$  and therefore, by Proposition 3.11,  $\Psi' \in \mathcal{W}(X, \mathcal{S})$  exists for which  $\Psi \subset \Psi'$ . In particular, we have

$$\forall i \in I, \forall k \in K_i : \rho_k^i \in \Psi',$$

so by Proposition 3.17 it follows that  $\Psi' \in \bigcap_{i \in I} \mathfrak{A}_I$  which proves that  $\bigcap_{i \in I} \mathfrak{A}_i \neq \emptyset$ , and this finishes the proof.  $\square$

Propositions 3.29, 3.33 and 3.34 justify

**Definition 3.35.** If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ , we call

$$w_{(X, \mathcal{S})} : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

the  $\mathcal{S}$ -Wallman-Shanin type compactification of  $(X, \mathcal{R})$ . When  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , we will write  $\mathcal{W}(X) \doteq \mathcal{W}(X, \mathcal{R})$ , respectively  $\mathcal{W}(R) \doteq \mathcal{W}(\mathcal{R}, \mathcal{R})$  and  $w_X \doteq w_{(X, \mathcal{R})}$ , and we call

$$w_X : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X), \mathcal{W}(R))$$

the Wallman-type compactification of  $(X, \mathcal{R})$ .

**Proposition 3.36.** If  $(X, \mathcal{R}) \in |k\mathbf{AP}_1^{ws}|$  and  $\mathcal{S}$  is a WS base for  $(X, \mathcal{R})$ ,

$$w_{(X, \mathcal{S})} : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

is an isomorphism.

*Proof.* Since we already proved that

$$w_{(X, \mathcal{S})} : (X, \mathcal{R}) \longrightarrow (\mathcal{W}(X, \mathcal{S}), \mathcal{W}(\mathcal{R}, \mathcal{S}))$$

is an embedding, only surjectivity remains to be verified, so let  $\Phi \in \mathcal{W}(X, \mathcal{S})$ . Then since  $\Phi \subset \mathcal{S}_0$ ,

$$\{\{\varphi \leq \varepsilon\} \mid \varphi \in \Phi, \varepsilon \in \mathbf{R}_0^+\}$$

is a family of  $\mathcal{T}_{\mathcal{R}}$ -closed and nonempty subsets of  $X$ . Moreover, this family has the finite intersection property because, for every  $\varphi, \mu \in \Phi$ , we have that  $\varphi \vee \mu \in \Phi$ , whence for each  $\varepsilon, \kappa \in \mathbf{R}_0^+$ ,

$$\{\varphi \leq \varepsilon\} \cap \{\mu \leq \kappa\} \supset \{\varphi \vee \mu \leq \varepsilon \wedge \kappa\} \neq \emptyset.$$

By compactness of  $(X, \mathcal{R})$  it follows that

$$\bigcap_{\varphi \in \Phi} \bigcap_{\varepsilon \in \mathbf{R}_0^+} \{\varphi \leq \varepsilon\} \neq \emptyset,$$

so we can pick  $x$  in this intersection. This means that

$$\forall \varphi \in \Phi : \varphi(x) = 0,$$

i.e.,  $\Phi \subset \Phi_x$ , whence by maximality of  $\Phi$ ,  $\Phi = \Phi_x$ .  $\square$

We will end this section with the following corollary.

**Corollary 3.37.** *For  $(X, \mathcal{R}) \in |k\mathbf{AP}_1^{ws}|$  and any  $\Phi \subset \mathcal{R}_0$  which is closed with respect to taking finite suprema, we have that*

$$\bigvee \Phi \in \mathcal{R}_0.$$

*Proof.* We can assume that  $\Phi \neq \emptyset$ . First note that obviously  $\bigvee \Phi \in \mathcal{R}$  and that  $\mathcal{R}$  is a WS base for  $(X, \mathcal{R})$ . Since  $\Phi \subset \mathcal{R}_0$  is closed for the formation of finite suprema,  $\Phi$  is a basis for a zero-ideal over  $\mathcal{R}$ . Applying Proposition 3.11 yields that  $\Phi$  must be contained in a maximal zero-ideal over  $\mathcal{R}$  and, combining this with Proposition 3.36 for  $\mathcal{S} \doteq \mathcal{R}$ , we obtain that  $y \in X$  exists with  $\Phi \subset \Phi_y$ , so  $(\bigvee \Phi)(y) = 0$ .  $\square$

**4. Relation to the topological situation.** In this section we consider the case where  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  and  $\mathcal{S} \doteq \mathcal{R}$ . As a consequence of Propositions 3.5, 3.31 and 3.34, we have for every  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$  that  $(\mathcal{W}(X), \mathcal{W}(\mathcal{R}))$  belongs to  $|k\mathbf{AP}_1^{ws}|$ . If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , we write

$$w_X^* : (X, \mathcal{T}_{\mathcal{R}}) \longrightarrow (w(X), w(\mathcal{T}_{\mathcal{R}}))$$

for the ‘classical’ Wallman compactification of  $(X, \mathcal{T}_{\mathcal{R}})$ . We now want to take a closer look at the relation between the topological bicoreflection  $(\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})})$  of the Wallman-type compactification  $(\mathcal{W}(X), \mathcal{W}(\mathcal{R}))$  and the ‘classical’ Wallman compactification  $(x(X), w(\mathcal{T}_{\mathcal{R}}))$  and we also want to prove that our Wallman-type compactification coincides with the classical one for  $T_1$  topological spaces.

We start by proving the following proposition.

**Proposition 4.1.** *For any  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$ , the approach space*

$$(\mathcal{W}(X), \mathcal{W}(\mathcal{R}_{\mathcal{T}}))$$

*is topological.*

*Proof.* Fix  $\Phi \in \mathcal{W}(X)$  and  $\mathfrak{A} \subset \mathcal{W}(X)$  with

$$\delta_{\mathcal{W}(\mathcal{R}_{\mathcal{T}})}(\Phi, \mathfrak{A}) > 0.$$

If  $\mathfrak{A} = \emptyset$ ,

$$\delta_{\mathcal{W}(\mathcal{R}_{\mathcal{T}})}(\Phi, \mathfrak{A}) = \infty$$

and there is nothing to prove, so assume without loss of generality that  $\mathfrak{A} \neq \emptyset$ . Applying the transition formula (frame  $\longrightarrow$  distance) now yields that we can find  $\varphi \in \mathcal{R}_{\mathcal{T}}$  and  $\varepsilon \in \mathbf{R}_0^+$  with  $\hat{\varphi}|_{\mathfrak{A}} = 0$  and  $\hat{\varphi}(\Phi) > \varepsilon$ . Since  $\varphi \in \mathcal{R}_{\mathcal{T}}$ ,  $\varphi$  is l.s.c. with respect to  $\mathcal{T}$  and therefore  $\{\varphi \leq \varepsilon\}$  is  $\mathcal{T}$ -closed, whence

$$\mu \doteq \theta_{\{\varphi \leq \varepsilon\}} = \delta_{\mathcal{T}}(\cdot, \{\varphi \leq \varepsilon\}) \in \mathcal{R}_{\mathcal{T}}.$$

Let  $\Psi \in \mathfrak{A}$ . If

$$\hat{\mu}(\Psi) > 0$$

were true, it would follow from Proposition 3.18 that

$$\hat{\varphi}(\Psi) > \frac{\varepsilon}{2} > 0,$$

yielding a contradiction. We therefore have that

$$\hat{\mu}|_{\mathfrak{A}} = 0.$$

On the other hand, it also follows that  $\hat{\mu}(\Phi) = \infty$ . (If there would exist  $\beta \in \mathbf{R}^+$  with  $\hat{\mu}(\Phi) \leq \beta$ , then there would exist  $\varphi_1 \in \Phi$  with  $\mu \leq \beta + \varphi_1$ , which would imply that  $\varphi_1|_{\{\varphi > \varepsilon\}} = \infty$ , whence  $\varphi \leq \varphi_1 + \varepsilon$  and therefore  $\hat{\varphi}(\Phi) \leq \varepsilon$  would be true, giving a contradiction.) Again, applying the transition formula (frame  $\rightarrow$  distance) gives us that

$$\delta_{\mathcal{W}(\mathcal{R}_T)}(\Phi, \mathfrak{A}) \geq \hat{\mu}(\Phi) = \infty$$

which concludes the proof, since an approach space the distance function of which only takes the values 0 and  $\infty$  is topological.  $\square$

The next result parallels the classical fact that, when  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$  and  $\mathcal{B}$  is the separating base of all closed subsets of  $X$ , we have that

$$\forall B \in \mathcal{B} : \tilde{B} = \text{cl}_{w(\mathcal{T})}(w_X^*(B)).$$

**Proposition 4.2.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , then, for every  $C \subset X$  with  $C$   $\mathcal{T}_{\mathcal{R}}$ -closed, the following holds:*

$$\text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)) = \{\widehat{\delta(\cdot, C)} = 0\}.$$

*Proof.* Fix  $C \subset X$ ,  $C$   $\mathcal{T}_{\mathcal{R}}$ -closed. If  $C = \emptyset$ , there is nothing to prove since  $\delta(\cdot, \emptyset) = \infty$ , so we may assume that  $C \neq \emptyset$  without loss of generality. We start with proving the inclusion ‘ $\subset$ ,’ so take

$$\Phi \in \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)).$$

Then we have that

$$\delta_{\mathcal{W}(\mathcal{R})}(\Phi, w_X(C)) = 0.$$

On the other hand, we see that  $\widehat{\delta(\cdot, C)} \in \mathcal{W}(\mathcal{R})$  since  $\delta(\cdot, C) \in \mathcal{R}_0$  and because it follows from Proposition 3.20 that

$$\widehat{\delta(\cdot, C)}|_{w_X(C)} = 0,$$

we have that

$$\widehat{\delta(\cdot, C)} \leq \delta_{\mathcal{W}(\mathcal{R})}(\cdot, w_X(C)),$$

yielding that

$$\widehat{\delta(\cdot, C)}(\Phi) = 0.$$

To verify the converse inclusion, fix  $\Phi \in \mathcal{W}(X)$  with

$$\widehat{\delta(\cdot, C)}(\Phi) = 0$$

and take  $f \in \mathcal{W}(\mathcal{R})$  with

$$f|_{w_X(C)} = 0.$$

Then there exists  $\mathcal{S} \subset \mathcal{R}$ ,  $\mathcal{S} \neq \emptyset$ , such that

$$f = \bigvee_{\varphi \in \mathcal{S}} \hat{\varphi}$$

and it again follows from Proposition 3.20 that

$$\forall \varphi \in \mathcal{S} : \varphi|_C = 0,$$

which implies that

$$\forall \varphi \in \mathcal{S} : \varphi \leq \delta(\cdot, C).$$

Finally, this yields that

$$f = \bigvee_{\varphi \in \mathcal{S}} \hat{\varphi} \leq \widehat{\delta(\cdot, C)},$$

whence  $f(\Phi) = 0$ . We have now shown that

$$\delta_{\mathcal{W}(\mathcal{R})}(\Phi, w_X(C)) = \sup\{f(\Phi) \mid f \in \mathcal{W}(\mathcal{R}), f|_{w_X(C)} = 0\} = 0,$$

or equivalently, that

$$\Phi \in \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)). \quad \square$$

**Lemma 4.3.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , then*

$$\{\{\widehat{\delta(\cdot, C)} = 0\} \mid C \in 2^X, C \mathcal{T}_{\mathcal{R}}\text{-closed}\}$$

*is a base for the  $\mathcal{T}_{\mathcal{W}(\mathcal{R})}$ -closed subsets of  $\mathcal{W}(X)$ .*

*Proof.* We have proved in Proposition 3.32 that

$$\{\{\hat{\rho} = 0\} \mid \rho \in \mathcal{R}_0\} \cup \{\emptyset\}$$

is a base for the  $\mathcal{T}_{\mathcal{W}(\mathcal{R})}$ -closed subsets of  $\mathcal{W}(X)$ . On the other hand, we have that

$$\emptyset = \{\widehat{\delta(\cdot, \emptyset)} = 0\}$$

and it follows from Proposition 3.18 that, for every  $\rho \in \mathcal{R}_0$ ,

$$\{\hat{\rho} = 0\} = \bigcap_{\varepsilon \in ]0, \infty]} \{\delta(\cdot, \{\rho \leq \varepsilon\}) = 0\}$$

and so we are done.  $\square$

The first question formulated at the beginning of this section is answered by:

**Proposition 4.4.** *If  $(X, \mathcal{R}) \in |\mathbf{AP}_1^{ws}|$ , then the following assertions are equivalent:*

(1)  $\forall B, C \in 2^X$ ,  $B, C \mathcal{T}_{\mathcal{R}}$ -closed:

$$\{\widehat{\delta(\cdot, B)} = 0\} \cap \{\widehat{\delta(\cdot, C)} = 0\} \subset \{\widehat{\delta(\cdot, B \cap C)} = 0\},$$

(2) *There exists a homeomorphism  $\Theta$  from  $(\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})})$  onto  $(w(X), w(\mathcal{T}_{\mathcal{R}}))$  which leaves  $X$  pointwise fixed, i.e., for which  $w_X^* = \Theta \circ w_X$ .*

*Proof.* We start with showing that (2) implies (1), so assume

$$\Theta : (\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})}) \longrightarrow (w(X), w(\mathcal{T}_{\mathcal{R}}))$$

is a homeomorphism such that  $w_X^* = \Theta \circ w_X$ . Fix  $B, C \in 2^X$  with  $B, C$   $\mathcal{T}_{\mathcal{R}}$ -closed. Then, using Proposition 4.2 and the fact that  $w_X^*$  is injective, it follows that

$$\begin{aligned} & \{\widehat{\delta(\cdot, B)} = 0\} \cap \{\widehat{\delta(\cdot, C)} = 0\} \\ &= \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(B)) \cap \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)) \\ &= \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(\Theta^{-1}(w_X^*(B))) \cap \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(\Theta^{-1}(w_X^*(C))) \\ &= \Theta^{-1}(\text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(B))) \cap \Theta^{-1}(\text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(C))) \\ &= \Theta^{-1}(\text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(B)) \cap \text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(C))) \\ &\stackrel{*}{=} \Theta^{-1}(\text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(B \cap C))) \\ &= \Theta^{-1}(\text{cl}_{w(\mathcal{T}_{\mathcal{R}})}(w_X^*(B \cap C))) \\ &= \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(\Theta^{-1}(w_X^*(B \cap C))) \\ &= \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(B \cap C)) \\ &= \{\widehat{\delta(\cdot, B \cap C)} = 0\}, \end{aligned}$$

where the equality (\*) follows from a result about the classical Wallman compactification which can be found in [12]. We now prove that (1) implies (2). Note that it follows from Propositions 4.2, 4.3 that

$$\{\text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)) \mid C \in 2^X, C \text{ } \mathcal{T}_{\mathcal{R}}\text{-closed}\}$$

is a base for the  $\mathcal{T}_{\mathcal{W}(\mathcal{R})}$ -closed subsets of  $\mathcal{W}(X)$ . On the other hand, it follows from (1) that, for every  $B, C \in 2^X$   $\mathcal{T}_{\mathcal{R}}$ -closed,

$$\begin{aligned} \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(B \cap C)) &= \{\widehat{\delta(\cdot, B \cap C)} = 0\} \\ &= \{\widehat{\delta(\cdot, B)} = 0\} \cap \{\widehat{\delta(\cdot, C)} = 0\} \\ &= \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(B)) \cap \text{cl}_{\mathcal{T}_{\mathcal{W}(\mathcal{R})}}(w_X(C)) \end{aligned}$$

and since it follows from Propositions 3.33, 3.29 and 3.34, that

$$w_X : (X, \mathcal{T}_{\mathcal{R}}) \longrightarrow (\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})})$$



is a  $T_1$  compactification of  $(X, \mathcal{T}_{\mathcal{R}})$ , (2) follows from a result which is stated in [12].  $\square$

*Remark 4.5.* Combining some results from [14] and [11], it was shown in [14] that when we use the notation  $d_E$  for the Euclidean metric on  $]0, 1[$ , the spaces  $(w(]0, 1[), w(\mathcal{T}_{d_E}))$  and  $\mathcal{W}(]0, 1[), \mathcal{T}_{\mathcal{W}(\mathcal{R}_{d_E}))}$  are not homeomorphic, whence Proposition 4.4 (1) is not always fulfilled.

To answer the second question we will now focus on a special kind of weakly symmetric  $T_1$  approach space.

**Definition 4.6.** For  $(X, \delta) \in |\mathbf{AP}|$ , we say that  $(X, \delta)$  is ‘semi-discrete’ if

$$\forall B \in 2_0^X, B \mathcal{T}_{\delta}\text{-closed} : \exists \varepsilon \in \mathbf{R}_0^+ : B^{(\varepsilon)} = B.$$

We use the notation  $sd\mathbf{AP}_1^{ws}$ , respectively  $sd\mathbf{AP}_1^s$ , for the full subcategory of  $\mathbf{AP}_1^{ws}$ , respectively  $\mathbf{AP}_1^s$ , formed by all semi-discrete objects.

If we identify a topological space with the corresponding approach object, we get

$$|\mathbf{TOP}_1| \subset |sd\mathbf{AP}_1^s| \subset |sd\mathbf{AP}_1^{ws}| \subset |\mathbf{AP}_1^{ws}|.$$

We now construct a homeomorphism from  $(\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})})$  onto  $(w(X), w(\mathcal{T}_{\mathcal{R}}))$  which leaves  $X$  pointwise fixed in the semi-discrete case, which at once gives us a better insight into the relation between these two ‘classical’ compactifications.

**Proposition 4.7.** *Let  $(X, \mathcal{R}) \in |sd\mathbf{AP}_1^{ws}|$  and  $\mathcal{C} \in w(X)$ . Then if we put*

$$\begin{aligned} \Phi(\mathcal{C}) \doteq \{ \varphi \in \mathcal{R} \mid \forall \varepsilon \in ]0, \infty], \forall \omega \in [0, \infty[ : \\ \exists C_\varepsilon^\omega \in \mathcal{C} : \varphi \wedge \omega \leq \delta(\cdot, C_\varepsilon^\omega) + \varepsilon \}, \end{aligned}$$

*we have that  $\Phi(\mathcal{C}) \in \mathcal{W}(X)$ .*

*Proof.* We start with verifying that  $\Phi(\mathcal{C})$  is indeed a zero ideal over  $\mathcal{R}$ . Since  $\mathcal{C} \neq \emptyset$ , it is clear that  $\Phi(\mathcal{C}) \neq \emptyset$ . On the other hand, because

$\emptyset \notin \mathcal{C}$ , it follows for every  $\varphi \in \Phi(\mathcal{C})$  if we take  $C_\varepsilon^\omega \in \mathcal{C}$  for all  $\varepsilon \in ]0, \infty]$ ,  $\omega \in [0, \infty[$  according to the definition of  $\Phi(\mathcal{C})$ , that

$$\forall \varepsilon \in ]0, \infty], \forall \omega \in [0, \infty[ : \left( \inf_{x \in X} \varphi(x) \right) \wedge \omega \leq \inf_{x \in X} \delta(x, C_\varepsilon^\omega) + \varepsilon = \varepsilon,$$

whence  $\varphi \in \mathcal{R}_0$  for all  $\varphi \in \Phi(\mathcal{C})$  and it is also clear that (I2) is fulfilled. Now let  $\varphi, \psi \in \Phi(\mathcal{C})$  and fix  $\varepsilon \in ]0, \infty]$ ,  $\omega \in [0, \infty[$ . Then  $B_\varepsilon^\omega$  and  $C_\varepsilon^\omega \in \mathcal{C}$  exist with

$$\varphi \wedge \omega \leq \delta(\cdot, B_\varepsilon^\omega) + \varepsilon$$

and

$$\psi \wedge \omega \leq \delta(\cdot, C_\varepsilon^\omega) + \varepsilon.$$

Since  $\mathcal{C}$  is a  $\mathcal{T}_\delta$ -closed filter,  $B_\varepsilon^\omega \cap C_\varepsilon^\omega \in \mathcal{C}$  and, because we also have that

$$\begin{aligned} (\varphi \vee \psi) \wedge \omega &= (\varphi \wedge \omega) \vee (\psi \wedge \omega) \\ &\leq \delta(\cdot, B_\varepsilon^\omega) \vee \delta(\cdot, C_\varepsilon^\omega) + \varepsilon \\ &\leq \delta(\cdot, B_\varepsilon^\omega \cap C_\varepsilon^\omega) + \varepsilon, \end{aligned}$$

it follows by the arbitrariness of  $\varepsilon, \omega$  that  $\varphi \vee \psi \in \Phi(\mathcal{C})$ . We now come to prove the maximality of  $\Phi(\mathcal{C})$ . Take  $\varphi \in \mathcal{R}_0$  such that

$$\forall \varphi' \in \Phi(\mathcal{C}) : \varphi \vee \varphi' \sim 0.$$

Because

$$\forall C \in \mathcal{C} : \delta(\cdot, C) \in \Phi(\mathcal{C}),$$

it follows that

$$\forall C \in \mathcal{C} : \varphi \vee \delta(\cdot, C) \sim 0,$$

whence

$$\forall C \in \mathcal{C}, \forall \varepsilon \in ]0, \infty] : \{\varphi \leq \varepsilon\} \cap C^{(\varepsilon)} \neq \emptyset,$$

which in its turn implies that

$$\forall C \in \mathcal{C}, \forall \varepsilon, \gamma \in ]0, \infty] : \{\varphi \leq \varepsilon\} \cap C^{(\gamma)} \neq \emptyset.$$

(To see that the last step is valid, we distinguish between two cases: if  $\gamma \leq \varepsilon$ , the conclusion follows from the fact that  $\{\varphi \leq \gamma\} \cap C^{(\gamma)} \neq \emptyset$  and  $\{\varphi \leq \gamma\} \subset \{\varphi \leq \varepsilon\}$  whereas for  $\gamma > \varepsilon$  it is true because

$\{\varphi \leq \varepsilon\} \cap C^{(\varepsilon)} \neq \emptyset$  and  $C^{(\varepsilon)} \subset C^{(\gamma)}$ .) Using the semi-discreteness of  $(X, \mathcal{R})$ , we obtain that

$$\forall C \in \mathcal{C}, \forall \varepsilon \in ]0, \infty] : C \cap \{\varphi \leq \varepsilon\} \neq \emptyset,$$

from which it follows by the maximality of  $\mathcal{C}$  and because all regular functions are l.s.c. with respect to the topological coreflection, that

$$\forall \varepsilon \in ]0, \infty] : \{\varphi \leq \varepsilon\} \in \mathcal{C}.$$

On the other hand, we have for every  $\varepsilon \in ]0, \infty]$  that  $(\varphi - \varepsilon) \vee 0 \in \mathcal{R}$  with  $((\varphi - \varepsilon) \vee 0)|_{\{\varphi \leq \varepsilon\}} = 0$ , yielding that

$$\forall \varepsilon \in ]0, \infty] : (\varphi - \varepsilon) \vee 0 \leq \delta(\cdot, \{\varphi \leq \varepsilon\}),$$

or equivalently that

$$\forall \varepsilon \in ]0, \infty] : \varphi \leq \delta(\cdot, \{\varphi \leq \varepsilon\}) + \varepsilon,$$

so we have proved that  $\varphi \in \Phi(\mathcal{C})$  and we are done.  $\square$

**Proposition 4.8.** *If  $(X, \mathcal{R}) \in |\text{sd}\mathbf{AP}_1^{ws}|$  and  $\Phi \in \mathcal{W}(X)$ , then*

$$\{\{\varphi \leq \varepsilon\} \mid \varphi \in \Phi, \varepsilon \in ]0, \infty]\}$$

*generates a maximal  $\mathcal{T}_{\mathcal{R}}$ -closed filter, which we denote by  $\mathcal{C}(\Phi)$ .*

*Proof.* First of all, note that the fact that  $\Phi \subset \mathcal{R}_0$  implies that  $\{\varphi \leq \varepsilon\}$  is a nonempty  $\mathcal{T}_{\mathcal{R}}$ -closed subset of  $X$  for every  $\varphi \in \Phi$  and every  $\varepsilon \in ]0, \infty]$ . Since, moreover,  $\Phi \neq \emptyset$  and for every  $\varphi, \psi \in \Phi$  and every  $\varepsilon, \gamma \in ]0, \infty]$  we have that  $\varphi \vee \psi \in \Phi$ ,  $\varepsilon \wedge \gamma \in ]0, \infty]$  and

$$\{\varphi \leq \varepsilon\} \cap \{\psi \leq \gamma\} \supset \{\varphi \vee \psi \leq \varepsilon \wedge \gamma\},$$

it is obvious that

$$\{\{\varphi \leq \varepsilon\} \mid \varphi \in \Phi, \varepsilon \in ]0, \infty]\}$$

generates a  $\mathcal{T}_{\mathcal{R}}$ -closed filter, which we denote by  $\mathcal{C}(\Phi)$ . To prove that  $\mathcal{C}(\Phi)$  is maximal, fix  $B \in 2_0^X$   $\mathcal{T}_{\mathcal{R}}$ -closed such that

$$\forall C \in \mathcal{C}(\Phi) : B \cap C \neq \emptyset.$$

Then surely

$$\forall \varphi \in \Phi, \forall \varepsilon \in ]0, \infty] : B \cap \{\varphi \leq \varepsilon\} \neq \emptyset$$

which implies that

$$\forall \varphi \in \Phi : \delta(\cdot, B) \vee \varphi \sim 0.$$

Since  $\Phi$  is a maximal zero ideal over  $\mathcal{R}$  and because  $\delta(\cdot, B) \in \mathcal{R}_0$ , it follows that  $\delta(\cdot, B) \in \Phi$ , whence, by definition of  $\mathcal{C}(\Phi)$ ,

$$\forall \varepsilon \in ]0, \infty] : B^{(\varepsilon)} = \{\delta(\cdot, B) \leq \varepsilon\} \in \mathcal{C}(\Phi)$$

and because  $(X, \mathcal{R})$  is semi-discrete, this proves that  $B \in \mathcal{C}(\Phi)$ .  $\square$

**Proposition 4.9.** *If  $(X, \mathcal{R}) \in |\text{sd}\mathbf{AP}_1^{\omega s}|$ , then we have that*

$$\forall \mathcal{C} \in w(X) : \mathcal{C} = \mathcal{C}(\Phi(\mathcal{C})).$$

*Proof.* Take  $\mathcal{C} \in w(X)$ . If  $C \in \mathcal{C}$ , we have that  $\delta(\cdot, C) \in \Phi(\mathcal{C})$ , and because, by semi-discreteness of  $(X, \mathcal{R})$ , there exists  $\varepsilon \in ]0, \infty]$  with  $C^{(\varepsilon)} = C$ , it follows that

$$C = C^{(\varepsilon)} = \{\delta(\cdot, C) \leq \varepsilon\} \in \mathcal{C}(\Phi(\mathcal{C})),$$

proving that  $\mathcal{C} \subset \mathcal{C}(\Phi(\mathcal{C}))$ , thus by maximality of  $\mathcal{C}$ , the equality follows.  $\square$

**Proposition 4.10.** *If  $(X, \mathcal{R}) \in |\text{sd}\mathbf{AP}_1^{ws}|$ , then we have that*

$$\forall \Phi \in \mathcal{W}(X) : \Phi = \Phi(\mathcal{C}(\Phi)).$$

*Proof.* Fix  $\Phi \in \mathcal{W}(X)$ . Since

$$\forall \varphi \in \Phi, \forall \varepsilon \in ]0, \infty] : \{\varphi \leq \varepsilon\} \in \mathcal{C}(\Phi)$$

and, as indicated in the proof of Proposition 4.7,

$$\forall \varphi \in \Phi, \forall \varepsilon \in ]0, \infty] : \varphi \leq \delta(\cdot, \{\varphi \leq \varepsilon\}) + \varepsilon,$$

it is clear that  $\Phi \subset \Phi(\mathcal{C}(\Phi))$  so, by the maximality of  $\Phi$ , we are done.  
 $\square$

**Proposition 4.11.** *If  $(X, \mathcal{R}) \in |\text{sd}\mathbf{AP}_1^{ws}|$ , we have that*

$$\Theta : (\mathcal{W}(X), \mathcal{T}_{\mathcal{W}(\mathcal{R})}) \longrightarrow (w(X), w(\mathcal{T}_{\mathcal{R}})) : \Phi \longrightarrow \mathcal{C}(\Phi)$$

*is a homeomorphism which leaves  $X$  pointwise fixed.*

*Proof.* It directly follows from Propositions 4.7, 4.8, 4.9 and 4.10 that  $\Theta$  is a bijection and that  $\Theta^{-1}(\mathcal{C}) = \Phi(\mathcal{C})$  for every  $\mathcal{C} \in w(X)$ . We now prove the following claim:

$$\forall C \in 2_0^X \text{ with } C \text{ } \mathcal{T}_{\mathcal{R}}\text{-closed, } \forall \Phi \in \mathcal{W}(X) : \widehat{\delta(\cdot, C)}(\Phi) = 0 \iff C \in \mathcal{C}(\Phi).$$

Therefore, fix  $C \in 2_0^X$  with  $C$   $\mathcal{T}_{\mathcal{R}}$ -closed and  $\Phi \in \mathcal{W}(X)$ . If  $\widehat{\delta(\cdot, C)}(\Phi) = 0$ , it follows from Proposition 3.18 that

$$\forall \varepsilon \in ]0, \infty] : \delta(\cdot, C^{(\varepsilon)}) = \delta(\cdot, \{\delta(\cdot, C) \leq \varepsilon\}) \in \Phi,$$

so by definition of  $\mathcal{C}(\Phi)$ , it follows that

$$\forall \varepsilon \in ]0, \infty] : \{\delta(\cdot, C^{(\varepsilon)}) \leq \varepsilon\} \in \mathcal{C}(\Phi).$$

By the triangular inequality, we have that  $\delta(\cdot, C) \leq \delta(\cdot, C^{(\varepsilon)}) + \varepsilon$  for all  $\varepsilon \in ]0, \infty]$ , whence

$$\forall \varepsilon \in ]0, \infty] : C^{(2\varepsilon)} = \{\delta(\cdot, C) \leq 2\varepsilon\} \supset \{\delta(\cdot, C^{(\varepsilon)}) \leq \varepsilon\},$$

whence

$$\forall \varepsilon \in ]0, \infty] : C^{(2\varepsilon)} \in \mathcal{C}(\Phi)$$

and because  $(X, \mathcal{R})$  is semi-discrete we can conclude that  $C \in \mathcal{C}(\Phi)$ . Conversely, if  $C \in \mathcal{C}(\Phi)$ , it is a direct consequence of Theorem 4.10 that  $\delta(\cdot, C) \in \Phi(\mathcal{C}(\Phi)) = \Phi$ , whence  $\widehat{\delta(\cdot, C)}(\Phi) = 0$  which finishes the verification of the claim. This shows that

$$\forall C \in 2^X \text{ with } C \text{ } \mathcal{T}_{\mathcal{R}}\text{-closed} : \Theta(\{\widehat{\delta(\cdot, C)} = 0\}) = \{\mathcal{C} \in w(X) \mid C \in \mathcal{C}\},$$

and, thanks to Proposition 4.3 and the fact that

$$\{\mathcal{C} \in w(X) \mid \mathcal{C} \in \mathcal{C} \mid \mathcal{C} \in 2^X, \mathcal{C} \mathcal{T}_{\mathcal{R}}\text{-closed}\}$$

is a base for the  $w(\mathcal{T}_{\mathcal{R}})$ -closed subsets of  $w(X)$ , we have shown that  $\Theta$  is a homeomorphism. On the other hand, we have that

$$\forall x \in X, \forall \varepsilon \in ]0, \infty] : \{x\}^{(\varepsilon)} = \{\delta(\cdot, \{x\}) \leq \varepsilon\} \in \mathcal{C}(\Phi_x)$$

and, using the semi-discreteness, we obtain that

$$\forall x \in X : \{x\} \in \mathcal{C}(\Phi_x)$$

whence  $w_X^*(x) \subset \mathcal{C}(\Phi_x)$  or, equivalently,  $\mathcal{C}(\Phi_x) = w_X^*(x)$  for all  $x \in X$ .  
□

We now immediately get that our compactification theory extends the topological one.

**Corollary 4.12.** *If  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$ , we have that*

$$\Theta : (\mathcal{W}(X), \mathcal{W}(\mathcal{R}_{\mathcal{T}})) \longrightarrow (w(X), \mathcal{R}_{w(\mathcal{T})}) : \Phi \longrightarrow \mathcal{C}(\Phi)$$

*is an isomorphism which leaves  $X$  pointwise fixed.*

*Proof.* This follows directly from the previous proposition and the fact that, for  $(X, \mathcal{T}) \in |\mathbf{TOP}_1|$ , the approach space  $(\mathcal{W}(X), \mathcal{W}(\mathcal{R}_{\mathcal{T}}))$  is topological. □

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