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SECOND-ORDER DIFFERENTIAL OPERATORS WITH INTEGRAL BOUNDARY CONDITIONS AND GENERATION OF ANALYTIC SEMIGROUPS

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ABSTRACT. Consider a second-order differential operator $Lu = u'' + q_1(x)u' + q_0(x)u$ with integral boundary conditions of the form

$$\int_a^b R_i(t)u(t)\,dt + \int_a^b S_i(t)u'(t)\,dt = 0, \quad i = 1, 2.$$

We study sufficient conditions on the functions R_i and S_i, $i = 1, 2$, such that the operator L is the generator of an analytic semigroup of operators on $L^p(a, b)$. The generation of analytic semigroups is proved by showing the estimate

$$\|R(\lambda : L)\| \leq \frac{M}{|\lambda|}$$

for the resolvent operator in a suitable sector of the complex plane. The motivation for this work is to generalize the results in [**3**], where nonseparated boundary conditions were considered.

1. Introduction. We consider a second-order differential operator of the form

$$l(u) = u'' + q_1(x)u' + q_0(x)u, \quad x \in (a, b), \tag{1.1}$$

where each $q_i(x)$ is a regular function with complex values. We can associate to $l(u)$ a variety of boundary conditions, in particular, the *nonseparated* ones:

$$\begin{cases} a_1u(a) + b_1u'(a) + c_1u(b) + d_1u'(b) = 0, \\ a_2u(a) + b_2u'(a) + c_2u(b) + d_2u'(b) = 0. \end{cases} \tag{1.2}$$

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The spectral theory of the natural operator L associated to problem (1.1)–(1.2) was initiated by Birkhoff [**1**]–[**2**] and continued by many other authors (see [**3**], [**6**] and the references therein).

In our paper [**3**] we proved that, for Birkhoff-regular boundary conditions, the operator L is the generator of an analytic semigroup of bounded linear operators on $L^p(a,b)$, for $1 \le p \le \infty$. Moreover, when $p \neq \infty$ this analytic semigroup is strongly continuous. In the present work we will try to generalize the results in [**3**] to the case of *integral boundary conditions*:

$$B_i(u) \equiv \int_a^b R_i(t)u(t)\,dt + \int_a^b S_i(t)u'(t)\,dt = 0, \quad i = 1,2 \tag{1.3}$$

where the functions R_i and S_i are in $L^\infty(a,b)$. This kind of boundary condition generalizes the nonseparated ones, at least in a formal sense: when $R_i = a_i\delta_a + c_i\delta_b$ and $S_i = b_i\delta_a + d_i\delta_b$ (δ_x being the Dirac delta function at the point x), condition (1.3) is of the form (1.2). Let L_p be the realization of problem (1.1)–(1.3) in $L^p(a,b)$, that is, $L_p u = l(u)$, with domain

$$D(L_p) = \{u \in W^{2,p}(a,b) : B_i(u) = 0,\ i = 1,2\}.$$

(Here $W^{2,p}(a,b)$ stands for the standard Sobolev space of order $(2,p)$.)

As is well-known [**5**, Chapter 2], for proving that L_p generates an analytic semigroup on $L^p(a,b)$ we must show that L_p is a sectorial operator, that is

(i) the resolvent set $\rho(L_p)$ contains a sector

$$\Sigma_{\delta,r} = \{\lambda \in \mathbf{C} : |\arg(\lambda - r)| < \delta,\ \lambda \neq r\}$$

for some $\pi/2 < \delta < \pi$ and $r \in \mathbf{R}$; and

(ii) there exists a constant M such that

$$\|R(\lambda : L_p)\| \le \frac{M}{|\lambda - r|}, \quad \forall\, \lambda \in \Sigma_{\delta,r}, \tag{1.4}$$

where $R(\lambda : L_p) = (\lambda I - L_p)^{-1}$ is the resolvent operator associated to λ, and the norm is the usual for bounded linear operators on $L^p(a,b)$.

When the domain $D(L_p)$ is dense in $L^p(a,b)$, the semigroup generated by L_p is strongly continuous [**5**, Proposition 2.1.4].

We briefly outline the structure of the paper. First of all, we localize the spectrum of L_p and prove (i). Then, in order to analyze the resolvent of L_p, for each $\lambda \in \Sigma_{\delta,r} \subset \rho(L_p)$ we express the associated resolvent operator in integral form:

$$R(\lambda : L_p)f = -\int_a^b G(\cdot, s; \lambda) f(s)\, ds, \quad f \in L^p(a,b), \tag{1.5}$$

where $G(x,s;\lambda)$ is the Green's function for the problem

$$\begin{cases} l(u) - \lambda u = f & \text{in} \quad (a,b), \\ B_i(u) = 0, & i = 1,2. \end{cases}$$

Following the technique in [**3**], we should bound (1.5) both in the spaces $L^1(a,b)$ and $L^\infty(a,b)$ by means of a suitable formula for $G(x,s;\lambda)$. Then, by interpolation, we would obtain similar bounds on $L^p(a,b)$ for $1 \le p \le \infty$. In $L^1(a,b)$ we prove bound (1.4) for a certain class of boundary conditions that we call *regular* in analogy with the Birkhoff-regular boundary conditions. Unfortunately, in $L^\infty(a,b)$ we do not arrive to bound (1.4); we give an example where it is shown that, even for regular boundary conditions, L_∞ cannot be a sectorial operator. We could then try to bound (1.5) directly in $L^p(a,b)$, but we do not get adequate bounds, so we restrict ourselves to the case $p=1$. For regular boundary conditions L_1 is a sectorial operator, so it generates an analytic semigroup on $L^1(a,b)$. As the domain $D(L_1)$ is not necessarily dense in $L^1(a,b)$, we cannot assume that this semigroup is strongly continuous.

However, for proving (ii) it will be necessary to make some additional regularity hypotheses on the functions R_i and S_i. First we will suppose that R_i and S_i are of class C^1 on $[a,b]$, and this will allow us to obtain bound (1.4) in $L^1(a,b)$. Then, by means of an approximation method, we will show that it is sufficient with supposing that R_i and S_i are only continuous in $[a,b]$.

Finally, in the Appendix, we consider a mixed problem with both nonseparated and integral boundary conditions:

$$\begin{cases} a_0 u(a) + b_0 u'(a) + c_0 u(b) + d_0 u'(b) = 0, \\ \displaystyle\int_a^b R(t) u(t)\, dt + \int_a^b S(t) u'(t)\, dt = 0, \end{cases} \tag{1.6}$$

where R and S are continuous functions. We prove that, for a certain class of boundary conditions, the operator M_1 associated to problem (1.1)–(1.6) is also the generator of an analytic semigroup on $L^1(a,b)$ that, in general, will not be strongly continuous.

2. Characteristic determinant and Green's function. Consider the differential system

$$\text{(2.1)} \qquad \begin{cases} l(u) = u'' + q_1(x)u' + q_0(x)u & \text{in} \quad (a,b), \\ B_i(u) = \displaystyle\int_a^b R_i(t)u(t)\,dt + \int_a^b S_i(t)u'(t)\,dt = 0 & i = 1,2, \end{cases}$$

where $q_1 \in C^1([a,b];\mathbf{C})$, $q_0 \in C([a,b];\mathbf{C})$ and R_i, S_i are in $L^\infty(a,b)$. In every space $L^p(a,b)$, $1 \le p \le \infty$, system (2.1) has a natural realization given by

$$L_p u = l(u), \qquad D(L_p) = \{u \in W^{2,p}(a,b) : B_i(u) = 0,\ i = 1,2\}.$$

Take an arbitrary $\lambda \in \mathbf{C}$ and let $u_1(x) \equiv u_1(x;\lambda)$ and $u_2(x) \equiv u_2(x;\lambda)$ be two solutions of the equation $l(u) = \lambda u$ with boundary conditions given, respectively, by $u_1(a) = 0$, $u_1'(a) = 1$ and $u_2(a) = 1$, $u_2'(a) = 0$. We define the *characteristic determinant* $\Delta(\lambda)$ of system (2.1) to be

$$\text{(2.2)} \qquad \Delta(\lambda) = \begin{vmatrix} B_1(u_1) & B_1(u_2) \\ B_2(u_1) & B_2(u_2) \end{vmatrix}.$$

It is easy to prove that the spectrum of L_p can be characterized as

$$\sigma(L_p) = \{\lambda \in \mathbf{C} : \Delta(\lambda) = 0\}.$$

As we are interested in operators for which condition (i) holds, we will suppose that $\Delta(\lambda)$ is not identically zero. For construction, the characteristic determinant is an entire function of λ, so the spectrum of L_p is as much a denumerable set with no finite accumulation points.

Let $\lambda \in \mathbf{C}$ be such that $\Delta(\lambda) \neq 0$, and define the function

$$\text{(2.3)} \qquad N(x,s;\lambda) = \begin{vmatrix} u_1(x) & u_2(x) & g(x,s;\lambda) \\ B_1(u_1) & B_1(u_2) & B_1(g)_x \\ B_2(u_1) & B_2(u_2) & B_2(g)_x \end{vmatrix}$$

where $x, s \in [a,b]$ and the symbol $B_i(g)_x$ means that the operation B_i is made over the function $g(x,s;\lambda)$ with respect to the variable x. Here $g(x,s;\lambda)$ is defined as follows

$$g(x,s;\lambda) = \pm\frac{1}{2}\,\frac{u_1(x)u_2(s) - u_1(s)u_2(x)}{u_1'(s)u_2(s) - u_1(s)u_2'(s)}, \tag{2.4}$$

where it takes the plus sign for $x > s$ and the minus sign for $x < s$. It is not difficult to prove that

$$G(x,s;\lambda) = \frac{N(x,s;\lambda)}{\Delta(\lambda)} \tag{2.5}$$

is Green's function for the problem

$$\begin{cases} l(u) - \lambda u = f & \text{in} \quad (a,b), \\ B_i(u) = 0 & i = 1,2. \end{cases}$$

This means that the resolvent operator associated to λ can be expressed as a Hilbert-Schmidt operator:

$$R(\lambda : L_p)f = -\int_a^b G(\cdot,s;\lambda)f(s)\,ds, \quad f \in L^p(a,b). \tag{2.6}$$

Formulae (2.2)–(2.6) are similar to those in [**3**], [**2**], [**6**], and they will be the key for obtaining bound (1.4).

3. Some simplifications on the original problem. We are going to introduce some modifications on the original problem in order to simplify the calculations to be made eventually.

By means of a linear change of variables, we can consider problem (2.1) on the interval $(0,1)$ instead of (a,b), so we begin with the system

$$\begin{cases} \tilde{l}(v) = v'' + \tilde{q}_1(x)v' + \tilde{q}_0(x)v & \text{in} \quad (0,1), \\ \tilde{B}_i(v) = \displaystyle\int_0^1 \tilde{R}_i(t)v(t)\,dt + \int_0^1 \tilde{S}_i(t)v'(t)\,dt = 0, & i = 1,2 \end{cases} \tag{3.1}$$

that has a natural realization $\tilde{L}_p$.

The next step is to eliminate the term $\tilde{q}_1 v'$ in $\tilde{l}(v)$, as it is made in [**6**, Chapter II]. Consider the C^2-diffeomorphism ϕ given by

$$\phi(t) = \exp\left(-\frac{1}{2}\int_0^t \tilde{q}_1(s)\,ds\right), \quad t \in [0,1],$$

and let $M_\phi : L^p(0,1) \to L^p(0,1)$ be multiplication by ϕ:

$$M_\phi u = \phi u, \quad u \in L^p(0,1).$$

We have that M_ϕ is a bounded linear operator with bounded inverse $M_\phi^{-1} = M_{\phi^{-1}}$, and M_ϕ also maps $W^{2,p}(0,1)$ one-to-one onto $W^{2,p}(0,1)$.

For $v \in W^{2,p}(0,1)$, define $u = M_\phi^{-1} v$, so $\tilde{l}(v) = \phi l(u)$, where $l(u) = u'' + q(x)u$ with $q = \tilde{q}_0 - \tilde{q}_1^2/4 - \tilde{q}_1'/2$. Then system (3.1) can be written as

$$(3.2) \quad \begin{cases} l(u) = u'' + q(x)u & \text{in} \quad (0,1), \\ B_i(u) = \displaystyle\int_0^1 R_i(t)u(t)\,dt + \int_0^1 S_i(t)u'(t)\,dt = 0, & i = 1,2 \end{cases}$$

where $R_i = \phi(\tilde{R}_i - (\tilde{S}_i\tilde{q}_1)/2)$ and $S_i = \phi\tilde{S}_i$. If L_p is the realization of system (3.2), then the following relation holds

$$\tilde{L}_p = M_\phi L_p M_\phi^{-1}.$$

From this equality it is easy to see that the resolvent sets of L_p and $\tilde{L}_p$ are equal, and

$$R(\lambda : \tilde{L}_p) = M_\phi R(\lambda : L_p) M_\phi^{-1}, \quad \forall\, \lambda \in \rho(L_p) = \rho(\tilde{L}_\rho).$$

As M_ϕ and M_ϕ^{-1} are bounded operators, it is then sufficient to prove bound (1.4) for L_p instead of $\tilde{L}_p$.

Finally we write the operator L_p as

$$L_p = T_p + Q_p,$$

where $T_p u = u''$ and $Q_p u = qu$ for every $u \in D(L_p) = D(T_p) = D(Q_p)$. The relation between the resolvents of L_p and T_p is given in the following result [**3**, Proposition 4.2].

Proposition 3.1. *If $\lambda \in \rho(T_p)$ is such that $\|R(\lambda : T_p)\| \le \|Q_p\|^{-1}/2$, then $\lambda \in \rho(L_p)$ and $\|R(\lambda : L_p)\| \le 2\|R(\lambda : T_p)\|$.*

If we prove bound (1.4) for the operator T_p, Proposition 3.1 allows us to deduce a similar bound for L_p, with the advantage that for T_p the calculations to be made in the following sections will be much simpler.

4. Bounds on the resolvent of the operator T_p. Due to the considerations above, we center our attention on the operator $T_p u = u''$, $D(T_p) = \{u \in W^{2,p}(0,1) : B_i(u) = 0,\ i = 1,2\}$, where

$$B_i(u) = \int_0^1 R_i(t)u(t)\,dt + \int_0^1 S_i(t)u'(t)\,dt, \quad i = 1,2.$$

Given an arbitrary $\delta \in (\pi/2, \pi)$, we consider the sector

$$\Sigma_\delta = \{\lambda \in \mathbf{C} : |\arg(\lambda)| < \delta,\ \lambda \ne 0\}.$$

For $\lambda \in \Sigma_\delta$, define ρ as the square root of λ with positive real part (thus $\rho \in \Sigma_{\delta/2}$). In Section 2 we constructed the characteristic determinant $\Delta(\lambda)$ and the Green's function $G(x,s;\lambda)$ for a general problem from a specific fundamental system of solutions of $l(u) = \lambda u$. For $\lambda \ne 0$, we can consider a simpler fundamental system of solutions of $u'' = \lambda u = \rho^2 u$, that it is given by $u_1(t) = e^{-\rho t}$ and $u_2(t) = e^{\rho t}$. Redefining (2.2)–(2.5) with this new fundamental system, it is straightforward to prove that formula (2.6) remains true for the operator T_p. In particular, we have that $\sigma(T_p) \setminus \{0\} = \{\lambda \in \mathbf{C} : \Delta(\lambda) = 0\}$. As $0 \notin \Sigma_\delta$, we will not take care of the value $\lambda = 0$.

Remark 4.1. A fundamental system for $\lambda = 0$ is given by $u_1(t) = 1$, $u_2(t) = t$. We then have that $\lambda = 0$ is an eigenvalue of T_p if and only if

$$\int_0^1 (tR_1(t) + S_1(t))\,dt \int_0^1 R_2(t)\,dt = \int_0^1 R_1(t)\,dt \int_0^1 (tR_2(t) + S_2(t))\,dt.$$

In the following we are going to deduce adequate formulae for $\Delta(\lambda)$ and $G(x,s;\lambda)$. First of all, for $i,j=1,2$, we have

$$B_i(u_j) = \int_0^1 R_i(t)e^{(-1)^j\rho t}\,dt + (-1)^j\rho\int_0^1 S_i(t)e^{(-1)^j\rho t}\,dt,$$

so we obtain from (2.2):

(4.1)
$$\begin{aligned}\Delta(\lambda) &= \left(\int_0^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\left(\int_0^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\\ &\quad - \left(\int_0^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\left(\int_0^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right).\end{aligned}$$

Formula (2.4) has the form:

$$g(x,s;\lambda) = \begin{cases} \dfrac{1}{4\rho}\left(e^{\rho(x-s)} - e^{\rho(s-x)}\right) & \text{if} \quad x>s,\\ \dfrac{1}{4\rho}\left(e^{\rho(s-x)} - e^{\rho(x-s)}\right) & \text{if} \quad x<s. \end{cases}$$

Thus we have

$$\begin{aligned}&B_i(g)_x\\ &\quad= \frac{e^{\rho s}}{4\rho}\left(\int_0^s (R_i(t)-\rho(S_i(t))e^{-\rho t}\,dt + \int_s^1 (-R_i(t)+\rho S_i(t))e^{-\rho t}\,dt\right)\\ &\qquad + \frac{e^{-\rho s}}{4\rho}\left(-\int_0^s (R_i(t)+\rho(S_i(t))e^{\rho t}\,dt + \int_s^1 (R_i(t)+\rho S_i(t))e^{\rho t}\,dt\right).\end{aligned}$$

After a long calculation, formula (2.3) can be written as

$$\begin{aligned}&N(x,s;\lambda)\\ &\quad= \varphi(x,s;\lambda) + \frac{e^{\rho(x+s)}}{2\rho}\left[\left(\int_0^s (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\right.\\ &\qquad\cdot\left(\int_s^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right) - \left(\int_0^s (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\\ &\qquad\qquad\left.\cdot\left(\int_s^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\right]\end{aligned}$$

$$+\frac{e^{-\rho(x+s)}}{2\rho}\bigg[\bigg(\int_0^s (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\bigg)\cdot\bigg(\int_s^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\bigg)-\bigg(\int_0^s (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\bigg)\cdot\bigg(\int_s^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\bigg)\bigg].$$

The function $\varphi(x,s;\lambda)$ is given by

$$\varphi(x,s;\lambda)=\begin{cases}\varphi_1(x,s;\lambda) & \text{if} \quad x>s,\\ \varphi_2(x,s;\lambda) & \text{if} \quad x<s,\end{cases} \tag{4.2}$$

where

$$\begin{aligned}\varphi_1(x,s;\lambda)&\\ =\frac{e^{\rho(x-s)}}{2\rho}&\bigg[\bigg(\int_0^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\bigg)\cdot\bigg(\int_0^s (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\bigg)-\bigg(\int_0^s (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\bigg)\cdot\bigg(\int_0^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\bigg)\bigg]\\ +\frac{e^{\rho(s-x)}}{2\rho}&\bigg[\bigg(\int_0^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\bigg)\cdot\bigg(\int_0^s (R_2(t)-\rho S_2(t))-e^{-\rho t}\,dt\bigg)-\bigg(\int_0^s (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\bigg)\cdot\bigg(\int_0^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\bigg)\bigg]\end{aligned}$$

and

$$\begin{aligned}\varphi_2(x,s;\lambda)&\\ =\frac{e^{\rho(x-s)}}{2\rho}&\bigg[\bigg(\int_s^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\bigg)\end{aligned}$$

$$
\begin{aligned}
&\cdot\left(\int_0^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)-\left(\int_0^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\int_s^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)\Bigg]\\
&+\frac{e^{\rho(s-x)}}{2\rho}\Bigg[\left(\int_s^1 (R_1(t)-\rho S_1(t))e^{-\rho t}\,dt\right)\\
&\cdot\left(\int_0^1 (R_2(t)+\rho S_2(t))e^{\rho t}\,dt\right)-\left(\int_0^1 (R_1(t)+\rho S_1(t))e^{\rho t}\,dt\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\cdot\left(\int_s^1 (R_2(t)-\rho S_2(t))e^{-\rho t}\,dt\right)\Bigg].
\end{aligned}
$$

We have that $|R_i(t)\pm\rho S_i(t)|\le\|R_i\|_\infty+|\rho|\|S_i\|_\infty$, $\forall\, t\in[0,1]$, where $\|\cdot\|_\infty$ is the supremum norm . Now we have

$$
\begin{aligned}
|N(x,s;\lambda)|\le|\varphi(x,s;\lambda)|&+\frac{e^{(x+s)\mathrm{Re}\,(\rho)}}{|\rho|}\\
&\cdot(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\
&\cdot\left(\int_0^s e^{-t\mathrm{Re}\,(\rho)}\,dt\right)\left(\int_s^1 e^{-t\mathrm{Re}\,(\rho)}\,dt\right)\\
&+\frac{e^{-(x+s)\mathrm{Re}\,(\rho)}}{|\rho|}(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\
&\cdot\left(\int_0^s e^{t\mathrm{Re}\,(\rho)}\,dt\right)\left(\int_s^1 e^{t\mathrm{Re}\,(\rho)}\,dt\right).
\end{aligned}
$$

Evaluating the integrals, we obtain

$$
\begin{aligned}
|N(x,s;\lambda)|\le{}&|\varphi(x,s;\lambda)|\\
&+\frac{e^{x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\
&\cdot(e^{s\mathrm{Re}\,(\rho)}-1)(e^{-s\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)})\\
&+\frac{e^{-x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\
&\cdot(1-e^{-s\mathrm{Re}\,(\rho)})(e^{\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}),
\end{aligned}
$$

where

$$
\begin{aligned}
&|\varphi_1(x,s;\lambda)| \\
&\quad\le \frac{e^{x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty) \\
&\qquad\cdot(1-e^{-\mathrm{Re}\,(\rho)})(1-e^{-s\mathrm{Re}\,(\rho)}) \\
&\qquad+\frac{e^{-x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty) \\
&\qquad\cdot(e^{\mathrm{Re}\,(\rho)}-1)(e^{s\mathrm{Re}\,(\rho)}-1)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\varphi_2(x,s;\lambda)| \\
&\quad\le \frac{e^{x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty) \\
&\qquad\cdot(1-e^{-\mathrm{Re}\,(\rho)})(e^{(1-s)\mathrm{Re}\,(\rho)}-1) \\
&\qquad+\frac{e^{-x\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^2}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty) \\
&\qquad\cdot(e^{\mathrm{Re}\,(\rho)}-1)(1-e^{(s-1)\mathrm{Re}\,(\rho)}).
\end{aligned}
$$

Bounds in $L^1(0,1)$. Suppose now that $p=1$. From (2.6) we have, for every $f\in L^1(0,1)$,

$$\|R(\lambda:T_1)f\|_{L^1(0,1)}\le\left(\sup_{0\le s\le 1}\int_0^1|G(x,s;\lambda)|\,dx\right)\|f\|_{L^1(0,1)}, \tag{4.3}$$

so we need to bound

$$\sup_{0\le s\le 1}\int_0^1|G(x,s;\lambda)|\,dx=\frac{1}{|\Delta(\lambda)|}\sup_{0\le s\le 1}\int_0^1|N(x,s;\lambda)|\,dx.$$

First of all, we have from the inequalities previously calculated:

$$\begin{aligned}\int_0^1 |N(x,s;\lambda)|\,dx \le \int_0^1 |\varphi(x,s;\lambda)|\,dx &+ \frac{e^{\mathrm{Re}\,(\rho)-1}}{|\rho|(\mathrm{Re}\,(\rho))^3}\\ &\cdot(\|R_1\|_\infty + |\rho|\|S_1\|_\infty)(\|R_2\|_\infty + |\rho|\|S_2\|_\infty)\\ &\cdot(e^{s\mathrm{Re}\,(\rho)}-1)(e^{-s\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)})\\ &+\frac{1-e^{-\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^3}\\ &\cdot(\|R_1\|_\infty + |\rho|\|S_1\|_\infty)(\|R_2\|_\infty + |\rho|\|S_2\|_\infty)\\ &\cdot(1-e^{-s\mathrm{Re}\,(\rho)})(e^{\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}),\end{aligned}$$

so

$$\begin{aligned}\int_0^1 |N(x,s;\lambda)|\,dx \le \int_0^1 |\varphi(x,s;\lambda)|\,dx &+ \frac{2}{|\rho|(\mathrm{Re}\,(\rho))^3}\\ &\cdot(\|R_1\|_\infty + |\rho|\|S_1\|_\infty)(\|R_2\|_\infty + |\rho|\|S_2\|_\infty)\\ &\cdot(e^{\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}\\ &\quad+e^{-s\mathrm{Re}\,(\rho)}-e^{(1-s)\mathrm{Re}\,(\rho)}+e^{(s-1)\mathrm{Re}\,(\rho)}).\end{aligned}$$

We use (4.2) to write $\int_0^1 |\varphi(x,s;\lambda)|\,dx$ as follows

$$\int_0^1 |\varphi(x,s;\lambda)|\,dx = \int_0^s |\varphi_2(x,s;\lambda)|\,dx + \int_s^1 |\varphi_1(x,s;\lambda)|\,dx.$$

From the inequalities obtained for $|\varphi_i(x,s;\lambda)|$, we have

$$\begin{aligned}&\int_0^s |\varphi_2(x,s;\lambda)|\,dx\\ &\quad\le \frac{2}{|\rho|(\mathrm{Re}\,(\rho))^3}(\|R_1\|_\infty + |\rho|\|S_1\|_\infty)(\|R_2\|_\infty + |\rho|\|S_2\|_\infty)\\ &\qquad\cdot(e^{\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}+e^{-s\mathrm{Re}\,(\rho)}\\ &\qquad-e^{(1-s)\mathrm{Re}\,(\rho)}+e^{(s-1)\mathrm{Re}\,(\rho)})\end{aligned}$$

and

$$\begin{aligned}\int_s^1 &|\varphi_1(x,s;\lambda)|\,dx\\ &\le \frac{2}{|\rho|(\mathrm{Re}\,(\rho))^3}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\ &\quad\cdot(e^{\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}+e^{-s\mathrm{Re}\,(\rho)}\\ &\qquad -e^{(1-s)\mathrm{Re}\,(\rho)}+e^{(s-1)\mathrm{Re}\,(\rho)}).\end{aligned}$$

Adding up the formulae obtained, we get

$$\begin{aligned}\int_0^1 &|N(x,s;\lambda)|\,dx\\ &\le \frac{6}{|\rho|(\mathrm{Re}\,(\rho))^3}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty)\\ &\quad\cdot(e^{\mathrm{Re}\,(\rho)}-e^{-\mathrm{Re}\,(\rho)}-e^{s\mathrm{Re}\,(\rho)}+e^{-s\mathrm{Re}\,(\rho)}\\ &\qquad -e^{(1-s)\mathrm{Re}\,(\rho)}+e^{(s-1)\mathrm{Re}\,(\rho)}).\end{aligned}$$

As $\mathrm{Re}\,(\rho)>0$, the part depending on s in the right member above is negative, so

$$\begin{aligned}\sup_{0\le s\le 1}&\int_0^1 |N(x,s;\lambda)|\,dx\\ &\le \frac{6e^{\mathrm{Re}\,(\rho)}}{|\rho|(\mathrm{Re}\,(\rho))^3}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty).\end{aligned}$$

As $\mathrm{Re}\,(\rho)\ge|\rho|\cos(\delta/2)$, we have that

$$\begin{aligned}\sup_{0\le s\le 1}&\int_0^1 |N(x,s;\lambda)|\,dx\\ &\le \frac{6e^{\mathrm{Re}\,(\rho)}}{|\rho|^4\cos^3(\delta/2)}\,(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty).\end{aligned}$$

From (2.5) and (4.3), we deduce the following inequality

$$\begin{aligned}\|R(\lambda:T_1)\| &\le \frac{6}{\cos^3(\delta/2)}\,\frac{e^{\mathrm{Re}\,(\rho)}}{|\rho|^4|\Delta(\lambda)|}\\ &\quad\cdot(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)(\|R_2\|_\infty+|\rho|\|S_2\|_\infty),\end{aligned}$$

that can be written as

$$\|R(\lambda : T_1)\| \le \frac{H(\rho)}{|\rho|^2} = \frac{H(\rho)}{|\lambda|}, \tag{4.4}$$

where

$$H(\rho) = \frac{6}{\cos^3(\delta/2)} \frac{e^{\mathrm{Re}\,(\rho)}}{|\Delta(\rho^2)|} \left(\frac{\|R_1\|_\infty}{|\rho|} + \|S_1\|_\infty \right) \left(\frac{\|R_2\|_\infty}{|\rho|} + \|S_2\|_\infty \right). \tag{4.5}$$

The next step is to determine the cases for which the function $H(\rho)$ remains bounded as $|\rho| \to \infty$ with $|\arg(\rho)| < \delta/2$. It will then be necessary to bound $|\Delta(\lambda)|$ appropriately. However, formula (4.1) is not useful for this purpose, so we are going to make a regularity assumption on the functions R_i and S_i that leads to an improved formula for $\Delta(\lambda)$. Suppose that $R_i, S_i \in C^1([0,1]; \mathbf{C})$, so we can integrate by parts in (4.1) to get

$$\begin{aligned} \Delta(\rho^2) = e^\rho \Big[& S_2(0)S_1(1) - S_1(0)S_2(1) + \frac{1}{\rho^2}(R_1(0)R_2(1) \\ & - R_2(0)R_1(1)) + \frac{\Phi(\rho)}{\rho^2} + \frac{1}{\rho}(R_1(0)S_2(1) \\ & - R_2(0)S_1(1) - R_2(1)S_1(0) + R_1(1)S_2(0)) \Big], \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} \Phi(\rho) = {} & (R_2(0) - \rho S_2(0)) \\ & \cdot \int_0^1 (R_1'(t) + \rho S_1'(t)) e^{\rho(t-1)}\, dt - (R_1(0) - \rho S_1(0)) \\ & \cdot \int_0^1 (R_2'(t) + \rho S_2'(t)) e^{\rho(t-1)}\, dt + (R_1(1) - \rho S_1(1)) \\ & \cdot \int_0^1 (R_2'(t) + \rho S_2'(t)) e^{\rho(t-2)}\, dt - (R_2(1) - \rho S_2(1)) \\ & \cdot \int_0^1 (R_1'(t) + \rho S_1'(t)) e^{\rho(t-2)}\, dt + (R_2(1) + \rho S_2(1)) \\ & \cdot \int_0^1 (R_1'(t) - \rho S_1'(t)) e^{-\rho t}\, dt - (R_1(1) + \rho S_2(1)) \end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^1 (R_2'(t) - \rho S_2'(t)) e^{-\rho t}\, dt + (R_1(0) + \rho S_1(0)) \\
&\cdot \int_0^1 (R_2'(t) - \rho S_2'(t)) e^{-\rho(t+1)}\, dt - (R_2(0) + \rho S_2(0)) \\
&\cdot \int_0^1 (R_1'(t) - \rho S_1'(t)) e^{-\rho(t+1)}\, dt \\
&+ \left(\int_0^1 (R_2'(t) - \rho S_2'(t)) e^{-\rho t}\, dt \right) \left(\int_0^1 (R_1'(t) + \rho S_1'(t)) e^{\rho(t-1)}\, dt \right) \\
&- \left(\int_0^1 (R_1'(t) - \rho S_1'(t)) e^{-\rho t}\, dt \right) \left(\int_0^1 (R_2'(t) + \rho S_2'(t)) e^{\rho(t-1)}\, dt \right) \\
&+ 2\rho e^{-\rho} (R_2(0)S_1(0) - R_1(0)S_2(0) + R_2(1)S_1(1) - R_1(1)S_2(1)) \\
&+ [(R_1(1) - \rho S_1(1))(R_2(0) + \rho S_2(0)) \\
&\quad - (R_1(0) + \rho S_1(0))(R_2(1) - \rho S_2(1))] e^{-2\rho}.
\end{aligned}
$$

After a straightforward calculation we obtain the following inequality, valid for $\rho \in \Sigma_{\delta/2}$ with $\mathrm{Re}\,(\rho)$ sufficiently large:

$$
\begin{aligned}
|\Phi(\rho)| \le{}& \frac{1}{\cos(\delta/2)} \\
&\cdot \Bigg[\left(\frac{|R_2(0)| + |R_2(1)|}{|\rho|} + |S_2(0)| + |S_2(1)| \right) (\|R_1'\|_\infty + |\rho| \|S_1'\|_\infty) \\
&\quad + \left(\frac{|R_1(0)| + |R_1(1)|}{|\rho|} + |S_1(0)| + |S_1(1)| \right) (\|R_2'\|_\infty + |\rho| \|S_2'\|_\infty) \Bigg] \\
&+ 2|\rho| |R_2(0)S_1(0) - R_1(0)S_2(0) + R_2(1)S_1(1) - R_1(1)S_2(1)| e^{-\mathrm{Re}\,(\rho)} \qquad (2.7) \\
&+ \frac{2}{(\mathrm{Re}\,(\rho))^2} (\|R_1'\|_\infty + |\rho| \|S_1'\|_\infty)(\|R_2'\|_\infty + |\rho| \|S_2'\|_\infty) \\
&+ 2(\|R_1\|_\infty + |\rho| \|S_1\|_\infty)(\|R_2\|_\infty + |\rho| \|S_2\|_\infty) e^{-2\mathrm{Re}\,(\rho)},
\end{aligned}
$$

where we have used that $1 - e^{-2\mathrm{Re}\,(\rho)} < 1$ and $\mathrm{Re}\,(\rho) > |\rho| \cos(\delta/2)$.

There are now several cases to analyze, depending on the functions R_i and S_i.

Case A. Suppose $\|S_1\|_\infty \cdot \|S_2\|_\infty \ne 0$, that is, the functions S_1 and S_2 are not identically zero. We must consider two subcases, depending on the coefficient $S_2(0)S_1(1) - S_1(0)S_2(1)$ in (4.6).

Case A.1. $S_2(0)S_1(1) - S_1(0)S_2(1) \neq 0$. From (4.6) we have for $|\rho|$ sufficiently large

$$\begin{aligned}|\Delta(\rho^2)| \geq e^{\mathrm{Re}\,(\rho)}\bigg[&|S_2(0)S_1(1) - S_1(0)S_2(1)|\\ &- \frac{1}{|\rho|^2}\,|R_1(0)R_2(1) - R_2(0)R_1(1)|\\ &- \frac{1}{|\rho|}\,|R_1(0)S_2(1) - R_2(0)S_1(1)\\ &- R_2(1)S_1(0) + R_1(1)S_2(0)| - \frac{|\Phi(\rho)|}{|\rho|^2}\bigg].\end{aligned}$$

Take $\mathrm{Re}\,(\rho) > r_0$, where $r_0 > 0$ is a constant to be chosen later. Then we can write

$$\begin{aligned}|\Delta(\rho^2)| \geq e^{\mathrm{Re}\,(\rho)}\bigg[&|S_2(0)S_1(1) - S_1(0)S_2(1)|\\ &- \frac{1}{r_0^2}\,|R_1(0)R_2(1) - R_2(0)R_1(1)|\\ &- \frac{1}{r_0}\,|R_1(0)S_2(1) - R_2(0)S_1(1)\\ &- R_2(1)S_1(0) + R_1(1)S_2(0)| - \frac{|\Phi(\rho)|}{r_0|\rho|}\bigg].\end{aligned}$$

From (4.7), we have

$$\begin{aligned}&\frac{|\Phi(\rho)|}{|\rho|}\\ &\quad\leq \frac{1}{\cos(\delta/2)}\bigg[\bigg(\frac{|R_2(0)|+|R_2(1)|}{r_0} + |S_2(0)| + |S_2(1)|\bigg)\\ &\qquad\cdot\bigg(\frac{\|R_1'\|_\infty}{r_0} + \|S_1'\|_\infty\bigg) + \bigg(\frac{|R_1(0)|+|R_1(1)|}{r_0} + |S_1(0)| + |S_1(1)|\bigg)\\ &\qquad\cdot\bigg(\frac{\|R_2'\|_\infty}{r_0} + \|S_2'\|_\infty\bigg)\bigg] + \frac{2}{r_0\cos(\delta/2)}\\ &\qquad\cdot\bigg(\frac{\|R_1'\|_\infty}{r_0} + \|S_1'\|_\infty\bigg)\bigg(\frac{\|R_2'\|_\infty}{r_0} + \|S_2'\|_\infty\bigg)\\ &\qquad+ \frac{1}{\cos(\delta/2)}\bigg(\frac{\|R_1\|_\infty}{r_0} + \|S_1\|_\infty\bigg)\bigg(\frac{\|R_2\|_\infty}{r_0} + \|S_2\|_\infty\bigg)\\ &\qquad+ 2|R_2(0)S_1(0) - R_1(0)S_2(0) + R_2(1)S_1(1) - R_1(1)S_2(1)|\\ &\quad\equiv C_0(r_0),\end{aligned}$$

where we have used that $e^{-\mathrm{Re}\,(\rho)} < 1$ and $|\rho| e^{-2\mathrm{Re}\,(\rho)} < (2\cos(\delta/2))^{-1}$ for $\mathrm{Re}\,(\rho) > 0$ with $|\arg(\rho)| < \delta/2$. We then have

$$\begin{aligned} |\Delta(\rho^2)| \geq e^{\mathrm{Re}\,(\rho)} \Bigg[& |S_2(0)S_1(1) - S_1(0)S_2(1)| \\ & - \frac{1}{r_0^2}\,|R_1(0)R_2(1) - R_2(0)R_1(1)| \\ & - \frac{1}{r_0}\,|R_1(0)S_2(1) - R_2(0)S_1(1) \\ & - R_2(1)S_1(0) + R_1(1)S_2(0)| - \frac{C_0(r_0)}{r_0}\Bigg]. \end{aligned}$$

We can now choose $r_0 > 0$ such that

$$\begin{aligned} & \frac{1}{r_0}\,|R_1(0)S_2(1) - R_2(0)S_1(1) - R_2(1)S_1(0) + R_1(1)S_2(0)| \\ & \qquad + \frac{1}{r_0^2}\,|R_1(0)R_2(1) - R_1(1)R_2(0)| + \frac{C_0(r_0)}{r_0} \\ & \qquad\qquad < \frac{|S_2(0)S_1(1) - S_1(0)S_2(1)|}{2}. \end{aligned}$$

Then, for $\mathrm{Re}\,(\rho) > r_0$, we have

$$|\Delta(\rho^2)| \geq \frac{|S_2(0)S_1(1) - S_1(0)S_2(1)|}{2} e^{\mathrm{Re}\,(\rho)}.$$

From (4.5) we deduce the following bound, valid for every $\mathrm{Re}\,(\rho) > r_0$ with $|\arg(\rho)| < \delta/2$

$$\begin{aligned} H(\rho) & \leq \frac{12}{\cos^3(\delta/2)|S_2(0)S_1(1) - S_1(0)S_2(1)|} \\ & \qquad \cdot \left(\frac{\|R_1\|_\infty}{|\rho|} + \|S_1\|_\infty\right)\left(\frac{\|R_2\|_\infty}{|\rho|} + \|S_2\|_\infty\right) \\ & \leq \frac{12}{\cos^3(\delta/2)|S_2(0)S_1(1) - S_1(0)S_2(1)|} \\ & \qquad \cdot \left(\frac{\|R_1\|_\infty}{r_0} + \|S_1\|_\infty\right)\left(\frac{\|R_2\|_\infty}{r_0} + \|S_2\|_\infty\right). \end{aligned}$$

This proves that $H(\rho)$ remains bounded as $|\rho| \to \infty$ in the sector $\Sigma_{\delta/2}$.

Case A.2. Suppose that $S_2(0)S_1(1)-S_1(0)S_2(1)=0$. Now $H(\rho)$ will not be bounded in any case. For proving this, note that, from (4.6), we have

$$|\Delta(\rho^2)| \leq \frac{e^{\mathrm{Re}\,(\rho)}}{|\rho|}\bigg[|R_1(0)S_2(1)-R_2(0)S_1(1)-R_2(1)S_1(0)+R_1(1)S_2(0)| + \frac{|R_1(0)R_2(1)-R_1(1)R_2(0)|}{|\rho|}+\frac{|\Phi(\rho)|}{|\rho|}\bigg].$$

In Case A.1 we proved that

$$\frac{|\Phi(\rho)|}{|\rho|} \leq C_0(r_0)$$

for every $\mathrm{Re}\,(\rho) > r_0$, so

$$\begin{aligned}|\Delta(\rho^2)| &\leq \frac{e^{\mathrm{Re}\,(\rho)}}{|\rho|}\bigg[|R_1(0)S_2(1)-R_2(0)S_1(1)-R_2(1)S_1(0)+R_1(1)S_2(0)| + \frac{|R_1(0)R_2(1)-R_1(1)R_2(0)|}{r_0}+\frac{C_0(r_0)}{r_0}\bigg]\\ &\equiv C_1\frac{e^{\mathrm{Re}\,(\rho)}}{|\rho|},\end{aligned}$$

where C_1 is a constant. From (4.5) we obtain

$$H(\rho) \geq \frac{6}{C_1\cos^3(\delta/2)}(\|R_1\|_\infty+|\rho|\|S_1\|_\infty)\bigg(\frac{\|R_2\|_\infty}{|\rho|}+\|S_2\|_\infty\bigg),$$

where the second member goes to infinity as $|\rho|\to\infty$.

The remaining cases are treated as before, so we simply state the results.

Case B. Suppose that $\|S_1\|_\infty=0$ and $\|S_2\|_\infty\neq 0$. If $R_1(0)S_2(1)+R_1(1)S_2(0)\neq 0$, we have the following bound, valid for $\mathrm{Re}\,(\rho) > r_0$ with $|\arg(\rho)| < \delta/2$

$$H(\rho) \leq \frac{12\|R_1\|_\infty}{\cos^3(\delta/2)|R_1(0)S_2(1)+R_1(1)S_2(0)|}\bigg(\frac{\|R_2\|_\infty}{r_0}+\|S_2\|_\infty\bigg).$$

If $R_1(0)S_2(1) + R_1(1)S_2(0) = 0$, then $H(\rho)$ is not bounded.

Case C. Suppose that $\|S_1\|_\infty \neq 0$ and $\|S_2\|_\infty = 0$. Then $H(\rho)$ remains bounded if and only if $R_2(0)S_1(1) + R_2(1)S_1(0) \neq 0$.

Case D. Now $\|S_1\|_\infty = \|S_2\|_\infty = 0$, that is, the functions S_1 and S_2 are identically zero (then $\|R_1\|_\infty \cdot \|R_2\|_\infty \neq 0$). If $R_1(0)R_2(1) - R_1(1)R_2(0) \neq 0$, we have

$$H(\rho) \leq \frac{12\|R_1\|_\infty \|R_2\|_\infty}{\cos^3(\delta/2)|R_1(0)R_2(1) - R_2(0)R_1(1)|},$$

which proves that $H(\rho)$ is bounded. If $R_1(0)R_2(1) - R_1(1)R_2(0) = 0$, $H(\rho)$ will not be bounded.

The preceding study of cases for $H(\rho)$ motivates the following definition, in which we only impose to the coefficients of the boundary conditions to be continuous.

Definition 4.1. Consider the boundary conditions

$$B_i(u) \equiv \int_0^1 R_i(t)u(t)\,dt + \int_0^1 S_i(t)u'(t)\,dt = 0, \quad i = 1, 2$$

where R_i and S_i are in $C([0,1];\mathbf{C})$. We say that $\{B_1, B_2\}$ are *regular* if one of the following conditions holds:

- $S_1(0)S_2(1) - S_1(1)S_2(0) \neq 0$;
- $S_1 \equiv 0$ and $R_1(0)S_2(1) + R_1(1)S_2(0) \neq 0$;
- $S_2 \equiv 0$ and $R_2(0)S_1(1) + R_2(1)S_1(0) \neq 0$;
- $S_1 \equiv 0$, $S_2 \equiv 0$ and $R_1(0)R_2(1) - R_1(1)R_2(0) \neq 0$.

Example 4.1. Consider the particular case in which $R_i = S_i'$ for $i = 1, 2$. Then the boundary conditions $\{B_1, B_2\}$ can be written as nonseparated ones

$$\begin{cases} B_1(u) \equiv \displaystyle\int_0^1 (S_1(t)u(t))'\,dt = -S_1(0)u(0) + S_1(1)u(1) = 0, \\ B_2(u) \equiv \displaystyle\int_0^1 (S_2(t)u(t))'\,dt = -S_2(0)u(0) + S_2(1)u(1) = 0. \end{cases}$$

It is a simple exercise to see that the integral conditions $\{B_1, B_2\}$ are regular if and only if they are Birkhoff-regular (as nonseparated boundary conditions).

For regular boundary conditions with $R_i, S_i \in C^1([0,1]; \mathbf{C})$, we have proved the existence of two positive constants r_0 and M_0 such that if $|\rho| > r_0$ and $|\arg(\rho)| < \delta/2$ then $H(\rho) \leq M_0$. This means that $\lambda = \rho^2$ belongs to $\rho(T_1)$ and

$$\|R(\lambda : T_1)\| \leq \frac{M_0}{|\lambda|}.$$

Define $r = r_0^2/\sin(\delta)$ and $M = M_0(1 + 1/\sin(\delta))$. Then the sector $\Sigma_{\delta,r}$ is contained in $\rho(T_1)$ and

$$\|R(\lambda : T_1)\| \leq \frac{M}{|\lambda - r|}, \quad \forall\, \lambda \in \Sigma_{\delta,r}.$$

This proves the following result.

Theorem 4.1. *Let $\{B_1, B_2\}$ be regular boundary conditions, and suppose that $R_i, S_i \in C^1([0,1]; \mathbf{C})$, $i = 1, 2$. Let T_1 be the differential operator in $L^1(0,1)$ defined as $T_1 u = u''$, $D(T_1) = \{u \in W^{2,1}(0,1) : B_1(u) = B_2(u) = 0\}$. Then T_1 is the generator of an analytic semigroup $\{e^{tT_1}\}_{t \geq 0}$ of bounded linear operators on $L^1(0,1)$ that, in general, will not be strongly continuous.*

Remark 4.2. In Section 5 we will see that the condition $R_i, S_i \in C^1([0,1]; \mathbf{C})$ in Theorem 4.1 can be dropped.

Bounds in $L^\infty(0,1)$. As we commented in the introduction, we would like to obtain bound (1.4) for the L^∞-realization T_∞ in order to use interpolation for proving (ii) in every space $L^p(0,1)$, $1 \leq p \leq \infty$, as it was made in [**3**] for the case of nonseparated boundary conditions. However, even for regular boundary conditions, we cannot obtain the desired kind of bounds in $L^\infty(0,1)$ as the following example shows.

Example 4.2. Consider the boundary conditions

$$\begin{cases} B_1(u) \equiv \displaystyle\int_0^1 u'(t)\,dt = 0, \\ B_2(u) \equiv \displaystyle\int_0^1 e^t u(t)\,dt = 0. \end{cases}$$

We have that $R_1 \equiv 0$, $S_1 \equiv 1$, $R_2(t) = e^t$ and $S_2 \equiv 0$. As $R_2(0)S_1(1) + R_2(1)S_1(0) = 1 + e \neq 0$, the conditions are regular. Note also that R_i and S_i belong to $C^1([0,1];\mathbf{C})$, so we have stronger conditions on the coefficients than mere regularity.

Fix $M > 0$ and take $f_0 \equiv 1$. If $\lambda = \rho^2 \in \Sigma_{\delta,r}$, for r sufficiently large we have

$$\begin{aligned} \|R(\lambda : T_\infty)\| &\geq \|R(\lambda : T_\infty) f_0\|_{L^\infty(0,1)} \\ &= \sup_{0\leq x\leq 1} |R(\lambda : T_\infty) f_0(x)| \\ &= \sup_{0\leq x\leq 1} \left| \int_0^1 G(x,s;\lambda)\,ds \right| \\ &= \sup_{0\leq x\leq 1} \left| \frac{1}{\Delta(\lambda)} \int_0^1 N(x,s;\lambda)\,ds \right|. \end{aligned}$$

The characteristic determinant is, in this case,

$$\Delta(\rho^2) = \frac{e^{-\rho} - 1}{\rho^2 - 1} \left[\rho(e{+}1)(e^\rho{-}1) - (e{-}1)(e^\rho{+}1)\right].$$

After a long calculation, we obtain

$$\int_0^1 N(x,s;\lambda)\,ds = (e-1)\,\frac{e^\rho - 1}{\rho(\rho^2-1)}\left[-\rho(1{+}e^{-\rho}) + (1{-}e^{-\rho})\right],$$

so

$$\begin{aligned} &\sup_{0\leq x\leq 1} \left| \frac{1}{\Delta(\rho^2)} \int_0^1 N(x,s;\lambda)\,ds \right| \\ &\qquad \geq \left| \frac{(e{-}1)(e^\rho - 1)}{\rho(e^{-\rho}{-}1)} \, \frac{(1 - e^{-\rho}) - \rho(1 + e^{-\rho})}{\rho(1{+}e)(e^\rho{-}1) - (e{-}1)(e^\rho{+}1)} \right|. \end{aligned}$$

The second member can be made greater than $M/|\rho|^2$, taking $|\rho|$ sufficiently large.

We have seen that for every $M > 0$ we can take $r > 0$ such that

$$\|R(\lambda : T_\infty)\| > \frac{M}{|\lambda|} > \frac{M}{|\lambda - r|}, \quad \forall\, \lambda \in \Sigma_{\delta,r}.$$

This proves that the operator T_∞ cannot be sectorial.

Remark 4.3. We could also try to directly bound T_p in the $L^p(0,1)$ norm, but we do not arrive to appropriate bounds.

5. Approximation. We are going to extend Theorem 4.1 to the case of regular boundary conditions, that is, we will drop the condition $R_i, S_i \in C^1([0,1]; \mathbf{C})$, $i = 1,2$. We will need the following result [**4**, Chapter 9].

Proposition 5.1. *Let $\{A_n\}_{n\in\mathbf{N}}$ be a family of linear operators on the Banach space X. Suppose that constants $r \in \mathbf{R}$, $M \geq 0$, and $\delta \in (\pi/2, \pi)$ exist such that*

$$(*) \qquad \begin{cases} \rho(A_n) \supset \Sigma_{\delta,r} = \{\lambda \in \mathbf{C} : \lambda \neq r,\ |\arg(\lambda - r)| < \delta\}; \\ \|R(\lambda : A_n)\| \leq \dfrac{M}{|\lambda - r|}, \quad \forall\, \lambda \in \Sigma_{\delta,r}. \end{cases}$$

If there is a $\lambda_0 \in \Sigma_{\delta,r}$ such that $R(\lambda_0 : A_n)f$ converges in X as $n \to \infty$ for every $f \in X$, then a unique operator A exists on X such that $\Sigma_{\delta,r} \subset \rho(A)$ and $R(\lambda : A_n)$ converges to $R(\lambda : A)$ strongly in X as $n \to \infty$ for every $\lambda \in \Sigma_{\delta,r}$. As a consequence, A satisfies condition $(*)$, so it is a sectorial operator on X.

Suppose that $R_i, S_i \in C([0,1]; \mathbf{C})$, $i = 1,2$, and that the boundary conditions $\{B_1, B_2\}$ are regular. For every $i = 1,2$, we can build two sequences of functions, $\{R_i^n\}_{n\in\mathbf{N}}$ and $\{S_i^n\}_{n\in\mathbf{N}}$ such that

1. $R_i^n, S_i^n \in C^1([0,1]; \mathbf{C})$ for every $n \in \mathbf{N}$.

2. The sequences $\{R_i^n\}, \{S_i^n\}, \{(R_i^n)'\}$ and $\{(S_i^n)'\}$ are uniformly bounded.

3. $\lim_{n\to\infty} \|R_i^n - R_i\|_\infty = 0$ and $\lim_{n\to\infty} \|S_i^n - S_i\|_\infty = 0$.

4. $R_i^n(0) = R_i(0)$, $R_i^n(1) = R_i(1)$, $S_i^n(0) = S_i(0)$ and $S_i^n(1) = S(1)$ for each $n \in \mathbf{N}$. If $S_i \equiv 0$, we take $S_i^n \equiv 0$ for every $n \in \mathbf{N}$.

For each $n \in \mathbf{N}$, define the operator $A_n u = u''$ with domain

$$D(A_n) = \{u \in W^{2,1}(0,1) : B_1^n(u) = B_2^n(u) = 0\},$$

where

$$B_i^n(u) = \int_0^1 R_i^n(t)u(t)\,dt + \int_0^1 S_i^n(t)u'(t)\,dt = 0, \quad i = 1, 2.$$

Note that condition 4 implies that the boundary conditions $\{B_1^n, B_2^n\}$ are regular, and they satisfy the same regularity condition as conditions $\{B_1, B_2\}$. As the coefficients R_i^n, S_i^n are of class 1, we have that every operator A_n is sectorial. Moreover, it is easily seen from 1–4 and the analysis of cases made in Section 4 that we can choose the constants $\delta \in (\pi/2, \pi)$, $M \geq 0$ and $r \in \mathbf{R}$ in a uniform way, so we have

$$\|R(\lambda : A_n)\| \leq \frac{M}{|\lambda - r|}, \quad \forall\, \lambda \in \Sigma_{\delta,r} \subset \rho(A_n), \quad \forall\, n \in \mathbf{N}.$$

This shows that condition $(*)$ in Proposition 5.1 holds for the operator $\{A_n\}$ with $X = L^1(0,1)$.

Let $\Delta(\lambda)$ and $\Delta_n(\lambda)$ be the characteristic determinants associated to the operators T_1 and A_n, respectively. As we stated in Section 2, the set of zeros of $\Delta(\lambda)$ is as much a denumerable set, so we can choose $\lambda_0 \in \Sigma_{\delta,r}$ such that $\Delta(\lambda_0) \neq 0$. As $\lambda_0 \in \rho(A_n)$, we also have that $\Delta_n(\lambda_0) \neq 0$.

In order to apply Proposition 5.1, we are going to see that $R(\lambda_0 : A_n)f$ converges to $R(\lambda_0 : T_1)f$ in $L^1(0,1)$ for every $f \in L^1(0,1)$. We begin with the following result

Lemma 5.2. *The numerical sequence $\{\Delta_n(\lambda_0)\}_{n\in\mathbf{N}}$ converges to $\Delta(\lambda_0)$.*

Proof. Let $\{u_1, u_2\}$ be a fundamental system of solutions for the differential equation $u'' = \lambda_0 u$. For $i, j = 1, 2$, we can write

$$B_i^n(u_j) - B_i(u_j) = \int_0^1 [R_i^n(t) - R_i(t)] u_j(t)\, dt + \int_0^1 [S_i^n(t) - S_i(t)] u_j'(t)\, dt,$$

so

$$|B_i^n(u_j) - B_i(u_j)| \le \|R_i^n - R_i\|_\infty \|u_j\|_{L^1(0,1)} + \|S_i^n - S_i\|_\infty \|u_j'\|_{L^1(0,1)}.$$

Condition 3 implies that the right member tends to zero as n goes to infinity, so $\lim_{n\to\infty} B_i^n(u_j) = B_i(u_j)$. From (2.2), we deduce the desired result. □

Now take $f \in L^1(0,1)$. If $G(x, s; \lambda_0)$ and $G_n(x, s; \lambda_0)$ are, respectively, the Green's functions associated to the operators T_1 and A_n, we have that

$$\|R(\lambda_0 : A_n)f - R(\lambda_0 : T_1)f\|_{L^1(0,1)} \le \|f\|_{L^1(0,1)} \sup_{0\le s\le 1} \int_0^1 |G_n(x, s; \lambda_0) - G(x, s; \lambda_0)|\, dx.$$

As we did in Lemma 5.2, it is easily seen from (2.3)–(2.5) that the second member in the above inequality goes to zero as $n \to \infty$. This proves the convergence of $R(\lambda_0 : A_n)f$ to $R(\lambda_0 : T_1)f$ in $L^1(0,1)$ for every $f \in L^1(0,1)$.

We can now apply Proposition 5.1 for obtaining a certain operator A that, in particular, verifies conditions $(*)$ so it is sectorial. From uniqueness, $A = T_1$.

We have the following result.

Theorem 5.3. *Let $\{B_1, B_2\}$ be regular boundary conditions, and let T_1 be the differential operator $T_1 : D(L_1) \subset L^1(0,1) \to L^1(0,1)$ defined as*

$$T_1 u = u'', \qquad D(T_1) = \{u \in W^{2,1}(0,1) : B_1(u) = B_2(u) = 0\}.$$

Then T_1 is the generator of an analytic semigroup $\{e^{tT_1}\}_{t\geq 0}$ of bounded linear operators on $L^1(0,1)$. In general this semigroup will not be strongly continuous.

Remark 5.1. Note that, in Definition 4.1 of regular boundary conditions, we only impose continuity to the coefficients of the boundary conditions.

Remark 5.2. For every $t \geq 0$, the operator e^{tT_1} can be obtained as the limit in $\mathcal{L}(L^1(0,1))$ of the operators e^{tA_n} with uniform convergence in every interval $[t_0, \infty)$, $t_0 > 0$.

6. Generation of analytic semigroups. In the previous section we showed that T_1 is a sectorial operator when the boundary conditions $\{B_1, B_2\}$ are regular. We are now going to generalize this result for the most general operators L_1 and $\tilde{L}_1$ defined in Section 3.

Remember that $L_1 = T_1 + Q_1$ where Q_1 was multiplication by q. As Q_1 is a bounded operator, we can choose r sufficiently large for

$$\|R(\lambda : T_1)\| \leq \frac{1}{2}\,\|Q_1\|^{-1}, \quad \forall\,\lambda \in \Sigma_{\delta,r}.$$

Now Proposition 3.1 assures that L_1 is a sectorial operator.

As we stated in Section 3, the resolvents of the operators L_1 and $\tilde{L}_1$ were related as follows

$$R(\lambda : \tilde{L}_1) = M_\phi R(\lambda : L_1) M_\phi^{-1}, \quad \forall\,\lambda \in \rho(L_1) = \rho(\tilde{L}_1).$$

As M_ϕ and M_ϕ^{-1} are bounded operators, we deduce that $\tilde{L}_1$ is also sectorial. It is straightforward to prove that the regularity of the boundary conditions does not depend on the changes introduced.

Finally, the linear change of variables for passing from (a,b) to $(0,1)$ does not affect the bounds obtained. We can then state our main result (note that some terms have been renamed).

Theorem 6.1. *Consider the differential system*

$$\begin{cases} l(u) = u'' + q_1(x)u' + q_0(x)u & \textit{in} \quad (a,b), \\ B_i(u) \equiv \displaystyle\int_a^b R_i(t)u(t)\,dt + \int_a^b S_i(t)u'(t)\,dt = 0 & i = 1, 2, \end{cases}$$

where $q_0, R_i, S_i \in C([a,b];\mathbf{C})$ *and* $q_1 \in C^1([a,b];\mathbf{C})$. *Suppose that the boundary conditions are regular, i.e., they verify one of the following conditions*:

- $S_1(a)S_2(b) - S_1(b)S_2(a) \neq 0$;
- $S_1 \equiv 0$ *and* $R_1(a)S_2(b) + R_1(b)S_2(a) \neq 0$;
- $S_2 \equiv 0$ *and* $R_2(a)S_1(b) + R_2(b)S_1(a) \neq 0$;
- $S_1 \equiv 0$, $S_2 \equiv 0$ *and* $R_1(a)R_2(b) - R_1(b)R_2(a) \neq 0$.

Define $L_1 : D(L_1) \subset L^1(a,b) \to L^1(a,b)$ *as*

$$L_1 u = l(u), \qquad D(L_1) = \{u \in W^{2,1}(a,b) : B_i(u) = 0,\ i = 1,2\}.$$

Then L_1 *is the generator of an analytic semigroup* $\{e^{tL_1}\}_{t\geq 0}$ *of bounded linear operators on* $L^1(a,b)$. *In general this semigroup will not be strongly continuous.*

Example 6.1. In Example 4.1 the domain $D(L_1)$ is dense in $L^1(0,1)$, so the analytic semigroup generated by L_1 is strongly continuous.

Let X be the closure of $D(L_1)$ in $L^1(a,b)$, and let $\overline{L}$ be the part of L_1 in X, that is, $\overline{L}u = L_1 u$ and $D(\overline{L}) = \{u \in D(L_1) : L_1 u \in X\})$. Then $\overline{L}$ verifies the hypotheses of Theorem 6.1 so it generates an analytic semigroup $\{e^{t\overline{L}}\}_{t\geq 0}$ on X; as the domain $D(\overline{L})$ is dense in X, the semigroup $\{e^{t\overline{L}}\}_{t\geq 0}$ is strongly continuous. Moreover, $e^{t\overline{L}}u = e^{tL_1}u$ for every $u \in X$.

7. Appendix. Consider the operator $l(u) = u'' + q_1(x)u' + q_0(x)u$, $x \in (a,b)$, together with the mixed boundary conditions given by

$$\begin{cases} V_1(u) \equiv a_0 u(a) + b_0 u'(a) + c_0 u(b) + d_0 u'(b) = 0, \\ V_2(u) \equiv \displaystyle\int_a^b R(t)u(t)\,dt + \int_a^b S(t)u'(t)\,dt = 0, \end{cases} \tag{7.1}$$

where $q_0, R, S \in C([a,b];\mathbf{C})$, $q_1 \in C^1([a,b];\mathbf{C})$ and the complex numbers a_0, b_0, c_0, d_0 are not simultaneously zero.

Let M_1 be the operator

$$M_1 u = l(u), \qquad D(M_1) = \{u \in W^{2,1}(a,b) : V_i(u) = 0,\ i = 1,2\}.$$

Following a similar plan as we did in Section 4, we obtain the following result.

Theorem 7.1. *If the boundary conditions* (7.1) *verify one of the following conditions*

- $b_0 S(b) - d_0 S(a) \neq 0$;
- $b_0 = d_0 = 0$ *and* $a_0 S(b) + c_0 S(a) \neq 0$;
- $S \equiv 0$ *and* $b_0 R(b) + d_0 R(a) \neq 0$;
- $S \equiv 0$, $b_0 = d_0 = 0$ *and* $a_0 R(b) - c_0 R(a) \neq 0$,

then M_1 *is the generator of an analytic semigroup of bounded linear operators* $\{e^{tM_1}\}_{t \geq 0}$ *on* $L^1(a,b)$. *In general, this semigroup will not be strongly continuous.*

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