ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 30, Number 3, Fall 2000

APPROACH GROUPS

R. LOWEN AND B. WINDELS

ABSTRACT. Any normed vector space X is a topological group with respect to the norm topology and the underlying group operation of the vector space. Although for the majority of applications it is sufficient to know that this operation $+ : X \times X \to X : (x, y) \mapsto x + y$ is continuous, stronger properties of this mapping can be shown. In fact, if $X \times X$ is equipped with the *sum* product metric, then addition becomes a contraction. Examples show that different well-known topological (semi-)groups can be equipped with a natural metric (or gauge of metrics) such that addition is contractive. This *approach group* structure is a canonical generalization of topological groups (or metric groups in the sense of Parthasarathy) and shares some of the important features with the classical concept. For instance, every approach group allows for a natural uniformization.

1. Introduction. For the convenience of the reader we briefly recall some definitions from Lowen and Windels [**6**].

A collection of ideals $(\mathcal{A}(x))_{x\in X}$ in $[0,\infty]$ is called an *approach system* on X if and only if for all $x \in X$ the following conditions are satisfied:

(A1) For all $\varphi \in \mathcal{A}(x) : \varphi(x) = 0$.

(A2) For all $\varphi \in [0,\infty]^X$: (for all $\varepsilon > 0$, for all $N < \infty$: there exists $\varphi_\varepsilon^N \in \mathcal{A}(x)$ such that $\varphi \wedge N \le \varphi_\varepsilon^N + \varepsilon) \Rightarrow \varphi \in \mathcal{A}(x)$.

(A3) For all $\varphi \in \mathcal{A}(x)$, for all $\varepsilon > 0$, for all $N < \infty$, there exists $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that for all $y, z \in X$: $\varphi(y) \wedge N \le \varphi_x(z) + \varphi_z(y) + \varepsilon$.

The pair $(X, (\mathcal{A}(x))_{x\in X})$ is called an *approach space*.

An approach space can also be described by means of a distance function $\delta : X \times 2^X \to [0,\infty]$ or by a limit operator $\lambda : \mathbf{F}(x) \to [0,\infty]^X$,

Received by the editors on April 28, 1999, and in revised form on September 7, 1999.
AMS *Mathematics Subject Classification.* 54H11, 54E15, 18B30, 54C05.
Key words and phrases. Topological group, approach space, approach group, uniformization, convergence group.
The second author is an Aspirant of the Foundation for Scientific Research Flanders.

satisfying the right conditions; see Lowen [**4**] for details.

A function $f : (X, (\mathcal{A}(x))_x) \to (Y, (\mathcal{A}'(y))_y)$ is called a *contraction* if and only if for all $x \in X$, for all $\varphi \in \mathcal{A}'(f(x)) : \varphi \circ f \in \mathcal{A}(x)$. The category of approach spaces and contractions is denoted by **Ap**.

2. Basic definitions and results. The usual approach product on $X \times X$ is not suitable in the current context. Therefore, we shall endow $X \times X$ with a so-called *additive product*.

Proposition 2.1. *Let* $(\mathcal{A}(x))_{x\in X}$ *be an approach system on a set* X*. Then*

$$\mathcal{B}(a,b) := \{\varphi : (x,y) \mapsto \varphi_1(x) + \varphi_2(y) \mid \varphi_1 \in \mathcal{A}(a), \varphi_2 \in \mathcal{A}(b)\}$$

defines an approach system on $X \times X$*.*

Proof. We only show that $\mathcal{B}(a,b)$ satisfies (A1) and (A3); the other axiom is obvious.

(A1) For every $\varphi \in \mathcal{B}(a,b)$, $\varphi(x,y) = \varphi_1(x) + \varphi_2(y)$ say, we have that $\varphi((a,b)) = \varphi_1(a) + \varphi_2(b) = 0 + 0 = 0$.

(A3) Let $\varphi \in \mathcal{B}(a,b)$, $\varphi(x,y) = \varphi_1(x) + \varphi_2(y)$, say. Let $\varepsilon > 0$ and $N < \infty$. Since $\varphi_1 \in \mathcal{A}(a)$ and $\varphi_2 \in \mathcal{A}(b)$, $(\varphi^1_z)_z \in \prod_{z\in X} \mathcal{A}(z)$ and $(\varphi^2_z)_z \in \prod_{z\in X} \mathcal{A}(z)$ exist such that for all $p, q \in X$:

$$\varphi_1(p) \wedge N \le \varphi^1_a(q) + \varphi^1_q(p) + \varepsilon/2$$

and

$$\varphi_2(p) \wedge N \le \varphi^2_b(q) + \varphi^2_q(p) + \varepsilon/2.$$

Define $(\varphi_{(z_1,z_2)})_{(z_1,z_2)} \in \prod_{(z_1,z_2)\in X\times X} \mathcal{B}((z_1,z_2))$ by $\varphi_{(z_1,z_2)}(x,y) = \varphi^1_{z_1}(x) + \varphi^2_{z_2}(y)$. Then for all $p_1, p_2, q_1, q_2 \in X$:

$$\begin{aligned}\varphi((p_1,p_2)) \wedge N &= (\varphi_1(p_1) + \varphi_2(p_2)) \wedge N \\ &\le \varphi_1(p_1) \wedge N + \varphi_2(p_2) \wedge N \\ &\le \varphi^1_a(q_1) + \varphi^1_{q_1}(p_1) + \varphi^2_b(q_2) + \varphi^2_{q_2}(p_2) + \varepsilon \\ &= \varphi_{(a,b)}(q_1,q_2) + \varphi_{(q_1,q_2)}(p_1,p_2) + \varepsilon. \qquad \square\end{aligned}$$

The approach structure defined in the previous lemma is called the *additive product*. If $X \times X$ is equipped with its additive product, we shall denote this by $X \otimes X$.

Analogous to the definition of a topological group, we define the following.

Definition 2.2. A triple $(X, \delta, +)$ is called an approach group if and only if

(a) (X, δ) is an approach space

(b) $(X, +)$ is a group

(c) $+ : X \otimes X \to X : (x, y) \mapsto x + y$ is a contraction

(d) $- : X \to X : x \mapsto -x$ is a contraction.

We now turn to some basic properties of approach groups.

Proposition 2.3. *Let $(X, \delta, +)$ be an approach group; then, for every $a \in X$,*

$$T_a : X \to X : x \mapsto x + a$$

is an isomorphism in $\mathbf{Ap}$.

Proof. If $a \in X$, then T_a is the composition of the map $X \to X \otimes X : x \mapsto (x, a)$ and addition, which are both contractions. In particular, T_{-a} is a contraction and is the inverse of T_a. □

If X is a group, then we denote the neutral element by zero, and further

$$\forall\, A, B \subset X : A - B = \{a - b \mid a \in A, b \in B\}$$

$$\forall\, \mathcal{F}, \mathcal{G} \in \mathbf{F}(X) : \mathcal{F} - \mathcal{G} = \{F - G \mid F \in \mathcal{F}, G \in \mathcal{G}\},$$

$A - x = A - \{x\}$ and $\mathcal{F} - x = \mathcal{F} - \dot{x}$.

As for topological groups, we have that the approach structure is completely determined by the structure around zero, in the sense of the following proposition.

Proposition 2.4. *Let* $(X, \delta, +)$ *be an approach group. Then*

(a) *for all* $x \in X : \mathcal{A}(x) = \{\varphi \circ T_{-x} \mid \varphi \in \mathcal{A}(0)\}$.

(b) *For all* $x \in X$*, for all* $A \subset X : \delta(x, A) = \delta(0, A - x)$.

(c) *For all* $x \in X$*, for all* $\mathcal{F} \in \mathbf{F}(X) : \lambda\mathcal{F}(x) = \lambda(\mathcal{F} - x)(0)$.

Proof. Let $x \in X$. Since T_{-x} is an isomorphism, we have that

(a) $\varphi \in \mathcal{A}(0) \Longleftrightarrow \varphi \circ T_{-x} \in \mathcal{A}(x)$,

(b) $\delta(0, A - x) = \delta(T_{-x}(x), T_{-x}(A)) = \delta(x, A)$ and

(c) $\lambda\mathcal{F}(x) = \lambda(T_{-x}(\mathcal{F}))(T_{-x}(x)) = \lambda(\mathcal{F} - x)(0)$. □

Proposition 2.5. *Let* $(X, \delta, +)$ *be an approach group, and let* $\mathcal{T}$ *be the topological coreflection of* δ*. Then* $(X, \mathcal{T}, +)$ *is a topological group.*

Proof. The fact that inversion is continuous follows from the fact that it is a contraction. To see that addition is continuous, it suffices to note that the topological coreflection of $X \times X$ and of $X \otimes X$ coincide. □

Corollary 2.6. *If* δ *is a topological approach space on* X*,* $\delta = \delta(\mathcal{T})$ *say, then the following are equivalent.*

(1) $(X, \delta, +)$ *is an approach group.*

(2) $(X, \mathcal{T}, +)$ *is a topological group.*

Proof. The fact that (1) $\Rightarrow$ (2) is exactly the previous proposition. Conversely, (2) $\Rightarrow$ (1) follows from the observations that **Top** is fully embedded in **Ap** and that if X is a topological approach space, then $X \otimes X$ is topological too. □

Denote by **ApGr** the category of approach groups and contractive group homomorphisms.

Corollary 2.7. *The category* **TopGr** *is a full coreflective subcategory of* **ApGr**.

3. Examples.

Example 3.1. (a) $(\mathbf{R}^n, \delta, +)$ with the usual distance δ and usual addition $+$ is an approach group.

(b) $(\mathbf{R}_0, \delta, \cdot)$ is not an approach group.

(c) If $(X, \odot)$ is a group and δ is the discrete distance on X, then $(X, \delta, \odot)$ is an approach group.

(d) If $(X, \odot)$ is a group and δ is the trivial distance on X, then $(X, \delta, \odot)$ is an approach group.

(e) Every topological group is an approach group.

Example 3.2. Any normed space is an approach group in the sense that if $\|\cdot\|$ is a norm on a vector space $(X, +)$ and $\delta_{\|\cdot\|}$ is the metric approach space induced by the metric $d(x, y) = \|y - x\|$, then $(X, \delta_{\|\cdot\|}, +)$ is an approach group.

This follows from the fact that for all $a, b, x, y \in X$:

$$\begin{aligned} d(a+b, x+y) &= \|(a+b) - (x+y)\| \\ &\leq \|a - x\| + \|b - y\| = d(a, x) + d(b, y) \end{aligned}$$

and

$$\begin{aligned} d(-a, -x) &= \|-a + x\| \\ &= \|a - x\| \\ &= d(a, x). \end{aligned}$$

Example 3.3. If X is a normed space and δ_w is the weak distance, i.e., for all $x \in X$:

$$\mathcal{A}(x) = \left\{ \sup_{f \in F} |f(x) - f(\cdot)| \mid F \text{ is a finite subset of } \{f \in X^* \mid \|f\| \leq 1\} \right\},$$

then $(X, \delta_w, +)$ is an approach group.

Indeed, for all $F \in 2^{X^*}$, for all $a, b, x, y \in X$:

$$\begin{aligned} \sup_{f \in F} |f(a+b) - f(x+y)| &= \sup_{f \in F} |f(a) - f(x) + f(b) - f(y)| \\ &\leq \sup_{f \in F} |f(a) - f(x)| + \sup_{f \in F} |f(b) - f(y)| \end{aligned}$$

and thus $+$ is a contraction. On the other hand, we have that for all $F \in 2^{X^*}$, for all $a, x \in X$:

$$\sup_{f\in F} |f(-a) - f(-x)| = \sup_{f\in F} |-f(a) + f(x)| = \sup_{f\in F} |f(a) - f(x)|$$

and consequently $-$ is a contraction.

Contractivity is typical for addition-like operations. However, approach groups exist which carry a multiplicative operator.

Example 3.4. Let $\mathcal{O}_2$ denote the set of all orthogonal 2×2 matrices. As a set, $\mathcal{O}_2$ can be seen as a subset of $\mathbf{R}^4$ and inherits the Euclidean metric. In fact,

$$d\left(\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}, \begin{bmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{bmatrix}\right) = 2\sqrt{1-\cos(\alpha-\beta)}.$$

It is an amusing trig exercise to check that for all $A, B, C \in \mathcal{O}_2$: $d(A\cdot C, B\cdot C) = d(A,B)$ and $d(A,B) = d(A^{-1}, B^{-1})$.

Now $(\mathcal{O}_2, \delta(d), \cdot)$ is an approach group. Indeed, for all $A, B, X, Y \in \mathcal{O}_2$ we have

$$\begin{aligned} d(AB, XY) &\leq d(AB, XB) + d(XB, XY) \\ &= d(A, X) + d(B, Y), \end{aligned}$$

and therefore multiplication is contractive. On the other hand, since $d(A^{-1}, X^{-1}) = d(A, X)$, inversion is contractive too.

Example 3.5. Let (X, d) be a metric space. Denote by

$$\mathrm{IT}\,(X) := \{f : X \to X \mid f \text{ is an isometric bijection}\}.$$

Let $\mathcal{X}$ be a collection of subsets of X such that

(1) $\cup\mathcal{X} = X$.

(2) For all $A, B \in \mathcal{X} : A \cup B \in \mathcal{X}$.

(3) For all $A \in \mathcal{X}$, for all $f \in \mathrm{IT}\,(X) : f(A) \in \mathcal{X}$.

On the set $\mathrm{IT}\,(X)$ we shall consider the approach structure of $\mathcal{X}$-convergence, that is, the approach structure δ generated by the pseudo-metrics

$$d_F : \mathrm{IT}\,(X) \times \mathrm{IT}\,(X) \longrightarrow [0,\infty] : (f,g) \longmapsto \sup_{x\in F} d(f(x),g(x))$$

for every $F \in \mathcal{X}$ (see [**6**]). Then $(\mathrm{IT}\,(X), \delta, \circ)$ is an approach group. Indeed, for every $x \in X$ and $f,g,p,q \in \mathrm{IT}\,(X)$ we have

$$\begin{aligned} d(fg(x), pq(x)) &\le d(fg(x), pg(x)) + d(pg(x), pq(x)) \\ &= d(f(g(x)), p(g(x))) + d(g(x), q(x)), \end{aligned}$$

and thus for all $F \in \mathcal{X}$:

$$d_F(fg,pq) \le d_{g(F)}(f,p) + d_F(g,q),$$

whence composition is a contraction. On the other hand, we have for all $f,p \in \mathrm{IT}\,(X)$:

$$\begin{aligned} d(f^{-1}(x), p^{-1}(x)) &= d(f^{-1}(x), p^{-1}ff^{-1}(x)) \\ &= d(pf^{-1}(x), ff^{-1}(x)) \\ &= d(f(f^{-1}(x)), p(f^{-1}(x))), \end{aligned}$$

and, therefore, for all $F \in \mathcal{X}$:

$$d_F(f^{-1}, p^{-1}) = d_{f^{-1}(F)}(f,p),$$

which means that inversion is a contraction as well.

4. Uniformization of approach groups. Recall from [**6**] that an approach uniformity (X,Γ) is a set X together with an ideal Γ of functions from $X \times X$ into $[0,\infty]$, satisfying the following conditions:

(AU1) For all $\gamma \in \Gamma$, for all $x \in X : \gamma(x,x) = 0$.

(AU2) For all $\xi \in [0,\infty]^{X\times X}$: $(\forall\, \varepsilon > 0,\ \forall\, N < \infty : \exists\, \gamma_\varepsilon^N \in \Gamma$ such that $\xi \wedge N \le \gamma_\varepsilon^N + \varepsilon) \Rightarrow \xi \in \Gamma$.

(AU3) For all $\gamma \in \Gamma, \forall\, N < \infty, \exists\, \gamma^N \in \Gamma$ such that $\forall\, x,y,z \in X$: $\gamma(x,z) \wedge N \le \gamma^N(x,y) + \gamma^N(y,z)$.

(AU4) For all $\gamma \in \Gamma : \gamma^s \in \Gamma$,

where $\gamma^s(x,y) = \gamma(y,x)$. If (X,γ) is an approach uniformity then we call the approach structure given by $\mathcal{A}(x) = \{\gamma(x,\cdot) \mid \gamma \in \Gamma\}$, the *underlying approach structure* of Γ, or that Γ and $(\mathcal{A}(x))_x$ are compatible.

As for topological groups, which are always uniformizable, we can associate with every approach group a natural approach uniformity.

Proposition 4.1. *Let $(X,\delta,+)$ be an approach group. Then*

$$\Gamma := \langle \{\gamma : X \times X \to [0,\infty] : (x,y) \longmapsto \varphi(x-y) \mid \varphi \in \mathcal{A}(0)\}\rangle$$

is an approach uniformity on X, compatible with δ.

Proof. Since $\mathcal{A}(0)$ is an ideal, Γ is an ideal too. Also we have

(AU1) For every $\gamma \in \Gamma$, $\gamma(x,y) = \varphi(x-y)$, say, and $x \in X$, we have that $\gamma(x,x) = \varphi(x-x) = \varphi(0) = 0$.

(AU3) Let $\gamma \in \Gamma$, $\gamma(x,y) = \varphi(x-y)$, say, $N < \infty$ and $\varepsilon > 0$. Since addition is contractive, some $\varphi_\varepsilon^N \in \mathcal{A}(0)$ exists such that for all $x, y \in X$:

$$\varphi(x+y) \wedge N \le \varphi_\varepsilon^N(x) + \varphi_\varepsilon^N(y) + \varepsilon.$$

Put $\gamma_\varepsilon^N(x,y) = \varphi_\varepsilon^N(x-y)$. Then, for all $x, y, z \in X$:

$$\begin{aligned}\gamma(x,z) \wedge N &= \varphi(x-y) \wedge N \\ &\le \varphi_\varepsilon^N(x-z) + \varphi_\varepsilon^N(z-y) + \varepsilon \\ &= \gamma_\varepsilon^N(x,z) + \gamma_\varepsilon^N(z,y) + \varepsilon.\end{aligned}$$

(AU4) follows from the fact that if $\varphi \in \mathcal{A}(0)$, then $\varphi(-\cdot) : x \mapsto \varphi(-x) \in \mathcal{A}(0)$.

Finally, the underlying approach structure of Γ is given by

$$\begin{aligned}\mathcal{A}'(x) &= \{\gamma(x,\cdot) \mid \gamma \in \Gamma\} \\ &= \{\gamma(\cdot,x) \mid \gamma \in \Gamma\} \\ &= \{\varphi \circ T_{-x} \mid \varphi \in \mathcal{A}(0)\} \\ &= \mathcal{A}(x). \qquad \square\end{aligned}$$

Proposition 4.2. *Let $(X, \mathcal{T}, +)$ be a topological group. If Γ is the uniformization of $(X, \delta(\mathcal{T}), +)$ and $\mathcal{U}$ is the* **Unif**-*uniformization of $(X, \mathcal{T}, +)$, then $\Gamma = \Gamma(\mathcal{U})$.*

Proof. Immediate. □

The next two sections consider two natural modifications of the concept of approach groups. In Section 5 we shall no longer require the existence of inverse elements. In Section 6 we will drop the triangular inequality-like axiom for the approach structure.

5. Approach semi-groups.

Proposition 5.1. *A triple $(X, \delta, +)$ is called an approach semi-group if and only if*

(a) (X, δ) *is an approach space.*

(b) $(X, +)$ *is a semi-group.*

(c) $+ : X \otimes X \to X : (x, y) \mapsto x + y$ *is a contraction.*

Example 5.2. Let X be a compact metrizable topological (additive) group. Let the collection $\mathcal{M}(X)$ of probability measures on X be equipped with the weak approach structure δ, see [**6**], generated by the pseudo-metrics

$$d_C(P, Q) = \sup_{f \in C} \left| \int f \, dP - \int f \, dQ \right| \qquad P, Q \in \mathcal{M}(X)$$

for every finite subset C of the set $\mathcal{C}(X, [0, 1])$ of continuous functions on X into $[0, 1]$. Let convolution on $\mathcal{M}(X)$ be denoted by $*$. In order to prove that $(\mathcal{M}(X), \delta, *)$ is an approach semi-group, it is sufficient to show that $* : \mathcal{M}(X) \otimes \mathcal{M}(X) \to \mathcal{M}(X)$ is a contraction.

Let $\mathcal{A}$ be the algebra generated by the collection

$$\{F : X \times X \to \mathbf{R} : (x, y) \mapsto F_1(x) \cdot F_2(y) \mid F_1, F_2 \in \mathcal{C}(X, [0, 1])\}.$$

Clearly, $\mathcal{A}$ contains all constant maps and separates points in $X \times X$. By the Stone-Weierstrass theorem, $\mathcal{A}$ is uniformly dense in $C^*(X \times X)$.

Consequently, and since $+$ is continuous on X, we have that for any continuous function $f \in \mathcal{C}(X,[0,1])$ and for every $\varepsilon > 0$ there exist (without loss of generality) $f_1^\varepsilon, f_2^\varepsilon \in C^*(X)$ such that $\sup_{x,y\in X} |f_1^\varepsilon(x) \cdot f_2^\varepsilon(y) - f(x+y)| < \varepsilon/2$.

Now let $\varepsilon > 0$ and $C \in 2^{\mathcal{C}(X,[0,1])}$. Consider the finite sets $C_1 = \{f_1^\varepsilon \mid f \in C\}$ and $C_2 = \{f_2^\varepsilon \mid f \in C\}$. For every $f \in C$ and for every $P, Q, \mu, \nu \in \mathcal{M}(X)$, we see that by definition of convolution and by the Fubini theorem

$$\begin{aligned}
&\left| \int f\, d(P * Q) - \int f\, d(\mu * \nu)\right| \\
&= \left| \int f(x+y) d(P \times Q)(x,y) - \int f(x+y)\, d(\mu \times \nu)(x,y)\right| \\
&\leq \left| \int (f_1^\varepsilon(x) f_2^\varepsilon(y))\, d(P \times Q)(x,y) - \int (f_1^\varepsilon(x) f_2^\varepsilon(y))\, d(\mu \times \nu)(x,y)\right| + \varepsilon \\
&\leq \left| \int f_1^\varepsilon(x)\, dP(x) - \int f_1^\varepsilon(x)\, d\mu(x)\right| \\
&\quad + \left| \int f_2^\varepsilon(y)\, dQ(y) - \int f_2^\varepsilon(y)\, d\nu(y)\right| + \varepsilon.
\end{aligned}$$

If we take the supremum over all $f \in C$, then we find that

$$d_C(P * Q, \mu * \nu) \leq d_{C_1}(P, \mu) + d_{C_2}(Q, \nu) + \varepsilon.$$

Therefore $(\mathcal{M}(X), \delta, *)$ is an approach semi-group.

6. Approach convergence groups.

6.1 Definitions. The aim of this section is to generalize the notion of *convergence groups* introduced in [**2**]. Recall from [**3**] that (X, λ) is called a *convergence approach space*, if $\lambda : \mathbf{F}(X) \to [0,\infty]^X$ satisfies the following conditions for every $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$:

(CAL1) For all $x \in X$: $\lambda \dot{x}(x) = 0$

(CAL2) $\mathcal{F} \subset \mathcal{G} \Rightarrow \lambda\mathcal{G} \leq \lambda\mathcal{F}$.

(CAL3) $\lambda(F \cap \mathcal{G}) \leq \lambda\mathcal{F} \vee \lambda\mathcal{G}$.

If (X, λ) and (Y, λ') are convergence approach spaces, then a function $f : (X, \lambda) \to (Y, \lambda')$ is called a *contraction* if and only if for all

$\mathcal{F} \in \mathbf{F}(X) : \lambda'(f(\mathcal{F}))\circ f \leq \lambda(\mathcal{F})$. The category of convergence approach spaces and contractions is denoted by **Cap**.

Lemma 6.1. *If $(X, (\mathcal{A}(x))_x$ is an approach space, then the approach limit on $X \otimes X$ is given by*

$$\lambda_\otimes \mathcal{F}(a,b) = \sup_{\varphi_a \in \mathcal{A}(a)} \sup_{\varphi_b \in \mathcal{A}(b)} \inf_{F \in \mathcal{F}} \sup_{(x,y) \in \mathcal{F}} (\varphi_a(x) + \varphi_b(y)).$$

Proof. This is a combination of the definition of the additive product and Proposition 1.8.30 in [**5**]. □

For a convergence approach space (X, λ) too, we can consider the sets

$$\mathcal{A}(x) = \{\varphi \mid \forall\, \mathcal{F} \in \mathbf{F}(X) : \inf_{F \in \mathcal{F}} \sup_{y \in F} \varphi(y) \leq \lambda \mathcal{F}(x)\}.$$

Lemma 6.2. *Let (X, λ) be a convergence approach space. The map $\lambda_\otimes$ defined in Lemma* 6.1 *defines a convergence approach structure on $X \times X$.*

We call this structure again the *additive* product on $X \times X$, and we denote this by $X \otimes X$. We have for all $\mathcal{F} \in \mathbf{F}(X \times X)$:

$$\lambda_\otimes \mathcal{F} \leq \lambda(\mathrm{pr}_1 \mathcal{F}) + \lambda(\mathrm{pr}_2 \mathcal{F}),$$

but the inequality is strict in general, which is shown in the following example. In particular, contractivity of $+$ yields for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X)$:

$$\lambda(\mathcal{F} + \mathcal{G}) \leq \lambda(\mathcal{F}) + \lambda(\mathcal{G}).$$

Example 6.3. On $\mathbf{R}^2$ with the usual Euclidean approach structure, consider the filter $\mathcal{F} = \text{stack}\ (\{(x, 1-x) \mid 0 \leq x \leq 1\})$. Then

$$\lambda_\otimes \mathcal{F}(0,0) = \inf_{F \in \mathcal{F}} \sup_{(x,y) \in F} (d(0,x) + d(0,y)) = 1$$

and

$$\lambda(\mathrm{pr}_1\mathcal{F})(0) + \lambda(\mathrm{pr}_2\mathcal{F})(0) = \inf_{F_1 \in \mathrm{pr}_1\mathcal{F}} \sup_{x \in F_1} d(0,x) + \inf_{F_2 \in \mathrm{pr}_2\mathcal{F}} \sup_{y \in F_2} d(0,y) = 2.$$

Definition 6.4. A triple $(X, \lambda, +)$ is called an approach convergence group if and only if

(a) (X, λ) is a convergence approach space

(b) $(X, +)$ is a group

(c) $+ : X \otimes X : (x, y) \mapsto x + y$ is a contraction

(d) $- : X \to X : x \mapsto -x$ is a contraction.

6.2 Examples.

Proposition 6.5. *Every approach group is a convergence approach group.*

Proof. Follows from Lemma 6.2. □

Proposition 6.6. *Every convergence group is a convergence approach group.*

Proof. Analogous to the proof of Proposition 2.5. □

6.3. Uniformization of (approach) convergence groups. A convergence group admits a uniform convergence structure, which was shown in Cook and Fischer [**1**]. Unfortunately, this particular uniformization is not an extension of the classical uniformization of topological groups (see Example 6.7). Therefore, the uniformization of approach convergence structures presented in this section shall not be required to extend the concept in [**1**]. Instead we shall suggest an alternative uniformization for ordinary convergence groups.

Example 6.7. Let us start from **R** equipped with the usual topology and addition. This topological group is uniformized by the usual uniformity $\mathcal{U}$ on $\mathbf{R}$. Now suppose the associated uniform convergence structure $\mathbf{L} = \{\mathcal{G} \mid \mathcal{G} \supset \mathcal{U}\}$ coincides with the uniformization of $(\mathbf{R}, \lambda, +)$; that is, the u.c.s. generated by $\mathbf{L}' = \{\mathcal{F} \times \mathcal{F} \mid \mathcal{F} - \mathcal{F} \to 0\}$. Then there are $\mathcal{F}_1, \dots, \mathcal{F}_n$ such that

$$\mathcal{F}_i - \mathcal{F}_i \longrightarrow 0 \quad \text{and} \quad \mathcal{U} \supset \bigcap_{i=1}^{n} \mathcal{F}_i \times \mathcal{F}_i.$$

Let $\varepsilon < \infty$ and choose $F_i \in \mathcal{F}_i$ such that $F_i - F_i \subset [-\varepsilon, \varepsilon]$. Then for all $i \in \{1, \dots, n\}$: $\operatorname{diam} F_i \leq \varepsilon$, and thus no $U \in \mathcal{U}$ can be covered by $\cup_{i=1}^{n} F_i \times F_i$.

We adopt the following notations for a group X. If $F \subset X \times X$ and $x \in X$, we denote the section of F in x by $F(x) = \{y \in X \mid (x, y) \in F\}$. If $\mathcal{F} \in \mathbf{F}(X \times X)$ then we denote the *global section* of $\mathcal{F}$ by $\mathcal{F}_X = \{\cup_{x \in X} F(x) - x \mid F \in \mathcal{F}\} \in \mathbf{F}(X)$.

Lemma 6.8. *Let X be a group. Then for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X \times X)$:*

(a) $(\mathcal{F} \cap \mathcal{G})_X = \mathcal{F}_X \cap \mathcal{G}_X$

(b) $(\mathcal{F} \times \mathcal{G})_X = \mathcal{G} - \mathcal{F}$

(c) $(\mathcal{F}^{-1})_X = -(\mathcal{F}_X)$,

(d) $(\mathcal{F} \circ \mathcal{G})_X \supset \mathcal{F}_X + \mathcal{G}_X$.

Proof. (a) follows from the observation that

$$\bigcup_{x \in X} (F \cup G)(x) - x = \bigcup_{x \in X} F(x) - x \cup \bigcup_{x \in X} G(x) - x.$$

(b) follows from the observation that for all $F \in \mathcal{F}$, for all $G \in \mathcal{G}$,

$$\bigcup_{x \in X} (F \times G)(x) - x = \bigcup_{x \in F} G - x = G - F.$$

(c) follows from the observation that $\cup_{x \in X} F^{-1}(x) - x =$

$-(\cup_{x\in X} F(x) - x)$ which holds since

$$\begin{aligned} y \in \bigcup_{x\in X} F^{-1}(x) - x &\Longleftrightarrow \exists x \in x : (x, x+y) \in F^{-1} \\ &\Longleftrightarrow \exists x \in X : (x+y, x) \in F \\ &\Longleftrightarrow \exists z \in X : (z, z-y) \in F \\ &\Longleftrightarrow \exists z \in X : y - z \in -F(z) \\ &\Longleftrightarrow y \in \bigcup_{z\in X} z - F(z). \end{aligned}$$

(d) follows from the observation that

$$\bigcup_{x\in X} (F\circ G)(x) - x \subset \bigcup_{x\in X} F(x) - x + \bigcup_{x\in X} G(x) - x.$$

Indeed, if $y \in \cup_{x\in X}(F\circ G)(x) - x$, then for some $x \in X$ we have that $(x, x+y) \in F\circ G$, which means that $p \in X$ exists such that $(x,p) \in G$ and $(p, x+y) \in F$. Let $y_1 = y - o + x$ and $y_2 = p - x$. Then

$$\begin{aligned} y_1 + y_2 &= (y - p + x) + (p - x) = y \\ (p, p+y_1) &= (p, x+y) \in F \\ (x, x+y_2) &= (x, p) \in G, \end{aligned}$$

and therefore,

$$\begin{aligned} &y \in \{y_1 + y_2 \mid \exists\, p, x \in X : (p, p+y_1) \in F, (x, x+y_2) \in G\} \\ &\quad = \Bigl(\bigcup_{p\in X} F(p) - p\Bigr) + \Bigl(\bigcup_{x\in X} G(x) - x\Bigr). \qquad \square \end{aligned}$$

Recall from [**8**] that an *approach uniform convergence structure* on the set X is a map η: $\mathbf{F}(X\times X) \to [0,\infty]$ such that for all $x \in X$, for all $\mathcal{F}, \mathcal{G} \in \mathbf{F}(X\times X)$:

(AUCS1) $\eta(\dot{x}\times\dot{x}) = 0$

(AUCS2) $\mathcal{F} \subset \mathcal{G} \Rightarrow \eta(\mathcal{G}) \le \eta(\mathcal{F})$

(AUCS3) $\eta(\mathcal{F}\cap\mathcal{G}) \le \eta(\mathcal{F}) \vee \eta(\mathcal{G})$

(AUCS4) $\eta(\mathcal{F}^{-1}) = \eta(\mathcal{F})$

(AUCS5) $\eta(\mathcal{F} \circ \mathcal{G}) \leq \eta(\mathcal{F}) + \eta(\mathcal{G})$.

If (X, η) is an approach uniform convergence group, then $\lambda\mathcal{F}(x) = \eta(F \times \dot{x})$ defines a convergence approach structure on X. We call λ the underlying convergence structure of η, or say that η and λ are compatible.

With every approach convergence group, we can associate an approach uniform convergence space in a natural way.

Proposition 6.9. *Let $(X, \lambda, +)$ be an approach convergence group. Then*

$$\eta : \mathbf{F}(X \times X) \longrightarrow [0, \infty] : \mathcal{F} \longmapsto \lambda\mathcal{F}_X(0)$$

is an approach uniform convergence structure on X, compatible with λ.

Proof. First we show that η is an approach uniform convergence structure.

(AUCS1) Let $x \in X$. By Lemma 6.8(b) and (CAL1), $\eta(\dot{x} \times \dot{x}) = \lambda\dot{0}(0) = 0$.

(AUCS2) Suppose $\mathcal{F} \subset \mathcal{G}$. Then $\mathcal{F}_X \subset \mathcal{G}_X$ and therefore, by (CAL2), $\eta(\mathcal{G}) = \lambda\mathcal{G}_X(0) \leq \lambda\mathcal{F}_X(0) = \eta(\mathcal{F})$.

(AUCS3) By Lemma 6.8(a) and (CAL3) we have

$$\begin{aligned}\eta(\mathcal{F} \cap \mathcal{G}) &= \lambda(\mathcal{F} \cap \mathcal{G})_X(0) = \lambda(\mathcal{F}_X \cap \mathcal{G}_X)(0) \\ &\leq \lambda\mathcal{F}_X(0) \vee \lambda\mathcal{G}_X(0) = \eta(F) \vee \eta(G).\end{aligned}$$

(AUCS4) By Lemma 6.8(c) and contractivity of inversion, we have $\eta(\mathcal{F}^{-1}) = \lambda(\mathcal{F}^{-1})_X(0) = \lambda(-\mathcal{F}_X)(0) = \lambda\mathcal{F}_X(0) = \eta(\mathcal{F})$.

(AUCS5) By Lemma 6.8(d), (CAL2) and contractivity of $+$, we have that

$$\begin{aligned}\eta(\mathcal{F} \circ \mathcal{G}) &= \lambda(\mathcal{F} \circ \mathcal{G})_X(0) \leq \lambda(\mathcal{F}_X + \mathcal{G}_X)(0) \\ &\leq \lambda\mathcal{F}_X(0) + \lambda\mathcal{G}_X(0) = \eta(\mathcal{F}) + \eta(\mathcal{G}).\end{aligned}$$

Second we show that the underlying **Cap**-structure of η is λ. Let λ' denote the underlying structure of η. Then by Lemma 6.8(d) we see that for all $\mathcal{F} \in \mathbf{F}(X)$ for all $x \in X$:

$$\begin{aligned}\lambda'\mathcal{F}(x) &= \eta(\mathcal{F} \times \dot{x}) = \eta(\dot{x} \times \mathcal{F}) \\ &= \lambda(\dot{x} \times \mathcal{F})_X(0) = \lambda(\mathcal{F} - x)(0) = \lambda\mathcal{F}(x). \qquad \square\end{aligned}$$

This concept generalizes the uniformization of approach groups discussed in Section 4.

Proposition 6.10. *Let $(X, \lambda, +)$ be an approach group. If η_1 is the* a.u.c.s. *induced by the approach uniformity Γ uniformizing X, and if η_2 is the* a.u.c.s. *uniformizing X as an approach convergence group, then $\eta_1 = \eta_2$.*

Proof. Let $\mathcal{A}(0)$ be the approach system of 0 with respect to λ. Then for all $\mathcal{F} \in \mathbf{F}(X \times X)$:

$$\begin{aligned}\eta_1(\mathcal{F}) &= \min\{\varepsilon \mid \mathcal{F} \supset \{\{\gamma < \alpha\} \mid \gamma \in \Gamma, \alpha > \varepsilon\}\} \\ &= \min\{\varepsilon \mid \forall\, \alpha > \varepsilon,\ \forall\, \varphi \in \mathcal{A}(0),\ \exists\, F \in \mathcal{F}, \\ &\qquad \forall\, (x,y) \in F : \varphi(y - x) < \alpha\} \\ &= \sup_{\varphi \in \mathcal{A}(0)} \inf_{F \in \mathcal{F}} \sup_{(x,y) \in \mathcal{F}} \varphi(y - x) \\ &= \sup_{\varphi \in \mathcal{A}(0)} \inf_{F_X \in \mathcal{F}_X} \sup_{z \in F_X} \varphi(z) \\ &= \lambda\mathcal{F}_X(0) = \eta_2(\mathcal{F}). \qquad \square\end{aligned}$$

As we stated earlier, our construction does not generalize the uniformization of convergence groups discussed in [**1**]: the latter is coarser in general. Therefore, we suggest the following modification.

Proposition 6.11. *Let $(X, \tau, +)$ be a convergence group. Then the collection*

$$\mathbf{L} = \{\mathcal{F} \in \mathbf{F}(X \times X) \mid F_X \to 0\}$$

is a uniform convergence structure compatible with $(X, \tau, +)$. Moreover, if $(X, \tau, +)$ is a topological group, then $\cap\{\mathcal{F} \mid \mathcal{F} \in \mathbf{L}\}$ is the classical uniformization of $(X, \tau, +)$.

Proof. The first part of the proposition is an easy consequence of Proposition 6.9. The second part is just the combination of Proposition 5.1 and Proposition 6.10. □

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University of Antwerp RUCA, Groenenborgerlaan 171, B-2020 Antwerp, Belgium
E-mail address: lowen@ruca.ua.ac.be

University of Antwerp RUCA, Groenenborgerlaan 171, B-2020 Antwerp, Belgium
E-mail address: windels@ruca.ua.ac.be