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FOURIER-TYPE MINIMAL EXTENSIONS IN REAL L_1-SPACE

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ABSTRACT. Let $V \subset L_1([0, 2\pi]^n)$ be a finite dimensional, shift-invariant subspace. Fix $w \in V$. In this paper we present an answer to the problem when the Fourier-type operator $F_w : L_1([0, 2\pi]^n) \to V$ defined by

$$F_w(f) = f * w$$

is the only minimal extension of its restriction to V. Also the case of supremum norm will be considered and [**5**, p. 243] will be generalized.

0. Introduction. Let π_k denote the space of all trigonometric polynomials of degree less than or equal to k. Let $C_0(2\pi)$ denote the space of all continuous, real valued 2π-periodic functions. It is well known by the result of Lozinski [**10**] that the Fourier projection defined by

$$(F_k f)t = (f * D_k)t = (1/2\pi) \int_0^{2\pi} f(s) D_k(t - s)\, ds,$$

where $D_k t = \sum_{j=-k}^{k} e^{ijt}$, has the minimal norm among all the projections of $C_0(2\pi)$ onto π_k (see also [**1**], [**11**]). Also, it has been shown in [**3**] that F_k is the only projection from $C_0(2\pi)$ onto π_k of minimal norm. The problem of the unique minimality of the Fourier projection in more general context has been widely studied in literature (see, e.g., [**2**], [**5**], [**7**], [**8**], [**9**]).

In this paper we study the problem of the unique minimality of the Fourier-type extensions in the case of the L_1-norm. More precisely, for $u \in \mathbf{R}^n$ and $f \in L_1 = L_1([0, 2\pi]^n)$, let

$$(I_u f)t = f(t + u). \tag{0.1}$$

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(We define $f(t+u) = f(r)$ where $r = (r_1, \dots, r_n) \in [0, 2\pi]^n$ and $r_j = (t_j + u_j) \bmod 2\pi$.) Let $V \subset L_1$ be a finite-dimensional, shift-invariant subspace, i.e., $I_u(V) \subset V$ for any $u \in \mathbf{R}^n$. Let $w \in V$. Set

$$(F_w f)(t) = (f * w)(t) = (1/2\pi)^n \int_{[0,2\pi]^n} f(s) w(t-s)\, ds. \tag{0.2}$$

Observe that if $V = \pi_k$ and $w = D_k$, then F_w is precisely the Fourier projection onto π_k. The main result of this paper, Theorem 1.6, is a characterization of those elements $w \in V$ for which the convolution operator F_w is the only extension of minimal norm of

$$R_w = F_w|_V : V \longrightarrow V \tag{0.3}$$

onto the whole L_1 (compare with [**8**, Theorem 1]). We also show that if F_w is not the only minimal extension of R_w, then the affine dimension of the set of all minimal extensions is infinite (Theorem 1.8). Also, some examples illustrating the possible applications of Theorems 1.6 and 1.8 will be given.

Now we will present a basic notation and some results and notions which will be of use later.

To the end of this paper, unless otherwise stated, $L_1 = L_1([0, 2\pi])^n)$, $L_\infty = L_\infty([0, 2\pi]^n)$, $C = C([0, 2\pi]^n)$ and V will be a finite-dimensional shift-invariant subspace of L_1. All linear spaces considered in this paper are real. The symbol $\int f(s)\, ds$ will stand for $(1/2\pi)^n \int_{[0,2\pi]^n} f(s)\, ds$. For any normed space X by $S(X)$ we denote the unit sphere in X. The symbol $\mathcal{L}(X, V)$ ($\mathcal{L}(X)$ if $X = V$) stands for the space of all linear, continuous mappings from a normed space X into a normed space V. Let V be a finite dimensional subspace of a Banach space X. Fix $A \in \mathcal{L}(V)$. Then

$$\mathcal{P}_A(X, V) = \{P \in \mathcal{L}(X, V) : P|_V = A\}. \tag{0.4}$$

An element $P_0 \in \mathcal{P}_A(X, V)$ is called a *minimal extension* if

$$\|P_0\| = \inf\{\|P\| : P \in \mathcal{P}_A(X, V)\}. \tag{0.5}$$

The symbol $\mathrm{Min}_A(X, V)$ will stand for the set of all minimal extensions of A. Observe that if $A = \mathrm{id}|_V$, then $\mathcal{P}_A(X, V)$ is exactly the set of all linear projections from X onto V and each minimal extension is exactly a minimal projection.

Lemma 0.1 (See, e.g., [**6**, 1.2]). *Let* $(S, \sum, \mu)$ *be a measure space. Let* $P \in \mathcal{L}(L_1(S))$, $P = \sum_{j=1}^{k} u_j(\cdot)v_j$, *where* $v_j \in L_1(S)$, $u_j \in L_\infty(S)$ *for* $j = 1, \ldots, n$. *Put*

$$x_s^P = \sum_{j=1}^{k} u_j(s)v_j. \tag{0.6}$$

Then

$$\|P\| = \max\{P^s : s \in S\},$$

where

$$P^s = \int_S \left| \sum_{j=1}^{k} u_j(s)v_j(t) \right| dt = \int_S |x_s^P(t)|\, dt. \tag{0.7}$$

Lemma 0.2. *Let* $w \in V$. *Then* $F_w \circ I_u = I_u \circ F_w$ *for any* $u \in \mathbf{R}^n$.

Proof. Note that for any $t, u \in \mathbf{R}^n$ and $f \in L_1$,

$$\begin{aligned}(I_u \circ F_w f)t = (F_w f)(t+u) &= \int f(s)w(t+u-s)\, ds \\ &= \int f(u+s)w(t-s)\, ds = (F_w \circ I_u f)t,\end{aligned}$$

as required.

Lemma 0.3. *Let* $V \subset L_1$, *and let* $w \in V$. *Then, for any* $s \in [0, 2\pi]^n$,

$$F_w^s = \int |w(t-s)|\, dt = \|F_w\| = \|w\|_1. \tag{0.8}$$

Proof. Note that $w = \sum_j \lambda_j v_j$, where $\{v_j\}$ is a fixed basis of V. Since V is an invariant subspace, for every $t, s \in [0, 2\pi]^n$, $w(t-s) =$

$\sum_j \lambda_j(s)v_j$, where $\lambda(s) \in L_\infty$. By (0.7),

$$F_w^s = \int \Big| \sum_j \lambda_j(s)v_j(t) \Big| \, dt = \int |w(t-s)| \, dt = \|w\|_1 = \|F_w\|,$$

as required.

Lemma 0.4 [**2**, 1.3]. *Let $w \in V$. If P and F_w are minimal extensions of R_w, then for all $s \in [0, 2\pi]^n$,*

$$\operatorname{sgn}(x_s^P) = \operatorname{sgn}(x_s^{F_w}). \tag{0.9}$$

Definition 0.5. Let $V \subset L_1$. V is said to be a smooth subspace if 0 is the only element of V vanishing on a set of positive measure.

Theorem 0.6 [**4**]. *Let T be a compact set without isolated points. If $L : C(T) \to C(T)$ is a compact operator, then*

$$\|I + L\| = 1 + \|L\|. \tag{0.10}$$

1. The results. First we prove some lemmas which will be of use later.

Lemma 1.1. *Let $w \in V$. Suppose that F_w is the only extension of R_w which commutes with each operator I_u (see Lemma 0.2). Then F_w is a minimal extension of its restriction to V, R_w, in $\mathcal{P}_{R_w}(L_1, V)$.*

Proof. The proof goes on the same line as the proof of the minimality of the Fourier projection from $C_0(2\pi)$ onto π_n (see [**10**]), but we present a sketch of it for the sake of completeness. Let $P \in \mathcal{P}_{R_w}(L_1, V)$ be a minimal extension of R_w. Let us define

$$Q = \int (I_u)^{-1} P I_u \, du, \tag{1.1}$$

Note that, by (0.1), each operator I_u is a linear isometry of L_1. Hence it is easy to check that Q is an extension of R_w of minimal norm.

Moreover, by the properties of the Haar measure, $I_u \circ Q = Q \circ I_u$ for any $u \in \mathbf{R}^n$. Hence $F_w = Q$, as required.

Lemma 1.2 (See, e.g., [**5**]). *Suppose that F_w is not the unique minimal extension of R_w. Then F_w is not an extreme point of $\mathrm{Min}_{R_w}(L_1, V)$.*

Proof. Suppose that $P \in \mathrm{Min}_{R_w}(L_1, V)$, $P \neq F_w$, exists. Then there is a $v \in L_1 \backslash V$ such that $Pv \neq F_w v$. By the proof of Lemma 1.1, for any set of positive measure $A \subset [0, 2\pi]^n$,

$$\begin{aligned} F_w &= \int (I_u)^{-1} P I_u \, du \\ &= m(A)\bigg(\int_A (I_u)^{-1} P I_u \, du\bigg)\Big/ m(A) \\ &\quad + (1 - m(A))\bigg(\int_{[0,2\pi]^n \backslash A} (I_u)^{-1} P I_u \, du\bigg)\Big/(1 - m(A)). \end{aligned}$$

Put

$$Q_1^A = \bigg(\int_A (I_u)^{-1} P I_u \, du\bigg)\Big/ m(A)$$

and

$$Q_2^A = \bigg(\int_{[0,2\pi]^n \backslash A} (I_u)^{-1} P I_u \, du\bigg)\Big/(1 - m(A)).$$

By Lemma 0.2, Q_1^A and Q_2^A are both minimal extensions of R_w for any set A of positive measure. Moreover, since $Pv \neq F_w v$, $Q_1^A v \neq F_w v$ for A being a sufficiently small neighborhood of $(0, \dots, 0)$ in $[0, 2\pi]^n$. Hence $Q_1^A \neq F_w$, which completes the proof.

Lemma 1.3. *Let W be a finite dimensional subspace of L_∞. Then a real valued function $f \in L_\infty$, $f \neq 0$, exists such that*

$$\int g(t) f(t) \, dt = 0 \tag{1.2}$$

for any $g \in W$.

Proof. Note that, in our case, $W \subset L_\infty \subset L_2([0,2\pi]^n)$. Hence the result follows easily from the Gram-Schmidt orthonormalization procedure.

Lemma 1.4. *Suppose $w \in S(V)$, $g \in V$ and $g/w \in L_\infty$. Assume additionally that V is a smooth subspace of L_1 (see Definition* 0.5*). Then, for any $\varepsilon \in \mathbf{R}$ with*

$$|\varepsilon| < (\|(g-w)/w\|_\infty)^{-1}, \tag{1.3}$$

$\operatorname{sgn}(w + \varepsilon(g-w)) = \operatorname{sgn}(w)$ *almost everywhere.*

Proof. Take any $t \in [0,2\pi]^n$ with $w(t) \neq 0$. By (1.3),

$$\begin{aligned} |\varepsilon \operatorname{sgn}(w(t))(g(t)-w(t))| &\leq |\varepsilon|\,|g(t)-w(t)| \\ &= |\varepsilon|(|g(t)-w(t)|/|w(t)|)|w(t)| \\ &\leq |\varepsilon|\,\|(g-w)/w\|_\infty |w(t)| < |w(t)|. \end{aligned} \tag{1.4}$$

Consequently, by (1.4),

$$\operatorname{sgn}(w(t))(w(t) + \varepsilon(g(t)-w(t))) > 0,$$

which completes the proof.

Definition 1.5. Let $V \subset L_1$, and let $w \in V$, $w \neq 0$. It is said that w is determined by its roots if and only if, for any $g \in V$, if $g/w \in L_\infty$, then g is a constant multiple of w. Now we will prove the main result of this section.

Theorem 1.6. *Let $V \subset L_\infty$ be a smooth subspace equipped with the L_1-norm, and let $w \in V$. Suppose that F_w satisfies the assumptions of Lemma* 1.1*. Then F_w is the only minimal extension of R_w if and only if w is determined by its roots.*

Proof. Without loss, to the end of this proof we can assume that $\|w\|_1 = 1$. Suppose that F_w is not the only minimal extension. By Lemma 1.2, F_w is not an extreme point in the set $\mathrm{Min}_{R_w}(L_1, V)$ of all minimal extensions of R_w. Hence, $F_w = (Q_1 + Q_2)/2$, where Q_1 and Q_2 are two different elements from $\mathrm{Min}_{R_w}(L_1, V)$. Put $L = (Q_1 - Q_2)/2 = \sum_{j=1}^{k} u_j(\cdot)v_j$, where $\{v_j : j = 1, \ldots, k\}$ is a fixed basis of V. Then $Q_1 = F_w + L$ and $Q_2 = F_w - L$. Consequently, by Lemma 0.3,

$$1 = \|F_w - L\| = \|F_w\| = \|F_w + L\|. \tag{1.5}$$

By Lemma 0.4,

$$\mathrm{sgn}\,(x_t^{F_w}) = \mathrm{sgn}\,(x_t^{F_w+L}) = \mathrm{sgn}\,(x_t^{F_w-L}) \tag{1.6}$$

for all $t \in [0, 2\pi]^n$, where $x_t^{(\cdot)}$ is defined by (0.6). Now fix $t \in [0, 2\pi]^n$ satisfying (1.6) with $x_t^L \neq 0$. Note that, by Lemma 0.3, for any $s \in [0, 2\pi]^n$, $x_t^{F_w}(s) = w(s - t)$. Put $h = I_t(x_t^L)$ and set $g = (w + h)$, where I_t is defined by (0.1). Since $x_t^L \neq 0$ and I_t is a linear isometry, $h \neq 0$. By (1.6), $\mathrm{sgn}\,(g) = \mathrm{sgn}\,(w) = \mathrm{sgn}\,(w - h)$ almost everywhere. Note that, by Lemma 0.1,

$$\|g\|_1 = (F_w + L)^t \leq \|F_w + L\| = 1$$

and

$$\|w - h\|_1 = (F_w - L)^t \leq \|F_w - L\| = 1.$$

Since $w = (g + (w - h))/2$ and $\|w\|_1 = 1$, $\|g\|_1 = 1$.

Now we show that $g/w \in L_\infty$. If not, then for every $k \in \mathbf{N}$ there is a measurable set A_k of positive measure, such that

$$|g(s)|/|w(s)| > k \tag{1.7}$$

for any $s \in A_k$. Put, for $s \in [0, 2\pi]^n$, $a_s = \mathrm{sgn}\,(w(s))$. By (1.7) we easily get

$$a_s h(s) > (k - 1)a_s w(s).$$

Hence,

$$a_s(w(s) - h(s)) < (2 - k)|w(s)| < 0.$$

Consequently, $\mathrm{sgn}\,(w-h) \neq \mathrm{sgn}\,(w)$ on A_k for $k > 2$; a contradiction with (1.6). Since w is determined by its roots, $g = cw$. Since $\mathrm{sgn}\,(g) = \mathrm{sgn}\,(w)$ almost everywhere and $\|g\|_1 = 1$, $g = w$ and, consequently, $h = 0$, a contradiction.

To prove the converse, suppose F_w is the only minimal extension of R_w. Take $g \in V$ such that $g/w \in L_\infty$. We will show that $g = cw$ for some $c \in \mathbf{R}$. Note that, by Lemma 1.4, for $\varepsilon > 0$ sufficiently small $\operatorname{sgn}(w + \varepsilon(g - w)) = \operatorname{sgn}(w)$ almost everywhere. Put

$$g_1 = (w + \varepsilon(g - w))/\|w + \varepsilon(g - w)\|_1$$

and $h = g_1 - w$. Since $\|g\|_1 = \|w\|_1 = 1$ and $\operatorname{sgn}(g_1) = \operatorname{sgn}(w)$, almost everywhere,

$$\int h(t)\operatorname{sgn}(w(t)) = 0. \tag{1.8}$$

Let

$$W = \operatorname{span}[v(\cdot)h(t - \cdot) : v \in V,\ t \in [0, 2\pi]^n]. \tag{1.9}$$

Since V is an invariant subspace, $\dim(W) < \infty$. Moreover, since $V \subset L_\infty$, $W \subset L_\infty$. By Lemma 1.3, we can find a real-valued function $f \in L_\infty$ such that $0 < \|f\|_\infty < (\|h/w\|_\infty)^{-1}$ which is orthogonal to v for any $v \in W$. Define $Q = F_w + L$ where

$$(L\phi)t = \int f(s)\phi(s)h(t - s)\, ds. \tag{1.10}$$

We show that Q is an extension of R_w of minimal norm. Note that $L\phi \in V$ for any $\phi \in L_1$ and, by the choice of f, $Lv = 0$ for any $v \in V$. Hence $Q : L_1 \to V$. To show that Q is minimal extension, let us observe that, by Lemmas 0.1, 0.3, 1.4 and (1.8),

$$\begin{aligned}
\|F_w\| \le \|Q\| &= \max_s [Q_s] \\
&= \max_s \left[\int |w(t-s) + f(s)h(t-s)|\, dt\right] \\
&= \max_s \left[\int \operatorname{sgn}(w(t-s))(w(t-s) + f(s)h(t-s))\, dt\right] \\
&= \max_s \left[\int |(w(t-s)|\, dt + f(s)\int \operatorname{sgn}(w(t-s))h(t-s)\, dt\right] \\
&= \max_s \left[\int |(w(t-s)|\, dt\right] = \|F_w\|.
\end{aligned}$$

Since F_w is the only minimal extension of R_w and V is smooth, $h = 0$. Hence, $g_1 = w$ and consequently $g = cw$ for some $c \in \mathbf{R}$. The proof is complete. $\square$

Remark 1.7. A complex version of Theorem 1.6 has been proved in [**9**] (see also [**8**, Theorem 1]). Observe that in the case of one variable if $w = D_k$ and $V = \pi_k$, then w is determined by its roots. By Theorem 1.6, the classical Fourier projection F_w is the only minimal projection with respect to $\|\cdot\|_1$.

Theorem 1.8. *Suppose $V \subset L_\infty$ and V is smooth. Let $w \in S(V)$. Put*

$$S_w = \operatorname{span}[\operatorname{Min}_{R_w}(L_1, V) - F_w].$$

If w is not determined by its roots, then $\dim(S_w) = \infty$.

Proof. If w is not determined by its roots, $g \in V$ exists such that $g/w \in L_\infty$ and g is not a constant multiple of w. Reasoning as in the proof of Theorem 1.6, we can construct $h \in V$ satisfying (1.8). Since g is not a constant multiple of w, $h \neq 0$. By Lemma 1.3 we can construct a linearly independent set $\{f_j : j \in \mathbf{N}\} \subset L_\infty$ orthogonal to the space W defined by (1.9) such that $0 < \|f_j\|_\infty < (\|h/w\|_\infty)^{-1}$. Let L_j be the operator associated with f_j and h by (1.10), and let $Q_j = F_w + L_j$. Reasoning as in the proof of Theorem 1.6, we can show that $Q_j \in \operatorname{Min}_{R_w}(L_1, V)$. Hence, to finish the proof, it is enough to show that $\{L_j : j \in \mathbf{N}\}$ is linearly independent. To do this, suppose that $\sum_{j=1}^k a_j L_j = 0$. This means that, for every $g \in L_1$, $t \in [0, 2\pi]^n$,

$$\int g(s)\bigg(\sum_{j=1}^k a_j f_j(s) h(t-s)\bigg)\,ds = 0. \tag{1.11}$$

By (1.11) and the Goldstine Theorem, for any $\phi \in (L_\infty)^*$,

$$\phi\bigg(\sum_{j=1}^k a_j f_j(s) h(t-s)\bigg) = 0,$$

and consequently, $\sum_{j=1}^{k} a_j f_j h(t-\cdot) = 0$. Since $h \neq 0$ and V is smooth, $\sum_{j=1}^{k} a_j f_j = 0$. Since f_j are linearly independent, $a_j = 0$ for $j = 1, \dots, k$. The proof is complete.

Now we consider a finite codimensional problem in L_∞. Put for $w \in V$,

$$(\mathcal{P}_{R_w}(L_1, V))^* = \{P^* : P \in \mathcal{P}_{R_w}(L_1, V)\}. \tag{1.12}$$

Theorem 1.9. *Let V satisfy the requirements of Theorem* 1.6. *Let Id denote the identity operator on L_∞. Then Id has the only element of best approximation in $(\mathcal{P}_{R_w}(L_1, V))^*$ if and only if w is determined by its roots.*

Proof. By Theorem 0.6, for any $P^* \in (\mathcal{P}_{R_w}(L_1, V))^*$,

$$\|(Id - P^*)|_C\| = 1 + \|P^*|_C\|. \tag{1.13}$$

Since C is weakly*-dense in L_∞, $\|P|_C\| = \|P\|$ for any $P \in \mathcal{P}_{R_w}(L_1, V)$. Consequently,

$$\|Id - P^*\| = 1 + \|P^*\|.$$

Hence an element P^* is the best approximation to Id if and only if P^* is an element of minimal norm in $(\mathcal{P}_{R_w}(L_1, V))^*$. Since $\|P\| = \|P^*\|$, this is equivalent to the fact that P is a minimal extension of R_w. Applying Theorem 1.6, we get the result.

Corollary 1.10. *Let $V = \operatorname{span}[v_1, \dots, v_k]$ satisfy the requirements of Theorem* 1.9. *If w is so chosen that $R_w = Id|_V$, then the only minimal projection from L_∞ or C onto $W = \cap_{j=1}^{k} \ker(v_j)$ exists* (*here we treat v_j as functionals on L_∞*) *if and only if w is determined by its roots.*

Remark 1.11. Let $n = 1$. Let $N \subset \mathbf{Z}$ be a finite set, symmetric with respect to zero. Put $V = \operatorname{span}[\sin(k\cdot), \cos(k\cdot) : k \in N]$. In this case Corollary 1.10 has been proved in [**5**, p. 243].

At the end of this paper we present two examples of invariant subspaces which fulfill the assumptions of Theorem 1.6 and some possible applications of Theorem 1.6.

Example 1.12. Put $n = 1$, and let $N \subset \mathbf{Z}$ be a finite subset. Put

$$V = \operatorname{span}[\cos(j\cdot), \sin(j\cdot) : j \in N].$$

Then it is easy to see that, for any $w \in V$, F_w satisfies the requirements of Theorem 1.6. In particular, if $w = \sum_{j \in N} 2\cos(j\cdot)$, if $j = 0$ the corresponding term of the sum should be one, then $R_w = id|_V$. In this case F_w is the only minimal projection if and only if w is determined by its roots. If

$$N = \{-4, -3, -2, 0, 2, 3, 4\},$$

then, by [**5**, Example 2], w is not determined by its roots. Hence, by Theorem 1.8, the set of all minimal projections has infinite affine dimension in this case.

Example 1.13. (Multivariate case.) Let N be a finite subset of $\mathbf{Z}^n$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in N$, set

$$G_\alpha = \left\{ f \in C : f(t) = \prod_{j=1,\dots,n} f_j(t_j) \right\},$$

where $f_j = \cos(\alpha_j\cdot)$ or $f_j = \sin(\alpha_j\cdot)$ for $j = 1, \dots, n$. Put $V = \operatorname{span}[G_\alpha : \alpha \in N]$. Then it is easy to see that V satisfies the requirements of Theorem 1.6.

Example 1.14. (Uniqueness.) Let $n = 1$ and $V = \pi_k$, the space of all trigonometric polynomials of degree $\leq k$. Let g be an algebraic polynomial of real variable of degree k. Set $w = g \circ \cos(\cdot)$. Then it is easy to see that $w \in \pi_k$. If g has k different roots in $(-1, 1)$, then w has $2k$ different roots in $(0, 2\pi)$. Since π_k is a Haar subspace on $(0, 2\pi)$, w is determined by its roots. Hence, by Theorem 1.6, F_w is the only minimal extension of R_w. Observe that the classical Chebyshev and Legendre polynomials of degree k have exactly k different roots in $(-1, 1)$.

Example 1.15. (Nonuniqueness.) Let $V = \pi_k$. Take any algebraic polynomial $g \neq 0$ of degree $\leq k$. Suppose that g has m roots $a_1, \dots, a_m \in (-1, 1)$, counting multiplicities and $m < k$. Let w be constructed as in Example 1.14. We show that w is not determined by its roots. To do this, let $b_1, \dots, b_{k-m+1}$ be different points, $|b_j| > 1$ for $j = 1, \dots, k-m+1$. Since $g \neq 0$, without loss, we can assume that $g(b_{k-m+1}) \neq 0$. Let g_1 be a polynomial of degree k satisfying $g_1(a_j) = 0$ for $j = 1, \dots, m$, $g_1(b_1) = 1$ and $g_1(b_j) = 0$ for $j = 2, \dots, k-m+1$. Set $w_1 = g_1 \circ \cos(\cdot)$. Then it is clear that $w_1/w \in L_\infty$ and w_1 is not a constant multiple of w. By Theorem 1.6, F_w is not the unique minimal extension. Moreover, by Theorem 1.8, $\dim(S_w) = \infty$.

Proposition 1.16. *Let $V \subset L_1([0,2\pi]^n)$ and $U \subset L_1([0,2\pi]^m)$ be two spaces satisfying the requirements of Theorem* 1.6. *Let $v \in V$ and $u \in U$. Put $W = V \otimes U \subset L_1([0,2\pi]^{n+m})$ and $w(t^1,t^2) = v(t^1)u(t^2)$. Then F_w is the only minimal extension of R_w if and only if F_v is the only minimal extension of R_v and F_u is the only minimal extension of R_u.*

Proof. Suppose that F_w is the only minimal extension. By Theorem 1.6, it is enough to show that u and v are determined by its roots. To do this, take any $g \in V$ such that $g/v \in L_\infty([0,2\pi]^n)$. Then $gu \in W$, $gu/w \in L_\infty([0,2\pi]^{n+m})$. By Theorem 1.6 applied to w, we get $gu = cw$ for some $c \in \mathbf{R}$. Consequently, $g = cv$, as required. The same reasoning applies to u.

Now suppose that both F_v and F_u are the only minimal extensions. Take any $g \in W$ with $g/w \in L_\infty([0,2\pi]^{n+m})$. Set

$$A = \{t^1 : v(t^1) \neq 0\}. \tag{1.14}$$

Since V is smooth, A is a set of full measure in $[0,2\pi]^n$. Fix any $t^1 \in A$. Note that $(g(t^1,\cdot)/v(t^1))/u(\cdot) \in L_\infty([0,2\pi]^m)$. Since F_u is the only minimal extension, by (1.14) and Theorem 1.6 applied to u,

$$g(t^1,t^2) = c(t^1)v(t^1)u(t^2) \tag{1.15}$$

for any $t^1 \in A$, $t^2 \in [0,2\pi]^m$. By (1.15), $c(t^1)v(t^1) \in V$.

Since $g/w \in L_\infty([0,2\pi]^{n+m})$, $c(t^1)v(t^1)/v(t^1) \in L_\infty([0,2\pi]^n)$. By Theorem 1.6 applied to v, $c(t^1)v(t^1) = cv(t^1)$ for any $t^1 \in [0,2\pi]^n$. Consequently, $c(t^1) = c$ and by (1.15) $g = cw$. Hence w is determined by its roots. By Theorem 1.6, F_w is the only minimal extension of R_w. The proof is complete. $\square$

Remark 1.17. By induction argument, Proposition 1.16 can be easily generalized to the case of tensor product of n spaces for any $n \in \mathbf{N}$.

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