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MONOTONICITY AND ROTUNDITY PROPERTIES IN BANACH LATTICES

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ABSTRACT. Some general results on geometry of Banach lattices are given. It is shown among others that uniform rotundity or rotundity coincide to uniform or strict monotonicity, respectively, on order intervals in positive cones of Banach lattices. Several equivalent conditions on uniform and strict monotonicity are also discussed. In particular, it is proved that in Banach function lattices uniform and strict monotonicity may be equivalently defined on orthogonal elements. It is then applied to show that p-convexification $E^{(p)}$ of E is uniformly monotone if and only if E possesses that property. A characterization of local uniform rotundity of Calderón-Lozanovskii spaces is also presented.

Introduction. In the following $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{R}_+$ stand for the sets of natural numbers, reals and nonnegative reals, respectively. The triple (T, Σ, μ) stands for a nonatomic, complete and σ-finite measure space. By $L^0 = L^0(\mu)$ we denote the space of all (equivalence classes of) Σ-measurable functions x from T to $\mathbf{R}$. By $E = (E, \leq, \| \, \|)$ we denote an abstract *Banach lattice* with a partial order $\leq$ (see [**2**], [**19**]) as well as a *Banach function space*, being a Banach sublattice of L^0 such that

(i) If $x \in L^0$, $y \in E$ and $|x| \leq |y|$, μ almost everywhere, then $x \in E$ and $\|x\| \leq \|y\|$.

(ii) There exists $x \in E$ such that $x(t) \neq 0$ for all $t \in T$.

The positive cone of E will be denoted by E_+. In the case of the counting measure space $(\mathbf{N}, 2^{\mathbf{N}}, \mu)$ where $\mu(A) = \mathrm{Card}\,(A)$ for every $A \subset \mathbf{N}$, a Banach function space E is called a *Banach sequence space.*

As usual, for every $x \in L^0$, $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$ is the support of x and χ_A is a characteristic function of $A \in \Sigma$. We denote by $S(E)$ and $B(E)$ the unit sphere and the unit ball in E, respectively. A Banach lattice E is said to be *order continuous* if, for every nonincreasing

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sequence (x_n) in E such that $x_n \downarrow 0$ μ-almost everywhere, $\|x_n\| \to 0$ holds.

We say that E is *strictly monotone* ($E \in$ (SM)) if, for every $x, y \in E_+$ with $x \leq y$ and $x \neq y$, we have $\|x\| < \|y\|$. E is said to be *uniformly monotone* ($E \in$ (UM)) if for any $\varepsilon \in (0,1)$ there is a $\delta(\varepsilon) \in (0,1)$ such that $\|y - x\| \leq 1 - \delta(\varepsilon)$, whenever $0 \leq x \leq y$, $\|y\| = 1$ and $\|x\| \geq \varepsilon$. E is called *lower locally uniformly monotone* ($E \in$ (LLUM)) if for any $x \in S(E_+)$ and $\varepsilon \in (0,1)$, $\delta(x,\varepsilon) \in (0,1)$ exists such that $\|x - y\| \leq 1 - \delta(x,\varepsilon)$ whenever $0 \leq y \leq x$ and $\|y\| \geq \varepsilon$. We say that E is *upper locally uniformly monotone* ($E \in$ (ULUM)) if for every $x \in S(E_+)$ and $\varepsilon > 0$, $\delta(x,\varepsilon) > 0$ exists such that $\|x + y\| > 1 + \delta(x,\varepsilon)$ whenever $y \geq 0$ and $\|y\| \geq \varepsilon$. For definitions and applications of monotonicity properties in Banach lattices, see [**1**], [**2**], [**3**], [**4**], [**10**] and [**18**].

For every (x_n) and x in L^0 we write $x_n \to x$ in L^0 to indicate *local convergence in measure*, i.e., for any $A \in \Sigma$ with $\mu(A) < \infty$, $(x_n - x)\chi_A \to 0$ in measure. We say that a Banach function space E has the *Kadec-Klee property with respect to local convergence in measure* ($E \in (\mathrm{H}_\mu)$), if for every (x_n) and x in E such that $x_n \to x$ in L^0 and $\|x_n\| \to \|x\|$, $\|x_n - x\| \to 0$ holds.

Let $X = (X, \|\ \|)$ denote a Banach space. Recall that X is said to be *rotund* ($X \in$ (R)) if for every $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ we have $\|x + y\| < 2$. X is said to be *uniformly rotund* ($X \in$ (UR)) if for every $\varepsilon \in (0,2)$, $\delta(\varepsilon) \in (0,1)$ exists such that if $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|(x+y)/2\| \leq 1 - \delta(\varepsilon)$. X is said to be *locally uniformly rotund* ($X \in$ (LUR)) if for any $x \in S(X)$ and $\varepsilon \in (0,2]$, $\delta(x,\varepsilon) \in (0,1)$ exists such that $\|(x+y)/2\| \leq 1 - \delta(x,\varepsilon)$ whenever $\|x - y\| \geq \varepsilon$ and $\|y\| \leq 1$.

It is not difficult to check that each condition of rotundity implies the appropriate condition of monotonicity. For instance, properties (UR) and (R) imply properties (UM) and (SM), respectively. In fact, we will prove stronger assertions.

For each kind of "uniform property" defined above, one can match a suitable modulus. The modulus of uniform monotonicity of E, for instance, is defined as follows

$$\delta(\varepsilon) = \inf\{1 - \|x - y\| : \|x\| = 1,\ \|y\| \geq \varepsilon,\ 0 \leq y \leq x\},$$

and obviously E is uniformly monotone if and only if $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. Analogously we introduce the other moduli (cf. [**6**]).

Recall that a Banach space $(X, \| \ \|)$ has *Kadec-Klee property* ($X \in$ (H)) if for any sequence $(x_n) \subset X$, $x \in X$, whenever $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ then $\|x_n - x\| \to 0$ (see [**6, 7**]).

A Banach function space E over the Lebesgue measure space $([0, \gamma), \Sigma, m)$, with $\gamma \in (0, \infty]$, is said to be *symmetric* if for every $x \in L^0$ and $y \in E$ with $m_x = m_y$ we have $x \in E$ and $\|x\| = \|y\|$. Recall that m_x denotes the distribution function of x, i.e., for all $\lambda \geq 0$,

$$m_x(\lambda) = m(\{t \in [0, \gamma) : |x(t)| > \lambda\}).$$

The decreasing rearrangement function of x is denoted by x^* and is defined by

$$x^*(t) = \inf\{\lambda > 0 : m_x(\lambda) \leq t\}$$

for $t \geq 0$. It is known that $m_{x^*} = m_x$ for any $x \in L^0$. For the basic properties of symmetric spaces we refer to [**17**] and [**19**].

A mapping $\varphi : \mathbf{R} \to \mathbf{R}_+$ is said to be an *Orlicz function* if $\varphi(0) = 0$, φ is convex, even and φ is not identically equal to zero. If E is a Banach function lattice and φ is an Orlicz function we say that φ satisfies the Δ_2^E-*condition*, see [**4**], whenever $K > 0$ and $u_0 \geq 0$ exist such that the inequality

$$\varphi(2u) \leq K\varphi(u)$$

is satisfied for all $|u| \geq u_0$ if $L^\infty \subset E$, for all $0 \leq |u| \leq u_0$ with $\varphi(u_0) > 0$ if $E \subset L^\infty$ and for all $u \in \mathbf{R}$ if neither $L^\infty \subset E$ nor $E \subset L^\infty$.

For any Orlicz function φ and any Banach function lattice $(E, \| \ \|)$, we define the Calderón-Lozanovskii space E_φ by

$$E_\varphi = \{x \in L^0 : \varphi \circ (\lambda x) \in E \text{ for some } \lambda > 0\},$$

where $\varphi \circ x(t) = \varphi(x(t))$ for any $t \in T$. The space E_φ is equipped with the norm

$$\|x\|_\varphi = \inf\{\lambda > 0 : \rho_\varphi(x/\lambda) \leq 1\},$$

where

$$\rho_\varphi(x) = \begin{cases} \|\varphi \circ x\| & \text{if } \varphi \circ x \in E \\ +\infty & \text{otherwise.} \end{cases}$$

When we consider the space E_φ, we always assume that the Banach function lattice E has the *Fatou property*, i.e., if $x_n \in E_+$ for $n \in \mathbf{N}$, $x \in (L^0)_+$, $x_n \uparrow x$ almost everywhere and $\sup_n \|x_n\| < +\infty$, then $x \in E$ and $\|x_n\| \uparrow \|x\|$. It is worthwhile to notice that E_φ is a special case of a general Calderón-Lozanovskii construction $\Psi(E, F)$ where E is a Banach function lattice and $F = L^\infty$ (cf. [**20**], [**4**]). If $\varphi(u) = u^p$, then E_φ is a p-convexification $E^{(p)}$ of E and by analogy E_φ is called a φ-convexification of E.

In this paper we prove a number of general results on geometric properties in Banach lattices. In particular, we show relations between monotonicity and rotundity properties. We prove that $E_+ \in$ (R) (respectively $E_+ \in$ (LUR)) is equivalent to $E \in$ (R) (respectively $E \in$ (LUR)) and that for symmetric separable spaces over a finite interval, (SM)-property is equivalent to (LLUM)-property. It is noted that monotonicity properties are equivalent to respective rotundity properties restricted to couples of compatible nonnegative elements. We also prove that strict monotonicity and uniform monotonicity in Banach function lattices can be equivalently considered only for orthogonal elements. It easily follows from this result that for $1 \leq p < \infty$, p-convexification $E^{(p)}$ of a Banach function lattice E is strictly or uniformly monotone if and only if E possesses the respective property. We then apply this result to get a characterization of uniform monotonicity of Lorentz spaces. Finally we prove that E_φ is locally uniformly rotund whenever E is uniformly monotone and φ is an Orlicz function strictly convex on $\mathbf{R}$ and satisfying the Δ_2^E-condition.

Results. We start with some general results which compare rotundity and monotonicity properties in Banach lattices.

Theorem 1. *Given a Banach lattice E the following hold true*:

(i) *If E_+ is rotund, then E is strictly monotone.*

(ii) *If E_+ is locally uniformly rotund, then E is upper and lower locally uniformly monotone.*

(iii) *If E_+ is uniformly rotund then E is uniformly monotone.*

(iv) *On the order intervals in the positive cone E_+ the inverse statement of each of the above is also true.*

Proof. (i) Suppose that E_+ is rotund, $x \in S(E_+)$ and $0 \le y \le x$, $y \ne x$. Then, by inequalities $y \le (y+x)/2 \le x$ we get

$$\|y\| \le \|(y+x)/2\| < 1.$$

(ii) Let E_+ be locally uniformly rotund, $x \in S(E_+)$, $\varepsilon > 0$ and $\delta_E(x,\cdot)$ the modulus of local uniform rotundity of E_+ at x. Now letting $0 \le y \le x$ with $\|y\| > \varepsilon$, $x - y \ge 0$ and $\|x-(x-y)\| = \|y\| \ge \varepsilon$. Thus by $E_+ \in$ (LUR), we obtain

$$\|(x+(x-y))/2\| = \|x - y/2\| \le 1 - \delta_E(x,\varepsilon),$$

whence $\|x-y\| \le 1-\delta_E(x,\varepsilon)$, i.e., $E \in$ (LLUM).

In order to show that $E \in$ (ULUM) we need to prove that for any $0 \ne x \in E_+$ and any sequence (x_n) in E such that $0 \le x \le x_n$ for $n \in \mathbf{N}$ and $\|x_n\| \to \|x\|$, $\|x_n - x\| \to 0$ holds. Let $0 \ne x$, $x_n \in E$, $0 \le x \le x_n$ and $\|x_n\| \to \|x\|$. Then

$$2\|x\| \le \|x + x_n\| \le \|x\| + \|x_n\| \to 2\|x\|,$$

whence $\|x + x_n\| \to 2\|x\|$. Consequently, by $E \in$ (LUR) we get $\|x_n - x\| \to 0$.

(iii) We apply the same arguments as in (ii).

(iv) Using the sequence definitions of (LUR)- and (ULUM)-properties, it is obvious that for sequences (x_n) in E_+ dominated from below by $x \in E_+$, the (LUR)-property means the same as property (ULUM).

Assume now that E is uniformly monotone, $0 \le y \le x$, $\|x\| = 1$ and $\|x-y\| \ge \varepsilon$. Then $\|(x-y)/2\| \ge \varepsilon/2$ and $0 \le (x-y)/2 \le x$. So, by $E \in$ (UM), we get $\|x - (x-y)/2\| \le 1 - \delta(\varepsilon/2)$, i.e., $\|(x+y)/2\| \le 1-\delta(\varepsilon/2)$, where $\delta(\cdot)$ denotes the modulus of uniform monotonicity of E.

This completes the proof, since the other properties may be proved analogously.

It is well known that $E \in$ (UR) is equivalent to $E_+ \in$ (UR) (cf. [**16**]). We will now show that the same holds true for the properties (R) and (LUR). The result for the rotundity seems also to be known, but for the sake of completeness we include the proof here.

Theorem 2. *A Banach function lattice is rotund if and only if E_+ is rotund.*

Proof. It is enough to show that $E_+ \in$ (R) implies that $E \in$ (R). Assume that $x, y \in S(E)$ and $x \neq y$. If $|x| \neq |y|$, then by $E_+ \in$ (R), $\|(x+y)/2\| \leq \|(|x|+|y|)/2\| < 1$.

In the case when $|x| = |y|$, by $x \neq y$, we have that the set $A = \{t : x(t) \neq y(t)\} = \{t : \operatorname{sgn}(x(t) \cdot y(t)) = -1\}$ has positive measure. Therefore, $x(t) + y(t) = 0$ for $t \in A$, and consequently

$$\left|\frac{x+y}{2}\right| \leq \frac{|x|+|y|}{2} \quad \text{and} \quad \left|\frac{x+y}{2}\right| \neq \frac{|x|+|y|}{2}.$$

Since, in view of Theorem 1, $E_+ \in$ (R) yields $E \in$ (SM), the above inequality gives $\|(x+y)/2\| < \|(|x|+|y|)/2\| \leq 1$, which finishes the proof.

Theorem 3. *A Banach function lattice E is locally uniformly rotund if and only if E_+ possesses the same property.*

Proof. We need only to show that local uniform rotundity of E_+ implies the same property for E. Assume that E_+ is locally uniformly rotund and that (x_n) and x in E satisfy the conditions $\|x_n\| \to \|x\| = 1$ and $\|x_n + x\| \to 2$. Then $\||x_n| + |x|\| \to 2$, whence we get $\||x_n| - |x|\| \to 0$ by $E_+ \in$ (LUR). We also have

$$\|x_n + x\| \leq \||x_n + x|/2 + (|x_n| + |x|)/2\| \leq 2,$$

and since $\|x_n + x\| \to 2$, we get

$$\||x_n + x|/2 + (|x_n| + |x|)/2\| \longrightarrow 2.$$

By virtue of $\||x_n| - |x|\| \to 0$, it yields

$$\||x_n + x|/2 + |x|\| \longrightarrow 2.$$

Thus, by $E_+ \in$ (LUR), $\||x_n + x|/2 - |x|\| \to 0$. Since $E_+ \in$ (LUR) implies that E is order continuous, by the well-known theorem of Davis, Ghoussoub and Lindenstrauss ([**5**, Theorem 1.2]) a lattice norm $\| \|_0$ exists in E equivalent to the original norm that is locally uniformly rotund. Therefore, $\||x_n + x|/2\|_0 \to \|x\|_0$ and $\|x_n\|_0 \to \|x\|_0$. Since the norm $\| \|_0$ is locally uniformly rotund, we get $\|x_n - x\|_0 \to 0$ and consequently $\|x_n - x\| \to 0$.

Theorem 4. *Let E be a separable symmetric space in which an equivalent symmetric norm $\| \|_0$ exists which is lower locally uniformly monotone. Then $(E, \| \|)$ is lower locally uniformly monotone if and only if it is strictly monotone.*

Proof. Under the assumptions on E it is enough to show that strict monotonicity implies lower local uniform monotonicity. Assume that (x_n) and x in E satisfy $0 \le x_n \le x$ and $\|x_n\| \to \|x\|$. Let (y_n^*) be an arbitrary subsequence of (x_n^*). By Helly's theorem (cf. [**21** Chapter 8, sect. 4]) a subsequence (z_n^*) of (y_n^*) and $z \in E$ exist such that $z^* = z$ and $z_n^* \to z$ pointwise. Since $x_n^* \le x^*$, we have $z_n^* \le x^*$ and $z \le x^*$. Separability of E implies that E is order continuous which yields $\|z_n^* - z\| \to 0$ (cf. [**2**]). Hence, $\|z_n^*\| \to \|z\|$. But (z_n^*) is a subsequence of (x_n^*) and $\|x_n^*\| = \|x_n\| \to \|x\|$, whence $\|z\| = \|x\|$. So, by the (SM)-property of E we get $z = x^*$, whence $\|z_n^* - x^*\| \to 0$. It follows that $\|x_n^* - x^*\| \to 0$. By the assumptions on x_n and x, $x_n \le (x_n + x)/2 \le x$ and so

$$0 \le (x_n + x)/2 \le x \quad \text{and} \quad \|(x_n + x)/2\| \to \|x\|.$$

Now applying the same procedure as above to $(x_n + x)/2$ and x, we get

$$\|(x_n + x)^*/2 - x^*\| \longrightarrow 0.$$

Therefore $\|(x_n + x)^*/2 - x^*\|_0 \to 0$ and so $\|(x_n + x)/2\|_0 \to \|x\|_0$. Since the norm $\| \|_0$ has the (LLUM)-property, this yields $\|x_n - x\|_0 \to 0$ and consequently $\|x_n - x\| \to 0$. This completes the proof.

It is shown in [**5**] that if E is an order continuous symmetric space over a bounded interval $[0, \gamma)$ in $\mathbf{R}_+$, then an equivalent norm $\| \|_0$ exists in E which is symmetric and locally uniformly rotund. So we get the following theorem as a consequence of Theorem 4.

Corollary 1. *If E is a symmetric space over a bounded interval $[0,\gamma)$ in $\mathbf{R}_+$ which is separable, then E is lower locally uniformly monotone if and only if it is strictly monotone.*

Theorem 5. (1) *In every Banach sequence lattice E property* (H_μ) *implies property* (H).

(2) *In every reflexive Banach sequence lattice E properties* (H_μ) *and* (H) *coincide.*

(3) *Each reflexive Banach function lattice E with property* (H) *has property* (H_μ).

Proof. (1) It follows by the fact that in Banach sequence lattices weak convergence implies pointwise convergence.

(2) We need only to show that (H) implies (H_μ). Assume that (x_n) and x are in E, $\|x_n\| \to \|x\|$ and $x_n \to x$ pointwise. By reflexivity of E, we assume without loss of generality that $y \in E$ exists such that $x_n \to y$ weakly. Hence $x_n \to y$ pointwise and so $y = x$, whence $x_n \to x$ weakly. Now by the Kadec-Klee property of E we get $\|x_n - x\| \to 0$ which finishes the proof.

(3) The proof is the same as the one of (2) under the observation that if $x_n \to x$ in L^0 and $x_n \to y$ weakly then $x = y$ (see Proposition 8 in [**12**]).

Theorem 6. *For any Banach function lattice E the following properties are equivalent.*

(i) *E is uniformly monotone.*

(ii) *For any $\varepsilon > 0$ there is $\sigma(\varepsilon) > 0$ such that $x, y \in E_+$, $\|x\| = 1$ and $\|y\| \geq \varepsilon$ imply $\|x + y\| \geq 1 + \sigma(\varepsilon)$.*

(iii) *For any $\varepsilon \in (0,1)$, there is an $\eta(\varepsilon)$ such that for any $x \in E_+$ with $\|x\| = 1$ and for any $A \in \Sigma$ such that $\|x\chi_A\| \geq \varepsilon$, $\|x\chi_{A'}\| \leq 1 - \eta(\varepsilon)$ holds where $A' = T\backslash A$.*

(iv) *For each $\varepsilon > 0$ there is a $p(\varepsilon) > 0$ such that, if $x, y \in E$, $\mu(\operatorname{supp} x \cap \operatorname{supp} y) = 0$, $\|x\| = 1$ and $\|y\| \geq \varepsilon$, then $\|x + y\| \geq 1 + p(\varepsilon)$.*

Proof. (i) $\Rightarrow$ (ii). At first observe that for each $0 < \varepsilon < b$ and $x, y \in E_+$ such that $y \le x$, $\|x\| \le b$ and $\|y\| \ge \varepsilon$, $\|x - y\| \le (1 - \delta(\varepsilon/b))\|x\|$ holds. Indeed, if $x \ne 0$, then $x/\|x\| \in S(E_+)$, $0 \le y/\|x\| \le x/\|x\|$ and $\|y/\|x\|\,\| \ge \varepsilon/b$. So, by uniform monotonicity of E, we get

$$\|x - y\| \le (1 - \delta(\varepsilon/b))\|x\|.$$

Assuming now that $x \in S(E_+)$, $y \ge 0$ and $\|y\| \ge \varepsilon$, define $z = x + y$. We can clearly assume that $\|z\| \le 2$. Then $1 = \|x\| = \|z - y\| \le (1 - \delta(\varepsilon/2))\|z\|$, and so

$$\|z\| = \|x + y\| \ge 1/(1 - \delta(\varepsilon/2)) \ge 1 + \sigma(\varepsilon),$$

where $\sigma(\varepsilon) = \min\{1, \delta(\varepsilon/2)/(1 - \delta(\varepsilon/2))\}$.

(ii) $\Rightarrow$ (iii). Assume that $x \ge 0$, $\|x\| = 1$, $A \in \Sigma$ and $\|x\chi_A\| \ge \varepsilon$. We will show that $\|x\chi_{A'}\| \le 1/(1 + (1/2)\sigma(\varepsilon))$. Suppose on the other hand that $\|x\chi_{A'}\| > 1/(1 + (1/2)\sigma(\varepsilon))$ and denote $z = x\chi_{A'}$. Then $z \ne 0$ and $\|x\chi_A/\|z\|\| \ge \varepsilon$. Thus, by (ii),

$$\left\| \frac{x}{\|z\|} \right\| = \left\| \frac{z}{\|z\|} + \frac{x\chi_A}{\|z\|} \right\| \ge 1 + \sigma(\varepsilon),$$

i.e., $\|x\| \ge (1 + \sigma(\varepsilon))\|z\| \ge (1 + \sigma(\varepsilon))/(1 + (1/2)\sigma(\varepsilon)) > 1$, a contradiction. So, $\|x\chi_{A'}\| \le 1/(1 + (1/2)\sigma(\varepsilon)) = 1 - \eta(\varepsilon)$, where $\eta(\varepsilon) = \sigma(\varepsilon)/(2 + \sigma(\varepsilon))$.

(iii) $\Rightarrow$ (i). Assume that $0 \le y \le x \in S(E)$, $\|y\| \ge \varepsilon$ and define

$$A = \{t \in T : y(t) \le (\varepsilon/2)x(t)\}.$$

Then $\|y\chi_A\| \le \varepsilon/2$ and so $\|x\chi_{A'}\| \ge \|y\chi_{A'}\| \ge \varepsilon/2$. Hence,

$$\begin{aligned} \|x - y\| &\le \|x - y\chi_{A'}\| \le \|x - (\varepsilon/2)x\chi_{A'}\| \\ &\le (1 - (\varepsilon/2))\|x\| + (\varepsilon/2)\|x - x\chi_{A'}\| \\ &\le 1 - (\varepsilon/2) + (\varepsilon/2)(1 - \eta(\varepsilon/2)) = 1 - (\varepsilon/2)\eta(\varepsilon/2), \end{aligned}$$

and consequently E is uniformly monotone.

It is clear that (ii) $\Rightarrow$ (iv) and we will finish by showing (iv) $\Rightarrow$ (iii). Assume that $x \in S(E_+)$, $A \in \Sigma$ and $\|x\chi_A\| \ge \varepsilon$. Define

$$z = \frac{x\chi_{A'}}{\|x\chi_{A'}\|} \quad \text{and} \quad \omega = \frac{x\chi_A}{\|x\chi_{A'}\|}.$$

Then supports of z and w are disjoint, $\|z\| = 1$ and $\|\omega\| \geq \varepsilon$. So by virtue of (iv), we get

$$\frac{1}{\|x\chi_{A'}\|} = \|z + \omega\| \geq 1 + p(\varepsilon),$$

whence

$$\|x\chi_{A'}\| \leq \frac{1}{1 + p(\varepsilon)} = 1 - \eta(\varepsilon),$$

where $\eta(\varepsilon) = p(\varepsilon)/(1 + p(\varepsilon))$.

Theorem 7. *Let E be an arbitrary Banach function lattice and φ an Orlicz function vanishing only at zero and satisfying the Δ_2^E-condition. Then E is uniformly monotone if and only if E_φ is uniformly monotone.*

Proof. The fact that uniform monotonicity of E yields the same property for E_φ has been proved in [**4**, Theorem 1]. So we will prove that $E_\varphi \in$ (UM) implies that $E \in$ (UM). By the Δ_2^E-condition, by virtue of Lemmas 2 and 3 in [**4**], the functions $\eta, q : (0, \infty) \to (0, \infty)$ exist such that for any $x \in E_\varphi$ and $\varepsilon > 0$,

$$\rho_\varphi(x) \geq 1 + \eta(\varepsilon) \quad \text{whenever} \quad \|x\|_\varphi \geq 1 + \varepsilon,$$

and

$$\|x\|_\varphi \geq q(\varepsilon) \quad \text{whenever} \quad \rho_\varphi(x) \geq \varepsilon.$$

Let p be the function from condition (iv) in Theorem 6 for E_φ in place of E. Assume that $x, y \in E_+$ have disjoint supports, $\|x\| = 1$ and $\|y\| \geq \varepsilon$. Define $\omega = \varphi^{-1} \circ x$, $z = \varphi^{-1} \circ y$. Then supports of z and ω are disjoint and

$$\rho_\varphi(\omega) = \|x\| = 1, \qquad \rho_\varphi(z) = \|y\| \geq \varepsilon.$$

Therefore, $\|\omega\|_\varphi = 1$ and $\|z\|_\varphi \geq q(\varepsilon)$. Consequently, $\|\omega + z\|_\varphi \geq 1 + p(q(\varepsilon))$, whence $\rho_\varphi(\omega + z) \geq 1 + \eta(p(q(\varepsilon)))$, i.e., $\|\varphi \circ (\omega + z)\| = \|x + y\| \geq 1 + \eta(p(q(\varepsilon)))$, which finishes the proof.

Corollary 2. *For any Banach function lattice E the following assertions are equivalent.*

(i) *E is uniformly monotone.*

(ii) *$E^{(p)}$ is uniformly rotund for any $p \in (1, \infty)$.*

(iii) *$E^{(p)}$ is uniformly rotund for some $p \in (1, \infty)$.*

(iv) *$E^{(p)}$ is uniformly monotone for some $p \in (1, \infty)$.*

(v) *$E^{(p)}$ is uniformly monotone for any $p \in (1, \infty)$.*

Proof. The implication (i) $\Rightarrow$ (ii) has been proved in [**11**, Corollary 9] while the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (ii) $\Rightarrow$ (v) are obvious. Finally, Theorem 7 applied for $\varphi(u) = |u|^p$ yields the remaining two implications (iv) $\Rightarrow$ (i) and (v) $\Rightarrow$ (i).

Note that Corollary 2 says among others that we cannot improve the monotonicity properties of a Banach function lattice E by p-convexification with $1 < p < \infty$, in opposition to rotundity properties (see [**4**] and [**11**]). It is not surprising in view of the fact that L^1 and l^1 are the spaces with the biggest modulus of monotonicity among the spaces L^p and l^p for $1 \le p < \infty$. The same concerns Theorem 7 and φ-convexification E_φ of E.

As a consequence of Corollary 2 we get a characterization of uniform convexity of Lorentz spaces. Recall that, given $1 \le p < \infty$ and a nonincreasing, locally integrable function $w : [0, \gamma) \to (0, \infty)$, the Lorentz space $\Lambda_{p,w}$ is defined as follows

$$\Lambda_{p,w} = \left\{ x \in L^0 : \|x\|_p = \left(\int_0^\gamma (x^*)^p w \right)^{1/p} < \infty \right\}.$$

For $p = 1$ it is denoted by Λ_w.Observe that $\Lambda_{p,w}$ is a p-convexification of Λ_w. We say that the weight w is *regular* if $\inf_{t \in (0,\gamma)} S(t)/S(t/2) > 1$, where $S(t) = \int_0^t w$.

Applying Corollary 2 and Halperin's result [**9**] that for $1 < p < \infty$, $\Lambda_{p,w}$ is uniformly rotund if and only if w is regular, we immediately obtain the following corollary proved earlier in [**10**].

Corollary 3. *The Lorentz space Λ_w is uniformly monotone if and only if w is regular.*

Theorem 8. *For any Banach function lattice E the following are equivalent.*

(i) *E is strictly monotone.*

(ii) *For every $x, y \in E\backslash\{0\}$, $\mu\{\operatorname{supp} x \cap \operatorname{supp} y\} = 0$, we have $\|x+y\| > \max(\|x\|, \|y\|)$.*

(iii) *For any $x \in E\backslash\{0\}$ and $A \in \Sigma$ such that $\|x\chi_A\| > 0$, $\|x\chi_{A'}\| < \|x\|$ holds.*

Proof. (i) $\Rightarrow$ (ii). We have $|x+y| \geq |x|$, $|x+y| \geq |y|$, $|x+y| \neq |x|$ and $|x+y| \neq y$, whence by (i), $\|x+y\| > \|x\|$ and $\|x+y\| > \|y\|$.

The implication (ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i). Assume that $0 \leq y \leq x$ and $y \neq 0$. We need to show that $\|x-y\| < \|x\|$. There exists $k \in \mathbf{N}$ such that $\mu(A) > 0$, where

$$A = \{t \in T : y(t) \geq (1/k)x(t)\}.$$

Then, by (iii),

$$\begin{aligned}\|x-y\| \leq \|x - y\chi_A\| &\leq \|x - (1/k)x\chi_A\| \\ &\leq (1-(1/k))\|x\| + (1/k)\|x - x\chi_A\| \\ &\leq (1-(1/k))\|x\| + (1/k)\|x\| = \|x\|\end{aligned}$$

holds. Thus the proof is complete. □

Theorem 9. *If E is a Banach function lattice which is uniformly monotone and φ is a strictly convex Orlicz function which satisfies the Δ_2^E-condition, then E_φ is locally uniformly rotund.*

Proof. We will give a proof only in the case when $L^\infty \hookrightarrow E_\varphi$. The remaining cases when $E \subset L^\infty$ or when neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$ are analogous to handle. It is known (see [**4**]) that if φ satisfies condition Δ_2^E, two functions $\sigma : \mathbf{R}_+ \to \mathbf{R}_+$ and $\beta : (0,1) \to (0,1)$ exist such that for all $\varepsilon > 0$

$$\rho_\varphi(x) \geq \sigma(\varepsilon) \quad \text{whenever} \quad \|x\|_\varphi \geq \varepsilon$$

and

$$\|x\|_\varphi \leq 1 - \beta(\varepsilon) \quad \text{whenever} \quad \rho_\varphi(x) \leq 1 - \varepsilon. \tag{1}$$

Recall that $\delta(\varepsilon)$ denotes the modulus of uniform monotonicity of E. In the case when $L^\infty \hookrightarrow E$, the Δ_2^E-condition means that φ satisfies the inequality $\varphi(2u) \le K\varphi(u)$ for some $K > 0$ and for large values $|u|$ only. For any $\varepsilon > 0$ we find $C_1 > 0$ such that

$$\rho_\varphi(2C_1\chi_T) = \|\varphi(2C_1)\| \le \sigma(\varepsilon)/16 \tag{2}$$

and then we choose $K > 0$ so that for all $u \in \mathbf{R}$

$$\varphi(2u) \le K\varphi(u) + \varphi(2C_1). \tag{3}$$

Let $\|x\|_\varphi = \|y\|_\varphi = 1$ and $\|x-y\|_\varphi \ge \varepsilon$. By virtue of the Δ_2^E-condition, $\rho_\varphi(x) = \rho_\varphi(y) = 1$ and $\rho_\varphi(x-y) \ge \sigma(\varepsilon)$. Since uniform monotonicity implies order continuity of E, $C_2 > 0$ exists such that

$$\rho_\varphi(x\chi_A) < (1/2)\delta(\sigma(\varepsilon)/32K) \tag{4}$$

where

$$A' = T\backslash A = \{t \in T : 1/C_2 \le |x(t)| \le C_2\}.$$

Now we can find a constant $C_3 > C_2$ satisfying

$$\varphi(C_2)/\varphi(C_3) \le (1/2)\delta(\sigma(\varepsilon)/32K).$$

Define

$$B = \{t \in T : |y(t)| > C_3\}$$

and

$$\begin{aligned} C = T\backslash(A \cup B) = \{t \in T : 1/C_2 \le |x(t)| \le C_2\} \\ \cap \{t \in T : |y(t)| \le C_3\}. \end{aligned}$$

Suppose that

$$\rho_\varphi((x-y)\chi_C) \le (3/4)\sigma(\varepsilon).$$

Then by $\rho_\varphi(x-y) \ge \sigma(\varepsilon)$, we have

$$\rho_\varphi((x-y)\chi_{A\cup B}) \ge \sigma(\varepsilon)/4. \tag{5}$$

By the definition of B, we get

$$\varphi(C_3)\|\chi_B\| = \|\varphi(C_3)\chi_B\| \le \|\varphi \circ y\chi_B\| \le 1,$$

whence

$$\|\chi_B\| \le 1/\varphi(C_3).$$

Since $|x(t)| \le C_2$ for $t \in B\backslash A$,

$$\rho_\varphi(x\chi_{B\backslash A}) \le \|\varphi \circ C_2\chi_B\| \le \varphi(C_2)/\varphi(C_3) \le (1/2)\delta(\sigma(\varepsilon)/32K).$$

Now by (4) and in view of the inequality $\delta(\varepsilon) \le \varepsilon$,

$$\begin{aligned}\rho_\varphi(x\chi_{A\cup B}) &\le \rho_\varphi(x\chi_A) + \rho_\varphi(x\chi_{B\backslash A}) \\ &\le \delta(\sigma(\varepsilon)/32K) \le \sigma(\varepsilon)/32K.\end{aligned} \tag{6}$$

Applying (2), (3), (5) and (6), we get

$$\begin{aligned}\frac{\sigma(\varepsilon)}{4} &\le \rho_\varphi((x-y)\chi_{A\cup B}) \\ &\le \frac{K}{2}\rho_\varphi(x\chi_{A\cup B}) + \frac{K}{2}\rho_\varphi(y\chi_{A\cup B}) + \rho_\varphi(2C_1\chi_T) \\ &\le \frac{K}{2}\rho_\varphi(y\chi_{A\cup B}) + \frac{K}{2}\cdot\frac{\sigma(\varepsilon)}{32K} + \frac{\sigma(\varepsilon)}{16} \\ &= \frac{K}{2}\rho_\varphi(y\chi_{A\cup B}) + \frac{5}{64}\sigma(\varepsilon).\end{aligned}$$

Therefore,

$$\|\varphi \circ y\chi_{A\cup B}\| = \rho_\varphi(y\chi_{A\cup B}) \ge \sigma(\varepsilon)/3K. \tag{7}$$

By $\rho_\varphi(x) = 1$ and (6), we have $\rho_\varphi(x\chi_C) \ge 1 - \delta(\sigma(\varepsilon)/32K)$. By $\rho_\varphi(y) = 1$, uniform monotonicity of E and (7), we get

$$\rho_\varphi(y\chi_C) = \|\varphi \circ y\chi_C\| = \|\varphi \circ y - \varphi \circ y\chi_{A\cup B}\| \le 1 - \delta(\sigma(\varepsilon)/3K).$$

Therefore we obtain

$$\begin{aligned}|\rho_\varphi(x\chi_C) - \rho_\varphi(y\chi_C)| &\ge 1 - \delta(\sigma(\varepsilon)/32K) - 1 + \delta(\sigma(\varepsilon)/3K) \\ &= \delta(\sigma(\varepsilon)/3K) - \delta(\sigma(\varepsilon)/32K) \\ &=: \gamma(\varepsilon),\end{aligned}$$

where $\gamma(\varepsilon) > 0$ since $\delta(\varepsilon)$ is strictly increasing. Now, following the proof of Lemma 5 in [**14**], there exists $\delta > 0$ depending only on x, ε, E and φ such that

$$\rho_\varphi((x-y)\chi_C) \geq \delta. \tag{8}$$

The proof of this fact can proceed in the same way as the proof of Lemma 5 in [**14**]. We can assume that $\delta \leq (3/4)\sigma(\varepsilon)$. Define

$$D = \{t \in C : |x(t) - y(t)| \geq (\delta/4)\max(|x(t)|, |y(t)|)\}.$$

By $(1/C_2) \leq \max\{|x(t)|, |y(t)|\} \leq C_3$ for $t \in D$ and strict convexity of φ on $\mathbf{R}$, applying Lemma 0.5 in [**15**] with $\delta/4$, $1/C_2$ and C_3 in place of ε, d_1 and d_2, respectively, one can find $p \in (0,1)$ depending only on ε, x and φ such that

$$\varphi\bigg(\frac{x(t)+y(t)}{2}\bigg) \leq \frac{1-p}{2}(\varphi(x(t)) + \varphi(y(t))),$$

for all $t \in D$, whence

$$\varphi \circ \left(\frac{x+y}{2}\right) \leq \frac{1}{2}(\varphi \circ x + \varphi \circ y) - \frac{P}{2}(\varphi \circ x\chi_D + \varphi \circ y\chi_D). \tag{9}$$

We also have

$$\varphi \circ (x-y)\chi_{C\setminus D} \leq (\delta/4)(\varphi \circ x\chi_{C\setminus D} + \varphi \circ y\chi_{C\setminus D}),$$

whence $\rho_\varphi((x-y)\chi_{C\setminus D}) \leq (\delta/2)$ and in view of (8) we get

$$\rho_\varphi((x-y)\chi_D) \geq \frac{\delta}{2}. \tag{10}$$

By the Δ_2^E-condition and strict monotonicity of φ on $\mathbf{R}$, there are positive constants C_4 and K_1 depending only on δ such that $\|\varphi \circ 2C_4\chi_T\| < \delta/4$ and

$$\varphi(2u) \leq K_1\varphi(u) + \varphi(2C_4)$$

for all $u \in \mathbf{R}$. So by (10) we get

$$\begin{aligned}\delta/2 &\leq \rho_\varphi((x-y)\chi_D) = \|\varphi \circ (x-y)\chi_D\| \\ &\leq K_1\|(\varphi \circ x\chi_D + \varphi \circ y\chi_D)/2\| + \|\varphi \circ 2C_4\chi_T\| \\ &\leq K_1\|(\varphi \circ x\chi_D + \varphi \circ y\chi_D)/2\| + \delta/4,\end{aligned}$$

whence

$$\|p(\varphi \circ x\chi_D + \varphi \circ y\chi_D)/2\| \geq p\delta/4K_1.$$

Since $(\varphi\circ x+\varphi\circ y)/2 \geq (\varphi\circ x\chi_D+\varphi\circ y\chi_D)/2 \geq 0$ and $\|(\varphi\circ x+\varphi\circ y)/2\| \leq 1$, we get by (9) and uniform monotonicity of E that

$$\rho_\varphi\bigg(\frac{x+y}{2}\bigg) \leq 1 - \delta\bigg(\frac{p\delta}{4K_1}\bigg),$$

whence in view of (1), we get

$$\bigg\|\frac{x+y}{2}\bigg\|_\varphi \leq 1 - \beta\bigg(\delta\bigg(\frac{p\delta}{4K_1}\bigg)\bigg),$$

which finishes the proof.

Notice that the assumption of the Δ_2^E-condition is necessary in Theorem 9, since otherwise E_φ contains an isomorphic copy of l^∞ (see [**11**]) and so E_φ is not locally uniformly rotund. Observe also that the conditions imposed on φ and w in the last theorem do not yield uniform rotundity of E_φ, although they are strong enough to guarantee its uniform monotonicity. Indeed, let $E = \Lambda_w$, where w is a regular weight, and let φ be an Orlicz function satisfying condition Δ_2 and not being uniformly convex. By the well-known criterion for uniform rotundity of the Orlicz-Lorentz space $\Lambda_{\varphi,w} = (\Lambda_w)_\varphi$ ([**16**]), E_φ is not uniformly rotund.

Corollary 4. (*see* [**15**]). *The Orlicz space L^φ over a nonatomic measure space is locally uniformly rotund whenever φ is strictly convex on* $\mathbf{R}$ *and it satisfies the suitable Δ_2 condition.*

Proof. We have $L^\varphi = (L^1)_\varphi$. Since $E = L^1$ is uniformly monotone, the result follows directly by Theorem 9.

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