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ON EQUAL SUMS OF SIXTH POWERS

AJAI CHOUDHRY

ABSTRACT. This paper provides a method of generating infinitely many integer solutions of the simultaneous equations $a^r + b^r + c^r = d^r + e^r + f^r$ where $r = 1, 2$ and 6. Several numerical solutions of this system of equations have also been obtained in this paper.

This paper deals with the simultaneous diophantine equations given by

$$a^r + b^r + c^r = d^r + e^r + f^r \tag{1}$$

where $r = 1, 2$ and 6. Numerical and parametric solutions of (1) with $r = 2$ and 6 have been obtained earlier by Subba Rao [**9**], Brudno [**2, 3**], Bremner [**1**], Choudhry [**4**] and Delorme [**5**]. It has been noted by Guy [**6**, p. 142] that all the known simultaneous solutions of (1) with $r = 2$ and 6 also satisfy (with appropriately chosen signs) the following three equations

$$\begin{aligned} a^2 + ad - d^2 &= f^2 + fc - c^2 \\ b^2 + be - e^2 &= d^2 + da - a^2 \\ c^2 + cf - f^2 &= e^2 + eb - b^2. \end{aligned} \tag{2}$$

Guy has asked the question whether there exists a counterexample which, while satisfying (1) for $r = 2$ and 6, does not satisfy the three equations given by (2). We also note that there exist solutions of (1) with $r = 6$ and $r \neq 2$. Lander, Parkin and Selfridge [**7**] gave one such numerical solution while Montgomery (as quoted by Guy [**6**, p. 142]) has listed 18 such solutions.

We will first obtain a numerical solution of (1) with $r = 1, 2$ and 6. This solution does not satisfy the three equations given by (2) and thus provides a counterexample asked for by Guy. Next we will use the

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computations for the numerical solution already obtained to show that there exist infinitely many integer solutions of (1) with $r = 1, 2$ and 6, and we also describe a method of generating these solutions.

To solve (1) with $r = 1, 2$ and 6, we write

$$\begin{aligned} a &= 2(\alpha+\beta)m + (\alpha-\beta+t)n, \\ b &= -2\alpha m + (\alpha+\beta+t)n, \\ c &= -2\beta m - (\alpha+\beta-t)n, \\ d &= -2(\alpha+\beta)m + (\alpha-\beta+t)n, \\ e &= 2\alpha m + (\alpha+\beta+t)n, \\ f &= 2\beta m - (\alpha+\beta-t)n. \end{aligned} \tag{3}$$

With these values of a, b, c, d, e and f, it is readily verified that equation (1) holds identically for $r = 1$ and 2. Further,

$$\begin{aligned} &a^6 + b^6 + c^6 - (d^6 + e^6 + f^6) \\ &\quad = 192\alpha\beta(\alpha+\beta)mn(m^2-n^2)[\{10(\alpha^2+\alpha\beta+\beta^2)t + 6\alpha^3 \\ &\qquad + 4\alpha^2\beta - 4\alpha\beta^2 - 6\beta^3\}m^2 + \{5t^3 + 5(\alpha-\beta)t^2 \\ &\qquad + 5(\alpha^2+\beta^2)t + \alpha^3 - \alpha^2\beta + \alpha\beta^2 - \beta^3\}n^2]. \end{aligned}$$

To obtain a nontrivial solution of (1) with $r = 1, 2$ and 6, we must find rational m and n satisfying the equation

$$\begin{aligned} &\{10(\alpha^2+\alpha\beta+\beta^2)t + 6\alpha^3 + 4\alpha^2\beta - 4\alpha\beta^2 - 6\beta^3\}m^2 \\ &+ \{5t^3 + 5(\alpha-\beta)t^2 + 5(\alpha^2+\beta^2)t + \alpha^3 - \alpha^2\beta + \alpha\beta^2 - \beta^3\}n^2 = 0. \end{aligned} \tag{4}$$

This will be possible if and only if there exist rational numbers α, β, s and t such that

$$\begin{aligned} s^2 = &-\{10(\alpha^2+\alpha\beta+\beta^2)t + 6\alpha^3 + 4\alpha^2\beta - 4\alpha\beta^2 - 6\beta^3\} \\ &\times \{5t^3 + 5(\alpha-\beta)t^2 + 5(\alpha^2+\beta^2)t + \alpha^3 - \alpha^2\beta + \alpha\beta^2 - \beta^3\} \end{aligned}$$

or,

$$\begin{aligned} s^2 = &-50(\alpha^2+\alpha\beta+\beta^2)t^4 - 20(4\alpha^3 + \alpha^2\beta - \alpha\beta^2 - 4\beta^3)t^3 \\ &- 20(4\alpha^4 + 2\alpha^3\beta + 3\alpha^2\beta^2 + 2\alpha\beta^3 + 4\beta^4)t^2 \\ &- 20(2\alpha^5 + \alpha^4\beta + \alpha^3\beta^2 - \alpha^2\beta^3 - \alpha\beta^4 - 2\beta^5)t \\ &- 2(3\alpha^6 - \alpha^5\beta - \alpha^4\beta^2 - 2\alpha^3\beta^3 - \alpha^2\beta^4 - \alpha\beta^5 + 3\beta^6). \end{aligned} \tag{5}$$

We now have to find rational numbers α, β, s and t satisfying (5). We must, however, exclude values of α, β and t that satisfy the relation

$$\alpha - \beta + 3t = 0$$

for then we obtain only a trivial solution of (1). It is easily found by trial that when

$$\alpha = 1, \qquad \beta = 7, \qquad s = 138600/529, \qquad t = 38/23, \tag{6}$$

equation (5) is satisfied and, with these values of α, β and t, equation (4) reduces to

$$-600(161m - 33n)(161m + 33n)/12167 = 0.$$

Taking $m = 33$ and $n = 161$, and using the values of α, β and t given by (6), we find from (3) that a solution of (1) with $r = 1, 2$ and 6 (after cancellation of common factors) is given by

$$\begin{aligned} a &= 43, \qquad b = -372, \qquad c = 371, \\ d &= 307, \qquad e = -405, \qquad f = 140. \end{aligned} \tag{7}$$

It is readily verified that this solution does not satisfy the equations (2).

We will now describe a method of generating infinitely many integer solutions of equation (1) with $r = 1, 2$ and 6. In equation (5), we fix $\alpha = 1$ and $\beta = 7$ so that (5) becomes

$$s^2 = -2850t^4 + 28200t^3 - 209100t^2 + 726000t - 666000. \tag{8}$$

It follows from the computations already carried out that $s = 138600/529$, $t = 38/23$, is a solution of (8). We now make the birational transformation

$$\begin{aligned} s &= 10576800Y/(19X^2), \\ t &= (74X - 20095920)/(19X), \end{aligned} \tag{9}$$

when (8) becomes

$$Y^2 = X^3 - 501771X^2 + 61855241760X - 11509611018422400 \tag{10}$$

TABLE OF SOLUTIONS OF $a^r + b^r + c^r = d^r + e^r + f^r, \;\; r = 1, 2, 6$

α	β	γ	δ	**a**	**b**	**c**	**d**	**e**	**f**
1	7	38	23	43	371	−372	140	307	−405
1	10	29	7	271	387	−562	178	461	−543
1	16	87	11	1195	1440	−2179	1229	1408	−2181
1	67	118	5	401	3292	−3519	300	3349	−3475
3	14	49	9	935	996	−1655	459	1388	−1571
3	19	144	19	2759	3270	−5165	1466	4317	−4919
4	5	5	7	83	211	−300	−124	−185	303
4	7	61	37	393	1084	−1351	140	1245	−1259
4	9	39	37	167	699	−764	271	627	−796
4	9	91	81	93	−1076	1115	535	809	−1212
4	19	121	13	421	1676	−1803	1049	1180	−1935
4	21	25	3	1587	1591	−2716	791	2259	−2588
4	39	9	1	251	852	−953	423	724	−997
7	11	76	87	86	3039	−3365	−1174	−2595	3529
11	34	121	13	3642	5569	−8587	−461	−7113	8198
12	−13	137	−27	31	2027	−2124	−300	−1945	2179
12	71	33	1	2091	3587	−4748	2423	3303	−4796
15	17	54	53	9540	32107	−40849	−5948	−33589	40335
16	63	21	1	536	673	−1077	120	−965	977
19	25	54	13	1619	7497	−9452	−3353	−6627	9644
24	47	19	3	381	1592	−2069	−1035	−1181	2120
28	99	21	1	1229	10105	−11940	−4119	−8804	12317
32	43	7	3	23	432	−479	−127	−393	496
41	66	49	3	183	−16190	17375	5854	13877	−18363
43	66	81	5	1322	5235	−6845	−3147	−4139	6998
48	91	9	1	109	728	−753	248	637	−801

Equation (10) represents an elliptic curve and the rational point on the curve (10) corresponding to the values $s = 138600/529$ and $t = 9$ which satisfy (8) is given by

$$X = 23110308/49, \qquad Y = 35910404244/343.$$

As this rational point on the elliptic curve (10) does not have integer coordinates, it follows from the Nagell-Lutz theorem [**8**, p. 56] on elliptic curves that this is not a point of finite order. Thus, there exist infinitely many rational points on the curve (10) and these can be obtained by applying the group law. These infinitely many rational points on (10) correspond to infinitely many rational solutions of equation (5) with $\alpha = 1$ and $\beta = 7$, and these solutions of (5) lead to rational values of m and n satisfying (4). Finally, using (3), infinitely many rational solutions of (1) with $r = 1, 2$ and 6 can be obtained. Solutions in integers are obtained by multiplying by a suitable constant.

While the method described above generates infinitely many integer solutions of (1) with $r = 1, 2$ and 6, the solutions obtained involve large integers. Solutions in smaller integers are more readily obtained by finding, by trial, values of α, β, s and t satisfying equation (5). For finding solutions of equation (5) by trial, we may take α and β to be integers on account of homogeneity, while we write $t = \gamma/\delta$ where γ and δ are integers. A computer search carried out for solutions of equation (5) in the range $4 \le (\alpha + |\beta| + \gamma + |\delta|) \le 200$ yielded a number of sets of values of α, β, γ and δ such that the righthand side of (5) becomes a perfect square and these, in turn, generated 26 distinct solutions of equation (1) with $r = 1, 2$ and 6. The values of $\alpha, \beta, \gamma, \delta$ and the corresponding solutions of (1) with the values of a, b, c, d, e and f suitably rearranged are given in the Table of Solutions. It may also be noted that a number of solutions of (1) are generated by several sets of values of α, β, γ and δ.

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Ambassador, Embassy of India, Kantari Street, Sahmarani Building, P.O. Box 113-5240, Beirut, Lebanon
E-mail address: ajaic203@yahoo.com.